

MATH204 Differential Equation

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Ordinary Differential Equation Systems with Constant Coefficients

Chapter 6

- Introduction
 - 1- The D operator
- Elimination Method.

The D operator

Definition

A **differential operator** is an operator defined as a function of the differentiation operator. It is helpful, as a matter of notation first, to consider differentiation as an abstract operation, accepting a function and returning another (in the style of a higher-order function in computer science).

Here we will use the following notations:

$$D = d/dt, D^2 = d^2/d^2t, \dots, D^n = d^n,$$

where t is the independent variable.

Simple Equivalents:

- D_u means $D_u \equiv \frac{du}{dt}$ but $uD \equiv u \frac{d}{dt}$.
- $D_y^2 \equiv D \times D_y \equiv \frac{d}{dt} \left(\frac{dy}{dt} \right)$.
- Similarly $D^2 \equiv \frac{d^2}{dt^2}$ and $D^3 \equiv \frac{d^3}{dt^3}$.

The following differential equation:

$$3\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 2y = 0,$$

may be expressed as:

$$(3D^2 + 7D + 2)y = 0$$

or

$$3D^2 + 7D + 2 = 0$$

Elimination Method

Definition

The elimination method consists in bringing the system of n differential equations into a single differential equation of order n .

In the elimination method you either add or subtract the system of differential equations into a single differential equation.

Example (1) Find the general solution of the system

$$\begin{cases} \frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} + \frac{dx}{dt} - 3\frac{dy}{dt} - x + 2y = 0, \\ \frac{dx}{dt} + 2\frac{dy}{dt} + 2x - 4y = 0. \end{cases} \quad (5)$$

Solution We first write the system (5) in its operator form

$$\begin{cases} (D^2 + D - 1)[x] + (D^2 - 3D + 2)[y] = 0, \\ (D + 2)[x] + (2D - 4)[y] = 0. \end{cases} \quad (6)$$

To eliminate x , we apply the operator $D^2 + D - 1$ to the second equation in (6) and $D + 2$ to the first one and subtract the first from the second, we get

$$((D^2 + D - 1)(2D - 4) - (D + 2)(D^2 - 3D + 2)) [y] = 0,$$

or

$$(D^3 - D^2 - 2D) [y] = 0 \Leftrightarrow y''' - y'' - 2y' = 0. \quad (7)$$

The characteristic equation for (7) is

$$m^3 - m^2 - 2m = 0$$

whose roots are $0, 2, -1$. Thus

$$y(t) = c_1 + c_2e^{2t} + c_3e^{-t}.$$

Substitution of this last expression in the second equation of (5) gives

$$x' + 2x = 4c_1 + 6c_3e^{-t}. \quad (8)$$

Equation (8) is a linear equation, we solve it to obtain

$$x(t) = 2c_1 + 6c_3e^{-t} + c_4e^{-2t}.$$

To eliminate the constant c_4 from the solution $x(t)$, we replace $x(t)$ and $y(t)$ in the first equation in (6) and we find $c_4 = 0$. Consequently

$$x(t) = 2c_1 + 6c_3e^{-t}.$$

Example (2) Solve the system

$$\begin{cases} x'' + y' - 3x' + 2x - y = 0, \\ x' + y' - 2x + y = 0. \end{cases} \quad (12)$$

Solution

We first write the system (12) in its operator form

$$\begin{cases} (D^2 - 3D + 2)[x] + (D - 1)[y] = 0, \\ (D - 2)[x] + (D + 1)[y] = 0. \end{cases} \quad (13)$$

To eliminate y , we apply $(D + 1)$ to the first equation in (13) and $(D - 1)$ to the second and subtract the first from the second, we get

$$(D^3 - 3D^2 + 2D)[x] = 0 \Leftrightarrow x''' - 3x'' + 2x' = 0. \quad (14)$$

The general solution of (14) is

$$x(t) = c_1 + c_2 e^{2t} + c_3 e^t.$$

From equation (2) in (12), we have

$$y' + y = c_3 e^t + 2c_1. \quad (15)$$

We solve the linear equation (15) to obtain

$$y(t) = 2c_1 + \frac{c_3}{2}e^t + c_4e^{-t}.$$

To eliminate the extraneous constant c_4 , we substitute for $x(t)$ and $y(t)$ in the first equation in (12) and find that $c_4 = 0$. Hence

$$y(t) = 2c_1 + \frac{c_3}{2}e^t.$$

Example (3) Find the general solution of the system

$$\begin{cases} \frac{1}{2}x''' - y'' = \cos t, \\ \frac{1}{2}x'' + x + y' = -\cos t \end{cases} \quad (19)$$

Solution We write the system (19) in the operator form

$$\begin{cases} \frac{1}{2}D^3[x] - D^2[y] = \cos t, \\ (\frac{1}{2}D^2 + 1)[x] + D[y] = -\cos t. \end{cases}$$

To eliminate y , we apply the operator D to the second equation and then sum both equations

$$(D^3 + D)[x] = \sin t + \cos t \Leftrightarrow x''' + x' = \sin t + \cos t. \quad (20)$$

By using the method of undetermined coefficients method, we find that the general solution of equation (20) is given by

$$x(t) = c_1 + c_2 \cos t + c_3 \sin t - \frac{t}{2}(\cos t + \sin t). \quad (21)$$

Substitution of the expression (21) in the second equation of (19) gives

$$y(t) = \left(\frac{3}{2} + \frac{c_3}{2} - \frac{t}{4}\right) \cos t + \left(-\frac{1}{4} - \frac{c_2}{2} + \frac{t}{4}\right) \sin t - c_1 t + c_4.$$

To eliminate the extraneous constant c_4 , we substitute for $x(t)$ and $y(t)$ in the second equation in (19) and find that $c_4 = 0$. Hence

$$y(t) = \left(\frac{3}{2} + \frac{c_3}{2} - \frac{t}{4}\right) \cos t + \left(-\frac{1}{4} - \frac{c_2}{2} + \frac{t}{4}\right) \sin t - c_1 t.$$