MATH204 Differential Equation

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Linear differential equations of higher order

Chapter 4

- General Solution of homogeneous linear differential equations
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Homogeneous Linear Differential Equations with Constant Coefficients

The linear differential equations with Constant Coefficients has the general form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0, \qquad (1)$$

which is a homogeneous linear DE with **constant real coefficients**, where each coefficient $a_i, 1 \leq i \leq n$ is real constant and $a_n \neq 0$.

Definition

The polynomial

$$f(m) = a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0, \qquad (2)$$

is called the characteristic polynomial for equation (1), and $f(m) = 0$ is called the characteristic equation of the linear differential equations with constant coefficients (1).

We conclude that if m is a root of equation (2), then

$$y = e^{mx}$$

is a solution of the differential equation (1). Also, Equation (2) has n roots.

Let us summarize the method to solve the differential equation (1):

(1) If all the roots of the characteristic equation are **real roots** then:

(i) If the roots are distinct (i.e. $m_1 \neq m_2 \neq m_3 \neq \cdots \neq m_n$), then the solution of the differential equation (1) is given by

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

(ii) If the roots are equal (i.e. $m_1 = m_2 = m_3 = \cdots = m_n$) (i.e. $m = m_i$ is a root of multiplicity n), then the solution of the differential equation (1) is given by

$$y = c_1 e^{mx} + c_2 x e^{mx} + c_3 x^2 e^{mx} + \cdots + c_n x^{n-1} e^{mx}$$

$$y = (c_1 + c_2 x + c_3 x^2 + \cdots + c_n x^{n-1}) e^{mx}$$

Examples

1- Solve the differential equation

$$y'' - y = 0.$$

2- Find the general solution of the differential equation

$$y''' - 6y'' + 11y' - 6y = 0 \ .$$

3- Solve the differential equation

$$y'' - 2y' + y = 0.$$

4- Solve the differential equation

$$y''' - 3y'' + 3y' - y = 0$$

Now we see the second case
(2) If the characteristic equation has **complex conjugate roots** such as

$$m = \alpha \mp i\beta$$

then he solution of the differential equation of second order is given by

$$y = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$

Remember:

$$1)\ \sqrt{-1} = i$$

$$2)\ x = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

to find the roots of Quadratic equation

$$ax^2 + bx + c = 0$$

Examples

1- Solve the differential equation

$$y'' + 4y' + 5y = 0.$$

2- Solve the differential equation

$$y^{(5)} - 3y^{(4)} + 4y''' - 4y'' + 3y' - y = 0.$$

3- Solve the initial value problem (IVP)

$$\begin{cases} y'' + y' + y = 0 \\ y(0) = 1 \quad , \quad y'(0) = \sqrt{3}. \end{cases}$$

Cauchy-Euler Differential Equation

A Cauchy-Euler differential equation is in the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = 0, \qquad (3)$$

where each coefficient $a_i, 1 \leq i \leq n$ are constants and $a_n \neq 0$ i.e. the coefficient $a_n x^n$ should never be zero. Equation (3) is on the interval either $(0, \infty)$ or $(-\infty, 0)$.

Euler differential equation is probably the simplest type of linear differential equation with variable coefficients.

The most common Cauchy-Euler equation is the second-order equation, appearing in a number of physics and engineering applications, such as when solving Laplace's equation in polar coordinates. It is given by the equation

$$x^2 \frac{d^2 y}{dx^2} + ax \frac{dy}{dx} + by = 0 \qquad (4)$$

To solve the Cauchy-Euler differential equation, we assume that $y = x^m$, where $x > 0$ and m is a root of a polynomial equation.

Example(1) Solve the Cauchy-Euler differential equation

$$x^2 \frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0.$$

Solution We substitute

$$y = x^m \implies y' = mx^{m-1} \implies y'' = m(m-1)x^{m-2}$$

in the differential equation, we obtain

$$x^2[m(m-1)x^{m-2}] + ax[mx^{m-1}] + bx^m = 0$$
$$x^m(m^2 - m) + amx^m + bx^m = 0$$
$$x^m[(m^2 - m) + am + b] = 0$$
$$x^m[m^2 + (1-a)m + b] = 0.$$

Since $x^m \neq 0$, then we have

$$m^2 + (1-a)m + b = 0$$

We then can solve for m. There are three particular cases of interest:
Case 1: Two distinct roots, m_1 and m_2. Thus, the solution is given by

$$y = c_1 x^{m_1} + c_2 x^{m_2}.$$

Case 2: One real repeated root, m. Thus, the solution is given by

$$y = c_1 x^{m} \ln(x) + c_2 x^{m}.$$

Case 3: Complex roots, $\alpha \pm i\beta$. Thus, the solution is given by

$$y = c_1 x^{\alpha} \cos\left(\beta \ln(x)\right) + c_2 x^{\alpha} \sin\left(\beta \ln(x)\right).$$

Example (2) Solve the Euler differential equation

$$2x^2y'' - 3xy' - 3y = 0. \qquad (5)$$

For $x > 0$.
Solution) We substitute

$$y = x^m \implies y' = mx^{m-1} \implies y'' = m(m-1)x^{m-2}$$

in the differential equation, we obtain

$$2x^2[m(m-1)x^{m-2}] - 3x[mx^{m-1}] - x^m = 0$$
$$x^m(2m^2 - 2m) - 3mx^m - 3x^m = 0$$
$$x^m[2m^2 - 2m - 3m - 3] = 0$$
$$x^m[2m^2 - 5m - 3] = 0.$$

Since $x^m \neq 0$, then we have

$$2m^2 - 5m - 3 = 0$$

So the roots of this equation are $m_1 = -\frac{1}{2}$, $m_2 = 3$ .Thus, from case 1 we have the solution is given by

$$y(x) = c_1 x^{-\frac{1}{2}} + c_2 x^3.$$

which is the general solution.

Example (3)
Find the general of the differential equation

$$x^2y'' - 3xy' + 13y = 0 \quad ; \quad x > 0.$$

Solution Substituting $y = x^m$ in the equation, we obtain

$$m(m-1) - 3m + 13 = m^2 - 4m + 13 = 0.$$

Then we have two complex roots $m = 3 \mp 3i$ (case 3), hence the the general of the differential equationis

$$y = c_1x^3\cos(3\ln x) + c_2x^3\sin(3\ln x) \quad ; \quad x > 0.$$

If we suppose $x < 0$, then the general of the differential equation is

$$y = c_1(-x)^3\cos(3\ln(-x)) + c_2(-x)^3\sin(3\ln(-x)) \quad ; \quad x < 0.$$

Example (4). Find the general solution of the differential equation

$$x^4y^{(4)} - 5x^3y''' + 3x^2y'' - 6xy' + 6y = 0 \quad ; \quad x > 0.$$

Solution Substituting $y = x^m$ in the equation, we obtain

$$m(m-1)(m-2)(m-3) - 5m(m-1)(m-2) + 3m(m-1) - 6m + 6 = 0.$$

This implies that

$$(m-1)(m-2)(m^2 - 8m + 3) = 0.$$

The roots of this equation are $m = 1$, $m = 2$, and $m = 4 \mp \sqrt{13}$, then the general solution of the differential equation is

$$y = c_1x + c_2x^2 + c_3x^{4+\sqrt{13}} + c_4x^{4-\sqrt{13}} \quad ; \quad x > 0.$$

Example (5) Find the general solution of the differential equation

$$x^5 y^{(5)} - 2x^3 y''' + 4x^2 y'' = 0 \quad ; \quad x < 0.$$

Solution Substituting $y = x^m$ in the equation, we obtain

$$m(m-1)(m-2)(m-3)(m-4) - 2m(m-1)(m-2) + 4m(m-1) = 0,$$

$$m(m-1)(m^3 - 9m^2 + 24m - 20) = m(m-1)(m-2)^2(m-5) = 0.$$

So the roots of this equation are $m = 0$, $m = 1$, $m = 2$ repeated two times and $m = 5$, then the general of the differential equation is

$$y = c_1 + c_2(-x) + c_3(-x)^2 + c_4(-x)^2 \ln(-x) + c_5(-x)^5.$$

General Solutions of Nonhomogeneous Linear DE

Nonhomogeneous linear n-th order ODE takes the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \qquad (6)$$

where $a_n(x),\ a_{n-1}(x),\ a_1(x)$ and $a_0(x)$ are functions of $x \in \mathrm{I} = (a, b)$, such that $a_n(x) \neq 0$ for all $x \in I$, and $g(x) \neq 0$.

Idea:

- Find the general solution y_c to the homogeneous equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

- Find a solution y_p to the nonhomogeneous equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

- The general solution $y = y_c + y_p$.

Undetermined coefficients

Let us take an example

Examples

1- Find the general solution of the differential equation :

$$y'' - y = -2x^2 + 5 + 2e^x. \tag{1}$$

2- Find only the form of particular solution of the differential equation :

$$y'' - 2y' - 3 = 3x^2e^x + e^{2x} + x\sin(x) + (2 + 3x). \tag{2}$$

3- Find the general solution of the differential equation :

$$y'' - 2y' + y = 2e^x - 3e^{-x}. \tag{3}$$

Variation of Parameters

This method is used to solve to determine the particular solution y_p of nonhomogeneous differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \quad (7)$$

If we have the nonhomogeneous differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x), \quad (8)$$

which has the particular solution

$$y_p = y_1 u_1 + y_2 u_2,$$

where y_1 and y_2 are the first and the second solution of the homogeneous differential equation, respectively.

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \quad (9)$$

Here we will explain the method to find u_1 and u_2. So, if we have y_1 & y_2 , then we will determine as below

$$W(x, y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1',$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ g(x) & y_2' \end{vmatrix} = -y_2 g(x),$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g(x) \end{vmatrix} = y_1 g(x).$$

Thus,

$$u_1' = \frac{W_1}{W}$$

and

$$u_2' = \frac{W_2}{W}.$$

Examples

1- Solve the differential equation

$$y'' + y = \csc x \quad ; \quad 0 < x < \pi.$$

2- Solve the differential equation

$$y'' - 4y' + 4y = (x+1)e^{2x}.$$

3- Solve the Differential equation

$$y'' - 3y' + 2y = \frac{1}{1+e^{-x}}.$$

4- Find the general solution of the differential equation

$$y''' + y' = \tan x \quad ; \quad 0 < x < \frac{\pi}{2}.$$

5- Find the solution of the initial value problem (IVP)

$$\begin{cases} 2x^2y'' + xy' - 3y = x^{-3} \quad ; \quad x > 0 \\ y(1) = 1 \quad , \quad y'(1) = -1. \end{cases}$$