MATH204 Differential Equation

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Linear differential equations of higher order

Chapter 4

- General Solution of homogeneous linear differential equations
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General Solution of homogeneous linear DEs

Definition

The general linear differential equations of order n is an equation that can be written

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = R(x), \quad (1)$$

where $a_n(x),\ a_{n-1}(x),\ a_1(x)$ and $a_0(x)$ are functions of $x \in \mathrm{I} = (a, b)$, and they are called **coefficients**.
Equation (1) is called homogeneous linear differential equation if the function $R(x)$ is zero for all $x \in (a, b)$.
If $R(x)$ is not equal to zero on I, the equation (1) is called non-homogeneous linear differential equation.

Initial-Value Problem (IVP)

An n-th order initial-value problem associate with (1) takes the form:
Solve:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = R(x),$$

subject to:

$$y(x_0) = y_0,\ y'(x_0) = y_1,\ y''(x_0) = y_2,\ \ \ldots,\ y^{n-1}(x_0) = y_{n-1}. \qquad (2)$$

Here (2) is a set of **initial conditions**.

Boundary-Value Problem (BVP)

Remark (Initial vs. Boundary Conditions):

Initial Conditions: all conditions are at the **same** $x = x_0$.
Boundary Conditions: conditions can be at **different** x.

Remark (Number of Initial/Boundary Conditions):

Usually a n-th order ODE requires n initial/boundary conditions to specify an unique solution.

Remark (Order of the derivatives in the conditions):

Initial/boundary conditions can be the value or the function of 0-th to $(n-1)$-th order derivatives, where n is the order of the ODE.

Example (Second-Order ODE)

Consider the following second-order ODE

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{d^{'}y}{dx^{'}} + a_0(x)y = R(x), \tag{3}$$

- IVP: solve (3) s.t. $y(x_0) = y_0; y'(x_0) = y1$.
- BVP: solve (3) s.t. $y(a) = y_0; y(b) = y1$.

Existence and Uniqueness of the Solution to an IVP

Theorem:

For the given linear differential equations of order n

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = R(x), \quad (4)$$

which is normal on an interval I. Subject to

$$y(x_0) = y_0,\ y'(x_0) = y_1,\ y''(x_0) = y_2,\ \ldots,\ y^{n-1}(x_0) = y_{n-1}. \quad (5)$$

If $a_n(x), a_{n-1}(x), \ldots, a_0(x)$ and $R(x)$ are all continuous on an interval I, $a_n(x)$ is not a zero function on I, and the initial point $x_0 \in I$, then the above IVP has a unique solution in I.

Example (1) Discuss the Existence of unique solution of IVP

$$\begin{cases} (x^2+1)y'' + x^2y' + 5y = \cos(x) \\ y(3) = 2 \quad , \quad y'(3) = 1. \end{cases}$$

Solution The functions

$$a_2(x) = x^2+1, a_1(x) = x^2, a_0(x) = 5.$$

and

$$R(x) = \cos(x).$$

are continuous on $I = \mathbb{R} = (-\infty \ , \ +\infty)$ and $a_2(x) \neq 0$ for all $x \in \mathbb{R}$, the point $x_0 = 3 \in I$. Then the previous Theorem assures that the IVP has a unique solution on $\mathbb{R}$.

Example(2) Find an interval I for which the initial values problem (IVP)

$$\begin{cases} x^2y'' + \frac{x}{\sqrt{2-x}}y' + \frac{2}{\sqrt{x}}y = 0 \\ y(1) = 0 \quad , \quad y'(1) = 1. \end{cases} .$$

has a unique solution around $x_0 = 1$.

Solution The function

$$a_2(x) = x^2,$$

is continuous on $\mathbb{R}$ and $a_2(x) \neq 0$ if $x > 0$ or $x < 0$. But $x_0 = 1 \in I_1 = (0 \ , \ \infty \ )$. The function

$$a_1(x) = \frac{x}{\sqrt{2-x}} \ ,$$

is continuous on $I_2 = (-\infty \ \ , 2)$ and the function

$$a_0(x) = .\frac{2}{\sqrt{x}},$$

is continuous on $I_1 = (0 \ , \ \infty \ )$

Then the (IVP) has a unique solution on $I_1 \cap I_2 = (0\ ,\ 2) = I$. We can take any interval $I_3 \subset (0\ ,\ 2)$ such that $x_0 = 1 \in I_3$. So I is that the largest interval for which the (IVP) has a unique solution.

Example(3) Find an interval I for which the IVP

$$\begin{cases} (x-1)(x-3)y'' + xy' + y = x^2 \\ y(2) = 1 \quad , \quad y'(2) = 0 \end{cases} .$$

has a unique solution about $x_0 = 2$.

Solution The functions

$$a_2(x) = (x-1)(x-3) \ , \ a_1(x) = x \ , a_0(x) = 1 \ , R(x) = x^2,$$

are continuous on $\mathbb{R}$. But $a_2(x) \neq 0$ if $x \in (-\infty \ , 1)$ or $x \in (1 \ , 3)$ or $x \in (3 \ , \infty)$. As $x_0 = 2$ so we take $I = (1 \ , \ 3)$. Then the IVP has a unique solution on $I = (1 \ , \ 3)$

Linear Dependence and Independence of Functions

Definition:

A set of functions $\{f_1(x), f_2(x). \dots, f_n(x)\}$ are **linearly dependent** on an interval I if $\exists\ c_1, c_2, \dots, c_n$ not all zero i.e. $(c_1, c_2, \dots, c_n) \neq (0, 0, \dots, 0)$ such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0, \quad \forall\ x \in I$$

that is, the linear combination is a zero function.
If the set of functions is not linearly dependent, it is **linearly independent**,
i.e. when $c_1, c_2, \dots, c_n$ all zero i.e. $(c_1, c_2, \dots, c_n) = (0, 0, \dots, 0)$.

Examples:

1- Show that $f_1(x) = \cos(2x)$, $f_2(x) = 1$, $f_3(x) = \cos^2(x)$ are linearly dependent on $\mathbb{R}$.

2- Show that $f_1(x) = 1$, $f_2(x) = \sec^2(x)$ and $f_3(x) = \tan^2(x)$ are linearly dependent on $(0 \ , \ \frac{\pi}{2})$.

3- Show that $f_1(x) = x$ and $f_2(x) = x^2$ are linearly independent on $I = [-1 \ , 1]$.

4- show that $f_1(x) = \sin(x)$, $f_2(x) = \sin(2x)$ are linearly independent on $I = [0 \ , \pi)$.

5- Show that $f_1(x) = x^2$ and $f_2(x) = x\,|x|$ are

(i) linearly dependent on $[0 \ , \ 1]$

(ii) linearly independent on $[-1 \ , \ 1]$

Criterion of Linearly Independent Solutions

Consider the homogeneous linear n-th order DE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0,$$

Given n solutions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$, we would like to test if they are independent or not.

Note: In Linear Algebra, to test if n vectors $\{v_1, v_2, \ldots, v_n\}$ are linearly independent, we can compute the determinant of the matrix.

$$\mathrm{V} := [v_1 \ v_2 \ \ldots \ v_n].$$

If the determinant of $\mathrm{V} = 0$, they are linearly dependent; if the determinant of $\mathrm{V} \neq 0$, they are linearly independent.

Definition

For n functions $W(f_1, f_2, \ldots, f_n)$ which are $n-1$ times differentiable on an interval I, the **Wronskian** $W(f_1, f_2, \ldots, f_n)$ as a function on I is defined by

$$W(f_1, f_2\ , ..., f_n) = \begin{vmatrix} f_1 & f_2 & ... & f_n \\ f_1' & f_2' & ... & f_n' \\ f_1'' & f_2'' & ... & f_n'' \\ ... & ... & ... & ... \\ f_1^{n-1} & f_2^{n-1} & .... & f_n^{n-1} \end{vmatrix}$$

To test the linear independence of n solutions $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ to

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0, \qquad (6)$$

we can use the following theorem.

Theorem

Let $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ be n solutions to the homogeneous linear DE (6) on an interval I. They are **linearly independent** on I

$$\Longleftrightarrow W(f_1, f_2\ , ..., f_n) := \begin{vmatrix} f_1 & f_2 & ... & f_n \\ f_1' & f_2' & ... & f_n' \\ f_1'' & f_2'' & ... & f_n'' \\ ... & ... & ... & ... \\ f_1^{n-1} & f_2^{n-1} & .... & f_n^{n-1} \end{vmatrix} \neq 0.$$

Examples

1- Prove that $f_1(x) = x^2$, $f_2(x) = x^2 \ln(x)$ are linearly independent on $(0 \ , \ \infty)$.

2- It is easy to see that the functions $y_1 = x$, $y_2 = x^2$, and $y_3 = x^3$ are solutions of the differential equation

$$x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0.$$

Show that y_1 , y_2 and y_3 are linearly independent on $(0, \infty)$.

Fundamental Set of Solutions

Definition

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0, \qquad (7)$$

Any set $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ of n linearly independent solutions to the homogeneous linear n-th order DE (7) on an interval I is called a **fundamental set of solutions**.

Theorem

Let $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ be a fundamental set of solutions to the homogeneous linear n-th order DE (7) on an interval I. Then the **general solution to** (7) is

$$y(x) = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x),$$

where $\{c_i \mid (i = 1, 2, \ldots, n)\}$ are arbitrary constants.

Examples

1- Verify that $y_1 = e^{2x}$ and $y_2 = e^{-3x}$ form a fundamental set of solutions of the differential equation

$$y'' + y' - 6y = 0.$$

and find the general solution.

2- It is easy to see that the functions

$$y_1 = e^{x}, y_2 = e^{2x}, \text{ and } y_3 = e^{3x}$$

are solutions of the differential equation

$$y''' - 6y'' + 11y' - 6y = 0.$$

Find the general solution of the differential equation.

3- Prove that

$$y_1 = x^3 e^{x}, \text{ and } y_2 = e^{x}.$$

are solutions of the differential equation

$$xy'' - 2(x+1)y' + (x+2)y = 0$$

where $x > 0$. Find also the general solution of the differential equation.

Reduction of order Method (when one solution is given)

It is employed when one solution $y_1(x)$ is known and a second linearly independent solution $y_2(x)$ is desired. The method also applies to n-th order equations.

Suppose that $y_1(x)$ is a non-zero solution of the equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0, \tag{8}$$

where $a_0(x),\ a_1(x)$ and $a_2(x)$ are continuous functions defined on interval I sauch that $a_2(x) \neq 0$ for all $x \in I$.

The method of reduction of order is used to obtain a second linearly independent $y_2(x)$ solution to this differential equation (8) using our one known solution.

We suppose that the solution of (8) is in the form

$$y = u(x)y_1,$$

where u is a function of x and which will be determined and satisfies a linear second-order differential equation (8) by using the following method

$$y = u(x)y_1 \Rightarrow y' = u'y_1 + y_1'u \Rightarrow y'' = u''y_1 + 2u'y_1' + y_1''u.$$

It is best to describe the procedure with a concrete example.

Example (1) If

$$y_1 = \frac{\sin x}{\sqrt{x}}.$$

is a solution of the differential equation

$$4x^2y'' + 4xy' + (4x^2 - 1)y = 0 \quad on \ 0 < x < \pi.$$

then find the general solution of the differential equation..

Solution The solution of the differential equation is of the form $y = u(x)y_1$ or

$$y = \frac{\sin x}{\sqrt{x}}u \ = (\sin x)\,(x)^{\frac{-1}{2}}\,u,$$

hence

$$y' = (\cos x)(x)^{\frac{-1}{2}}u - \frac{1}{2}\sin x(x)^{\frac{-3}{2}}u + \sin x(x)^{\frac{-1}{2}}u',$$

$$\begin{aligned} y'' \quad &= \quad -\sin x(x)^{\frac{-1}{2}}u - \cos x(x)^{\frac{-3}{2}}u + 2\cos x(x)^{\frac{-1}{2}}u' \\ &\quad + \frac{3}{4}\sin x(x)^{\frac{-5}{2}}u - \sin x(x)^{\frac{-3}{2}}u' + \sin x(x)^{\frac{-1}{2}}u'' \end{aligned}$$

we substitute y , y', and y'' in the arbitrary constant we obtain

$$4x^{\frac{3}{2}}\sin x u'' + \left(8x^{\frac{3}{2}}\cos x\right)u' = 0,$$

hence

$$\sin x u'' + 2\cos x u' = 0.$$

To solve this differential equation we put $w = u'$, then we have $w' = u''$. Then

$$\int \frac{dw}{w}dx + \int \frac{2\cos x}{\sin x}dx = 0,$$

hence

$$u' = w = \frac{c_1}{\sin^2 x},$$

where $c_1 \neq 0$ is an arbitrary constant. So we have $u = -c_1\cot x + c_2$, hence

$$y = y_1 u = \frac{\sin x}{\sqrt{x}}(-c_1\cot x + c_2),$$

or

$$y = c_3\frac{\cos x}{\sqrt{x}} + c_2\frac{\sin x}{\sqrt{x}},$$

finally we have

$$y = c_2 y_1 + c_3 y_2,$$

where $c_3 = -c_1$ and c_2 are arbitrary constants, is the general solution of the differential equation and we can prove that

$$y_1 = \frac{\sin x}{\sqrt{x}} \text{ and } y_2 = \frac{\cos x}{\sqrt{x}}$$

are linearly independent on solutions $(0, \pi)$.

General case of Equation (8)

Equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0,$$

can be written as the form

$$y'' + p(x)y' + q(x)y = 0, \tag{9}$$

where

$$p(x) = \frac{a_1(x)}{a_2(x)},$$

and

$$q(x) = \frac{a_0(x)}{a_2(x)}.$$

Also, let us suppose that y_1 is a known solution of (9) on I and $y_1(x) \neq 0$ for all $x \in I$.
Thus the second solution of (9) y_2 can be given from

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2}\, dx. \tag{10}$$