MATH204 Differential Equation

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Application of First Order Differential Equation

Chapter 3

- Orthogonal Trajectories.
- Growth and Decay.
- Radio active Decay.
- Newton's Law of cooling.

Orthogonal Trajectories

suppose that we have a family of curves given by

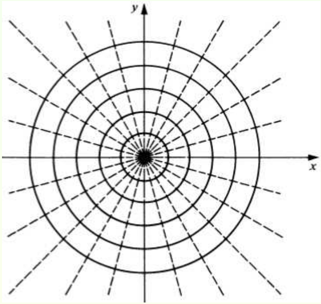
$$F(x, y, c) = 0, \tag{1}$$

and another family of curves given by

$$G(x, y, k) = 0, \tag{2}$$

such that at any intersection of a curve of the family $F(x, y, c)$ with a curve of the family $G(x, y, k)$, the tangents of the curves are perpendicular.

Therefore, are two families of curves that always intersect perpendicularly. **Example** Consider the family of circles represented by $x^2 + y^2 = c$, with center at the origin, and the family $y = kx$ of straight lines through the origin, are **orthogonal trajectories** of each other, as shown in the figure.



How to Find Orthogonal Trajectories

To find the orthogonal trajectories of the family

$$F(x, y, c) = 0, \tag{3}$$

Step1: Differentiate (3) implicitly with respect to x to get a relation of the form (3)

$$g\left(x, y, \frac{dy}{dx}, c\right); \tag{4}$$

Step2: Eliminate the parameter c from (3), and (4) to obtain the differential equation

$$F\left(x, y, \frac{dy}{dx}\right) \tag{5}$$

corresponding to the first family (3);
Step3: Replace $\frac{dy}{dx}$ by $\frac{-1}{\frac{dy}{dx}}$ in (5) to obtain the differential equation

$$H\left(x, y, \frac{dy}{dx}\right) \tag{6}$$

of the orthogonal trajectories (as shown in the figure below);

Step4: General solution of (6) gives the required orthogonal trajectories.

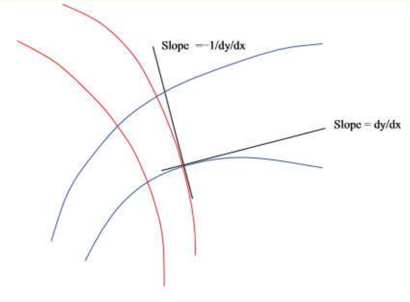


Figure: Orthogonal trajectories

Example 1

Find the orthogonal trajectories of family of straight lines through the origin.

Solution: The family of straight lines through the origin is given by

$$y = kx, \tag{7}$$

To find the orthogonal trajectories, we follow the previous four steps:
Step1: Differentiate (7) implicitly with respect to x, we obtain

$$\frac{dy}{dx} = k, \tag{8}$$

Step2: Eliminate the parameter k from (7), and (8), we obtain the differential equation

$$\frac{dy}{dx} = \frac{y}{x}, \tag{9}$$

This gives the differential equation of the family (7). **Step3:** Replacing $\frac{dy}{dx}$ by $\frac{-1}{\frac{dy}{dx}}$ in (9) we obtain

$$\frac{dy}{dx} = -\frac{x}{y}, \tag{10}$$

Step4: Solving differential equation (10), we obtain

$$x^2 + y^2 = c. \tag{11}$$

Thus, the orthogonal trajectories of family of straight lines through the origin is given by (11). Note that (11) is the family of circles with centre at the origin.

Examples

Find the orthogonal trajectories of the family

1- $cx^2 - y^2 = 1$

2- $y^2 = cx^3$

3- $x^3 + 3xy^2 = c$

Growth and Decay

In many natural phenomena, quantities grow or decay at a rate proportional to their size. For example, if $y = y(t)$ is the number of individuals in a population of animals or bacteria at time t, then it seems reasonable to expect that the rate of growth $y'(t)$ is proportional to the population $y(t)$; that is, $y'(t) = ky(t)$ for some constant k.
The mathematical model given by the equation $y'(t) = ky(t)$ can be predicted what actually happens fairly accurately under ideal conditions (unlimited environment, adequate nutrition, immunity to disease).
Also, we can see many examples in nuclear physics, chemistry and finance.

In general, if y(t) is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to its size y(t) at any time, then

$$\frac{dy}{dt} = ky, \tag{12}$$

where k is a constant, and Equation (12) is sometimes called the **law of natural growth** (if $k > 0$) or the **law of natural decay** (if $k < 0$). Thus, the law of Exponential Growth and Decay can be written as

$$y = ce^{kt},$$

c is the initial value and can be found from the initial condition $y(t_0) = y_0$ k is the constant of proportionality, which is can be found from an additional condition which might be given in the problem.
Note: If $k > 0$ the exponential growth occurs, and if $k < 0$ the exponential decay occurs.

To proof that let us take some initial time quantity is known and is $y(t_0) = y_0$. The differential equation

$$\frac{dy}{dt} = ky$$

is separable differential equation and we can solve it.

$$\frac{dy}{dt} = ky$$

$$\frac{dy}{y} = kdt$$

$$\int \frac{dy}{y} = \int kdt$$

$$\ln y = kt + c$$

$$e^{\ln y} = e^{kt+c}$$

$$y = e^c e^{kt}$$

$$y = c_1 e^{kt};\ c_1 = \pm e^c$$

Using the initial condition $y(0) = y_0$, i.e $t_0 = 0$, $y = y_0$

$$y_0 = c_1 e^0 \;\Rightarrow\; y_0 = c_1$$

$$y = y_0 e^{kt}.$$

To find the additional constant k we need additional condition which might be given in the problem.

Example 1

A certain culture of bacteria grows at rate proportional to its size. If the size doubles in 4 days, find the time required for the culture to increase to 10 times to its original size.

Solution

Examples

2- Use the fact that the world population was 2560 million in 1950 and 3040 million in 1960 to model the population of the world in the second half of the 20th century. (Assume that the growth rate is proportional to the population size.) What is the relative growth rate k? Use the model to estimate the world population in 1993 and to predict the population in the year 2020.

3- The population of a town at a rate proportional to the population size at any time. Its initial population of 1000 increases by 10% in 5 years. What will be the population after 50 years?
(Hint: $p(0) = 1000$, $p(5) = 1000 + (1000/10) = 1100$)

Radio active Decay

Example 3 A radio active material has an initial mass 100mg. After two years it is left to 75mg. Find the amount of the material at any time. What is the period of its half-life?

Solution

Newton's Law of cooling

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that this difference is not too large.

If we let $T(t)$ be the temperature of the object at time t and T_s be the temperature of the surroundings, then we can formulate Newton's Law of Cooling as a differential equation:

$$\frac{dT}{dt} = k(T - T_s), \tag{13}$$

where k is a constant of proportionality.

Examples

1- A glass of a hot water has an initial temperature $80°$C, placed in a room where the temperature is $30°$C. After one minute the water temperature drops to $70°$C. What will be the temperature after 3 minutes? At what time the water cools down to $40°$C?

2- A bottle of soda at room temperature $72°$F is placed in a refrigerator where the temperature is $44°$F. After half an hour the soda has cooled to $61°$F.
(a) What is the temperature of the soda after another half hour?
(b) How long does it take for the soda to cool to $50°$F?