MATH204 Differential Equation

Dr. Bandar Al-Mohsin

School of Mathematics, KSU

Chapter 2

- Integrating Factor
- The General Solution of a Linear Differential Equations
- Bernoulli's Equation

Integrating Factor

Consider a first order differential equation

$$M(x,y)dx + N(x,y)dy = 0, \tag{1}$$

where M, N and $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous on a certain region R in xy-plane. Suppose that the equation (1) is **not exact**, i.e

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

Definition: A function $\mu(x,y)$ is called an **integrating factor** of (1) if the differential equation

$$(\mu M)dx + (\mu N)dy = 0, \tag{2}$$

is **exact**, i.e

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}. \tag{3}$$

In other words, if the equation (1) is **not exact**, we can often make it so by multiplying throughout by an $\mu(x, y)$ and the finding $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$.
The integrating factors are able to be determined by solving

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

for $\mu \neq 0$ for all $(x, y) \in R$.
The integrating factor will be in one of the following forms

1. $\mu = \mu(x)$
2. $\mu = \mu(y)$
3. $\mu = \mu(x, y) = x^m y^n$

We can rewrite the equation (3) as follows:

$$N\mu_x - M\mu_y = (M_y - N_x)\mu \tag{4}$$

In general, it is very difficult to solve the equation (4). In this section we will only consider that μ is a one variable function (x or y, not both). There are two cases:

1. If μ depends on x $\mu_x = \frac{d\mu}{dx}$. Then $\mu_y = 0$, so the equation (4) becomes
$$\frac{1}{\mu}\mu_x = \frac{1}{\mu}\frac{d\mu}{dx} = \frac{M_y - N_x}{N},$$
so
$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}.$$
2. If μ depends on y ($\mu = \mu(y)$). Then $\mu_x = 0$, so the equation (4) becomes
$$\frac{1}{\mu}\mu_y = \frac{1}{\mu}\frac{d\mu}{dy} = \frac{N_x - M_y}{M},$$
so
$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}.$$

We summarize that for the differential equation:

$$M(x,y)dx + N(x,y)dy = 0, \tag{5}$$

as following

1. If $(M_y - N_x)/N$ is a function of x only, then the integrating factor for (5) is
$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}.$$
2. If $(N_x - M_y)/M$ is a function of y only, then the integrating factor for (5) is
$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}.$$

Examples:

1- Solve the following differential equations:

1. $xydx + (2x^2 + 3y^2 - 20)dy = 0; x \neq 0, y > 0.$
2. $(4xy + 3y^2 - x)dx + x(x + 2y)dy = 0, x(x + 2y) \neq 0.$

2- Find m, n such that

$$\mu(x, y) = x^m y^n,$$

is an integrating factor of the differential equation

$$(2y^2 + 4x^2y)dx + (4xy + 3x^3)dy = 0.$$

Exercies:

Solve the following differential equations:

1. $(x^2 + y^2 + 1)dx + x(x - 2y)dy = 0.$
2. $y(x + y + 1)dx + x(x + 3y + 2)dy = 0; y(x + y + 1) \neq 0$

The General Solution of a Linear Differential Equations

Consider the linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x), \qquad (6)$$

where P and Q are continuous function on the interval (a, b).
The integrating factor of the differential equation (6) is

$$\mu(x) = e^{\int P(x)dx}.$$

The general solution of equation (6) is given by

$$y\mu(x) = \int \mu(x)Q(x)dx + C.$$

Since $\mu(x) \neq 0$, for $x \in (a, b)$, then we can write

$$y\mu(x) = \int \mu(x)Q(x)dx + C,$$

$$y(x) = e^{-\int P(x)dx} \int \mu(x)Q(x)dx + Ce^{-\int P(x)dx}.$$

Examples:

Solve the following differential equations:

1. $x\frac{dy}{dx} + 2y = x^3$.
2. $(1 + x^2)\frac{dy}{dx} + xy + x^3 + x = 0$.
3. $(y - x + xy\cot x)dx + xdy = 0; 0 < y < \pi$ with initial value problem $y(\pi/2) = 0$.

Bernoulli's equation

The Bernoulli's equation is a first order differential equation, which can be written in the form

$$y' + P(x)y = Q(x)y^n, \tag{7}$$

where $n \in \mathbb{R}$.

1. If $n = 0$ then the equation (7) is a linear first order differential equation and we can solve it as we saw before.
2. If $n = 1$ then the equation (7) is becomes a differential equation with separable variables, and we can solve it by by separating the variables.

3. If $n \neq 0$ and $n \neq 1$ then the equation (7) can be written in the form

$$y^{-n}y' + P(x)y^{-n+1} = Q(x).$$

Now we let $u = y^{-n+1}$, then we have

$$\frac{du}{dx} = (-n+1)y^{-n}\frac{dy}{dx}$$

or

$$u' = (-n+1)y^{-n}y'.$$

$$\frac{1}{-n+1}u' + P(x)u = Q(x)$$

or

$$u' + (-n+1)P(x)u = (-n+1)Q(x),$$

which is a linear first order differential equation and we can solve it.

Examples:

Solve the following differential equations:

1. $\frac{dy}{dx} + 2xy = xe^{-x^2}y^3.$
2. $y(6y^2 - x - 1)dx + 2xdy = 0; x \neq 0.$
3. $\frac{dy}{dx} - \frac{1}{x}y = -2e^x y^2.$
4. $(2y^3 - x^3)dx + 2xy^2dy = 0; x \neq 0$ with **IVP** $y(1) = 1.$