MATH204 Differential Equation

Dr. Bandar Al-Mohsin

School of Mathematics, KSU

Chapter 2

- Equations with Homogeneous Coefficients
- Solving some differential equations by using appropriate substitution
- Exact Differential Equations

Equations With Homogeneous Coefficients:

Definition

A function $F(x, y)$ is called **homogeneous of degree** n if

$$F(tx, ty) = t^n F(x, y), \quad \text{for all } t > 0; t \in \mathbb{R}.$$

A first-order differential equation form

$$M(x, y)dx + N(x, y)dy = 0, \tag{1}$$

is said to be homogeneous if both coefficient functions M and N are homogeneous equations of the **same** degree.
In other words, (1) is homogeneous if

$$M(tx, ty) = t^n M(x, y) \text{ and } N(tx, ty) = t^n N(x, y).$$

Note (1) If $M(x, y)$ and $N(x, y)$ are both homogeneous of the same degree, then $\frac{M(x,y)}{N(x,y)}$ is homogeneous of degree zero. For example $f(x, y) = \frac{x^2-y^2}{x^2+y^2}$ is homogeneous of degree zero.

(2) The function $f(x,y) = x - 5y + \sqrt{x^2 + 3y^2}$, is homogeneous of degree one,
(3) The function $F(x,y) = x^7 \ln(x) - x^7 \ln(y)$, is homogeneous of degree 7,
(4) The functions

$$f(x,y) = x^2 + y^2 + \frac{x+y}{x-y} \quad \text{and } g(x,y) = 3x - 2y + e^{x-y},$$

are not homogeneous.

General Method

A first order differential equation $\frac{dy}{dx} = f(x, y)$ which can be written in the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

is called a **homogeneous differential equation**.

To solve the homogeneous differential equation:

by letting $u = \frac{y}{x}$, that is let $y = xu \Rightarrow \frac{dy}{dx} = x\frac{du}{dx} + u$,
the equation then becomes

$$x\frac{du}{dx} + u = F(u).$$

Hence

$$x\frac{du}{dx} = F(u) - u.$$

This equation is clearly separable, and can be solved as such.

Example

Solve the following differential equations:

1. $(x^2 - xy + y^2)dx - xydy = 0$.
2. $\frac{dy}{dx} + \frac{3xy+y^2}{x^2+xy} = 0$; $x \neq 0$ and $y \neq -x$.
3. $ydx + x(\ln(\frac{x}{y}) - 1)dy = 0$, $y(1) = e$.
4. $x\frac{dy}{dx} - y = \sqrt{x^2 + y^2}$; $x > 0$.

Solutions

Let us summarize the steps to follow:

(1) Recognize that your equation is an homogeneous equation; that is, you need to check that $f(tx, ty) = f(x, y)$, meaning that f(tx,ty) is independent of the variable t;
(2) Write out the substitution $u = y/x$;
(3) Through easy differentiation, find the new equation satisfied by the new function u. You may want to remember the form of the new equation:

$$x\frac{du}{dx} = F(u) - u;$$

(4) Solve the new equation (which is always separable) to find u;
(5) Go back to the old function y through the substitution $y = xu$; (6) If you have an **IVP**, use the initial condition to find the particular solution.

Exercise

Solve the following differential equations:

1. $(x^2 + y^2)dx - 2xydy = 0.$
2. $(x - y)dx + (2x + y)dy = 0.$
3. $2x^2y' - y(2x + y) = 0.$
4. $xdx + \sin^2\left(\frac{x}{y}\right)[ydx - xdy] = 0.$

Homogeneous Equations Requiring a Change of Variables:

Solving Some Differential Equations by Using Appropriate Substitution

If we have a differential equation of the form

$$\frac{dy}{dx} = f(ax + by),$$

we use the substitution $u = ax + by$, then we get

$$\frac{du}{dx} = a + b\frac{dy}{dx}.$$

Examples

Solve the following differential equations by using appropriate substitution:

1. $\frac{dy}{dx} = (-2x + y)^2 - 7, \quad y(0) = 0.$
2. $\frac{dy}{dx} = \frac{1-4x-4y}{x+y}$; $y \neq -x$
3. $\frac{dy}{dx} = \frac{x-y-3}{x+y-1}; \quad x + y - 1 \neq 0.$
4. $\frac{dy}{dx} = \frac{y(1+xy)}{x(1-xy)}; x > 0, y > 0, xy \neq 1.$ (Use the substitution $u = xy$)

Solutions

Exact Differential Equations:

A differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0, \tag{2}$$

is called **exact**, if there is a function F of x and y such that

$$dF(x,y) = M(x,y)dx + N(x,y)dy = 0. \tag{3}$$

Recall Recall that the total differential of a function $F(x,y)$ is

$$dF(x,y) = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy.$$

provided that the partial derivatives of the function F is exists.

Theorem

If $M, N, \frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous on a region R in $xy-$plane, then the differential equation (1) is exact if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \textit{on} \quad R$$

Example:

Prove that the following differential equations are exact and find their solutions

1. $(6x^2 + 4xy + y^2)dx + (2x^2 + 2xy - 3y^2)dy = 0$
2. $\left[\cos x \ln(2y - 8) + \frac{1}{x}\right] dx + \frac{\sin x}{y-4} dy$; $x \neq 0$ and $y > 4$.
3. $(e^{2y} - y\cos xy)dx + (2xe^{2y} - x\cos xy + 2y)dy = 0$