MATH204 Differential Equation

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Chapter 2

- Initial-Value Problems (IVP)
- Existence of a Unique Solution
- Separable Equations
- Exact Differential Equations
- Integrating Factor
- The General Solution of a Linear Differential Equations
- Bernoulliâs Equation.

First Order Differential Equation

Here we will start to study some methods which might use to solve first order differential equations.
Consider the equation of order one

$$F(x, y, y') = 0 \tag{1}$$

We suppose that the equation (1) can be written as the form

$$y' = \frac{dy}{dx} = f(x, y). \tag{2}$$

The equation (2) can be written as follows

$$M(x, y)dx + N(x, y)dy = 0, \tag{3}$$

where M and N are two functions of x and y.

Initial Value Problem (IVP)

We are interested in problems in which we seek a solution $y(x)$ of differential equation which satisfies some conditions imposed on the unknown $y(x)$ or its derivatives. On some interval I containing x_0, the problem

$$\text{Solve:} \qquad \frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$$
$$\text{Subject to:} \qquad y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1},$$

where $y_0, y_1, \ldots, y_{n-1}$ are arbitrary specified real constants, is clled an *initial-values problem (IVP)* and its $n-1$ derivatives at a single point x_0: $y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$ are called *initial conditions*.

Special cases:

First and second-order **IVPs**

$$\text{Solve:} \quad \frac{dy}{dx} = f(x, y)$$
$$\text{Subject to:} \quad y(x_0) = y_0.$$

$$\text{Solve:} \quad \frac{d^2y}{dx^2} = f(x, y, y')$$
$$\text{Subject to:} \quad y(x_0) = y_0, y'(x_0) = y_1.$$

Examples

Solve the initial value problem:

1- $y' = 10 - x, \quad y(0) = -1.$

2- $y' = 9x^2 - 4x + 5, \quad y(-1) = 0.$

3- $\frac{ds}{dt} = \cos t + \sin t, \quad s(t) = 1.$

4- $y'' = 2 - 6x, \quad y'(0) = 4, y(0) = 1.$

5- $y'' = x, \quad y(0) = 1, y'(0) = -1.$

6- $y' + 2xy^2 = 0, \quad y(0) = -1.$

Existence of a Unique Solution

Theorem: Consider a first order differential equation

$$\frac{dy}{dx} = f(x,y), \text{with the initial value } y(x_0) = y_0,$$

there exists a unique solution if

- $f(x,y)$ and $\frac{\partial f(x,y)}{\partial y}$ are continuous with in the region $\mathbb{R}^2$ of xy-plane.
- (x_0, y_0) be a point in the region $\mathbb{R}^2$

Examples:

Find the largest region of the xy-plane for which the following initial value problems have unique solutions:

(a) $\sqrt{x^2-4}y' = 1 + \sin(x)\ln(y)$, with initial condition $y(3) = 4$.

Solution

$$y' = \frac{1+\sin(x)\ln y}{\sqrt{x^2-4}} = f(x,y)$$

$$y' = \frac{1}{\sqrt{x^2-4}} + \frac{\sin(x)}{\sqrt{x^2-4}}\ln y; \quad y > 0 \text{ and } |x| > 2$$

$$\frac{\partial f}{\partial y} = \frac{\sin x}{\sqrt{x^2-4}}\frac{1}{y}.$$

Then f and $\frac{\partial f}{\partial y}$ are continuous on

$$R = \{(x,y) \in \mathbb{R}^2;\ |x| > 2,\ y > 0\}$$

$$R_1 = \{(x,y) \in \mathbb{R}^2;\ x > 2,\ y > 0\} \cup R_2 = \{(x,y) \in \mathbb{R}^2;\ x < -2,\ y > 0\}$$

As we see the point $(3,4) \in R_1 = \{(x,y);\ x > 2,\ y > 0\}$, so the largest region in xy-plane for which the **IVP** has a unique solution is R_1.

$$\text{(b)}\ \ \ln(x-2)\frac{dy}{dx} = \sqrt{y-2}, \text{with initial condition}\ \ y(\frac{5}{2}) = 4.$$

$$\text{(c)}\ \ \sqrt{x/y}y' = \cos(x+y);\ y \neq 0,\ \text{with initial condition}\ \ y(1) = 1.$$

Exercise Determine the largest region of the xy-plane for which the following initial value problem has a unique solution:

$$\frac{\partial y}{\partial x} = \frac{y+2x}{y-2x}, \text{with initial condition}\ \ y(1) = 0.$$

Separable Equations:

We begin to study the methods for solving the first-ordre differential equations. Consider a first-order differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0, \tag{4}$$

where M and N are two functions of x and y. Sometimes we can write the equation (4) as

$$F(x)dx + G(y)dy = 0, \tag{5}$$

which is said to be variables separable equation. We solve a variables separable equation by **separating** the variables and integrating.

$$\frac{dy}{G(y)} = f(x)\ dx$$

$$\int \frac{dy}{G(y)} = \int f(x)\ dx + c$$

Since we have one arbitrary constant in the solution, we have found the general solution of the variables separable equation.

Examples

Solve the following differential equations:

1- $\frac{dy}{dx} = 2x$

2- $\frac{dy}{dx} = 2xy$

3- $e^x \cos y \; dx + (1 + e^x) \sin y \; dy$

4- $2x(y^2 + y)dx + (x^2 - 1)ydy, \quad y \neq 0$

5- $(xy + x)dx = (x^2y^2 + x^2 + y^2 + 1)dy$