MATH204 Differential Equation

Dr. Bandar Al-Mohsin

School of Mathematics, KSU

Fourier series

Chapter 7

- Orthogonal Functions
- Fourier Series
- Even and Odd Functions
- Properties of symmetric functions
- Fourier Cosine and Sine Series
- Complex form of a Fourier Series

Orthogonal Functions

Firstly, we will introduce a tool called inner product to define orthogonal functions and sets of orthogonal functions.

Definition

The inner product of two functions f and g on the interval $[\alpha, \beta]$ is the scalar (real number)

$$(f, g) = \int_{\alpha}^{\beta} f(x)g(x)\ dx.$$

Definition

We say that The two functions f and g are orthogonal functions on the interval $[\alpha, \beta]$ if

$$(f, g) = \int_{\alpha}^{\beta} f(x)g(x)\ dx = 0.$$

Example (1) The two functions $f(x) = \cos x$ and $g(x) = \sin x$ are orthogonal on the interval $[-\pi, \pi]$ since

$$(f, g) = \int_{-\pi}^{\pi} \cos x . \sin x dx = 0.$$

Example (2) The two functions $f(x) = x$ and $g(x) = e^{|x|}$ are orthogonal on any symmetric interval $[-A, A]$, where A is a positive real constant. By using integration by parts, It can be easily checked that

$$(f, g) = \int_{-A}^{A} x e^{|x|} dx = 0.$$

Definition

We say that The set of functions $\{\varphi_1(x), \varphi_2(x), \varphi_3(x), \ldots, \varphi_n(x), \ldots\}$ is orthogonal on the interval $[\alpha, \beta]$ if

$$(\varphi_n(x), \varphi_m(x)) = \int_\alpha^\beta \varphi_n(x)\varphi_m(x)\ dx = 0, \quad n \neq m.$$

Definition

We define the norm (length) of function f in terms of the inner product as the quantity

$$||f|| = \sqrt{(\varphi_n, \varphi_n)} = \left(\int_\alpha^\beta \varphi_n^2(x)\ dx\right)^{1/2}.$$

Definition

If $\{\varphi_1(x), \varphi_2(x), \varphi_3(x), \ldots, \varphi_n(x), \ldots\}$ is an orthogonal set of function on the interval $[\alpha, \beta]$ with the property $||\varphi_n|| = 1$ for $n = 1, 2, \ldots,$ then the set $\{\varphi_n(x)\}_{n\geq 1}$ is said to be an orthonormal set on the interval.

$$(\varphi_n(x), \varphi_m(x)) = \int_\alpha^\beta \varphi_n(x)\varphi_m(x)\, dx = 0, \quad n \neq m.$$

Definition

A set of real-valued functions $\{\varphi_1(x), \varphi_2(x), \varphi_3(x), \ldots, \varphi_n(x), \ldots\}$ is said to be orthogonal with respct to weight function $w(x) > 0$ on the interval $[\alpha, \beta]$ if We define the norm (length) of function f in terms of the inner product as the quantity

$$(\varphi_n, \varphi_m)_{w(x)} = \int_\alpha^\beta w(x)\varphi_n(x)\varphi_m(x)\, dx = 0, \quad n \neq m.$$

Example (3) Show that the set of functions $\{1, \sin x, \cos x, \sin 2x, \cos 2x, ..., \sin mx, \cos mx, ..\}$ is orthogonal on the interval $[-\pi, \pi]$. Find the corresponding orthonormal set on $[-\pi, \pi]$.
We have to show that

$$\begin{aligned}(1, \sin nx) &= 0,\ (1, \cos nx) = 0,\ (\sin nx, \sin mx) = 0, \\ (\cos nx, \cos mx) &= 0, (\sin nx, \cos mx) = 0, \quad \forall n \neq m.\end{aligned}$$

$$\begin{aligned}(1, \sin nx) &= \int_{-\pi}^{\pi} \sin nx dx = -\frac{1}{n} \cos nx|_{-\pi}^{\pi} = 0, \\ (1, \cos nx) &= \int_{-\pi}^{\pi} \cos nx dx = \frac{1}{n} \sin nx|_{-\pi}^{\pi} = 0, \\ (\sin nx, \sin mx) &= \int_{-\pi}^{\pi} \sin nx \sin mx dx \\ &= \int_{-\pi}^{\pi} \frac{\cos(n-m)x - \cos(n+m)x}{2} dx = 0,\ n \neq m,\end{aligned}$$

$$
\begin{aligned}
(\cos nx, \cos mx) &= \int_{-\pi}^{\pi} \cos nx \cos mx dx \\
&= \int_{-\pi}^{\pi} \frac{\cos(n-m)x + \cos(n+m)x}{2} dx = 0, \; n \neq m, \\
(\sin nx, \cos mx) &= \int_{-\pi}^{\pi} \sin nx \overset{\pi}{\;} \cos mx \; dx \\
&= \int_{-\pi}^{\pi} \frac{\sin(n-m)x + \sin(n+m)x}{2} dx = 0.
\end{aligned}
$$

To determine the orthonormal set on $[-\pi, \pi]$, we have to divide each element by its norm.

$$\|1\|^2 = \int_{-\pi}^{\pi} dx = 2\pi,$$

$$\begin{aligned} \|\sin mx\|^2 &= \int_{-\pi}^{\pi} (\sin mx)^2 dx = \int_{-\pi}^{\pi} \frac{1 - \cos 2mx}{2} dx = \pi, \\ \|\cos mx\|^2 &= \int_{-\pi}^{\pi} (\cos mx)^2 dx = \int_{-\pi}^{\pi} \frac{1 + \cos 2mx}{2} dx = \pi. \end{aligned}$$

Hence the orthonormal set on $[-\pi, \pi]$:

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, ..., \frac{\sin mx}{\sqrt{\pi}}, \frac{\cos mx}{\sqrt{\pi}}, .. \right\}.$$

Example (4) Show that the functions

$$f(x) = 1, g(x) = 2x, h(x) = 4x^2 - 2$$

are orthogonal with respect to the weight function $w(x) = e^{-x^2}$ on the interval $(-\infty, \infty)$.

$$(1, 2x)_{w(x)} = \int_{-\infty}^{\infty} 2xe^{-x^2} dx = -\int_{-\infty}^{\infty} -2xe^{-x^2} dx = -\left. e^{-x^2} \right|_{-\infty}^{\infty} = 0,$$

$$\begin{aligned} (1, 4x^2 - 2)_{w(x)} &= \int_{-\infty}^{\infty} (4x^2 - 2)e^{-x^2} dx \\ &= -\int_{-\infty}^{\infty} 2xe^{-x^2} dx - 2\int_{-\infty}^{\infty} e^{-x^2} dx \\ &= -2xe^{-x^2}\Big|_{-\infty}^{\infty} + 2\int_{-\infty}^{\infty} e^{-x^2} dx - 2\int_{-\infty}^{\infty} e^{-x^2} dx \\ &= 0. \end{aligned}$$

In the same way and by integration by parts, we find that

$$(2x, 4x^2 - 2)_{w(x)} = 0.$$

Fourier Series

Theorem

Suppose that f and f' are piecewise continuous on the interval $[-T, T]$. Further, suppose that f is defined outside the interval $[-T, T]$ so that it is periodic with period $2T$. Then f has a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{T} + b_n \sin \frac{n\pi x}{T} \right).$$

Whose coefficients are given by

$$a_n = \frac{1}{T} \int_{-T}^{T} f(x) \cos \frac{n\pi x}{T} \, dx, \quad (n = 1, 2, \dots),$$

$$b_n = \frac{1}{T} \int_{-T}^{T} f(x) \sin \frac{n\pi x}{T} \, dx, \quad (n = 1, 2, \dots), \quad a_0 = \frac{1}{T} \int_{-T}^{T} f(x) \, dx.$$

Even and Odd Functions

Recall that if $f(x)$ is an even function then

$$f(-x) = +f(x).$$

Examples: $f(x) = x^4 - x^2$, $h(x) = \sqrt{2 + x^4}$ and $f(x) = e^{-|x|}$.

Recall that if $f(x)$ is an odd function then

$$f(-x) = -f(x)$$

Examples: $f(x) = x^3$, $f(x) = x$.

Two symmetry properties of functions will be useful in the study of Fourier series. A function $f(x)$ that satisfies $f(-x) = f(x)$ for all x in the domain of f has a graph that is symmetric with respect to the $y - axis$. This function is said to be even. For example:

$$f(x) = \sqrt{2 + x^4}, g(x) = e^{-|x|},$$

$$h(x) = \cos x + \ln(1 + x^2),$$

$$k(x) = \begin{cases} |\sin x|, & |x| \leq \pi \\ 0, & |x| > \pi \end{cases}.$$

A function f that satisfies $f(-x) = -f(x)$ for all x in the domain of f has a graph that is symmetric with respect to the origin. It is said to be an odd function. For example:

$$f(x) = e^{|x|} \sin x,$$

$$h(x) = \sqrt{1 + x^2} \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

$$k(x) = \begin{cases} x - 1, & 0 < x < 1, \\ x + 1, & -1 < x < 0, \\ 0, & |x| > 1 \end{cases},$$

$$M(x) = x^{1/3} - \sin x.$$

Properties of symmetric functions

- If $f(x)$ is an even piecewise continuous function on $[-L, L]$, then

$$\int_{-L}^{L} f(x)\ dx = 2\int_{0}^{L} f(x)\ dx$$

- If $f(x)$ is an odd piecewise continuous function on $[-L, L]$, then

$$\int_{-L}^{L} f(x)\ dx = 0$$

- For an even function, we have the Fourier coefficients

$$a_n = \frac{2}{T}\int_{0}^{T} f(x)\cos\frac{n\pi x}{T}\ dx, \quad (n = 1, 2, \dots),$$

$$a_0 = \frac{2}{T}\int_{0}^{T} f(x)\ dx,$$

and

$$b_n = 0, \quad (n = 1, 2, \dots)$$

- For an odd function, we have the Fourier coefficients

$$b_n = \frac{1}{T}\int_{-T}^{T} f(x)\sin\frac{n\pi x}{T}\;dx, \quad (n = 1, 2, \dots),$$

$$a_n = 0, \quad (n = 0, 1, 2, \dots)$$

- When n is an integer

$$\sin n\pi = 0 \;\; \text{and} \;\; \cos n\pi = (-1)^n.$$

Example (1) Assume that there is a Fourier series converging to the function

$$\begin{aligned} f(x) &= \begin{cases} -x, & -T \leq x < 0 \\ x, & 0 \leq x \leq T; \end{cases} \\ f(x+2T) &= f(x). \end{aligned}$$

Compute the Fourier series for the given function.
The Fourier series has the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{n\pi x}{T} + b_n \sin\frac{n\pi x}{T}).$$

Since $f(-x) = f(x)\ \forall x \in [-T, T]$, then f is even on $[-T, T]$, hence $b_n = 0$, $(n = 1, 2, ...)$.
We compute to find that

$$a_0 = \frac{2}{T}\int_0^T f(x)dx = T,$$

$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T f(x)\cos\frac{n\pi x}{T}dx, \quad (n = 1, 2, ...), \\ &= \frac{2}{T}\int_0^T x\cos\frac{n\pi x}{T}dx \\ &= \frac{2T}{(n\pi)^2}\left(\cos n\pi - 1\right), \quad (n = 1, 2, ...), \end{aligned}$$

Thus the Fourier series for the function f is given by

$$f(x) = \frac{T}{2} - \frac{4T}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{T}.$$

Observe that from the obtained Fourier series, we can deduce that

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

This follows from the fact that the Fourier series converges to $f(0) = 0$ at $x = 0$.

Example (2) Find a Fourier series to represent the function

$$f(x) = x - x^2$$

from $x = -\pi$ to $x = \pi$. Deduce that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{6}.$$

We write

$$x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

We have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)dx = \frac{-2}{3}\pi^2,$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\cos nx dx = -\frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx dx \\ &= \frac{4}{n^2}(-1)^{n+1}, \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx dx \\ &= \frac{2}{n}(-1)^{n+1}. \end{aligned}$$

Hence

$$x - x^2 = \frac{-2}{3}\pi^2 - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx.$$

By setting $x = 0$, we obtain

$$\frac{-2}{3}\pi^2 - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = 0.$$

From which it follows that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{6}.$$

Fourier Cosine and Sine Series

Sometimes it is possible to represent a function as a Fourier Cosine or Sine Series. To do this we use the properties of even and odd functions as defined previously. To determine a series we usually extend the interval of definition to create a new function that is either even or odd depending on the type of series required. If we require a Fourier cosine series then the new function created is chosen to be an even function. Similarly, If we require a Fourier sine series then the new function created is chosen to be an odd function.

For example, let $f(x)$ be defined on the interval $[0, L]$.

- If we require a Fourier cosine series then we create a new function created, $f_e(x)$, which is an even function over the interval $[-L, L]$. That is, we let

$$\begin{aligned} f_e(x) &= \begin{cases} f(x), & 0 < x < L, \\ f(-x), & -L \leq x \leq 0; \end{cases} \\ \text{with} \quad f_e(x+2L) &= f_e(x). \end{aligned}$$

- If we require a Fourier sine series then we create a new function created, $f_o(x)$, which is an odd function over the interval $[-L, L]$. That is, we let

$$f_o(x) = \begin{cases} f(x), & 0 < x < L, \\ -f(-x), & -L < x < 0, \end{cases}$$

and extending $f_o(x)$ to all x using the $2L$ periodicity.

Definition

Let $f(x)$ be piecewise continuous function on the interval $[0, L]$.

- The Fourier cosine series of $f(x)$ on $[0, L]$ is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L},$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} \, dx, \quad (n = 0, 1, 2, \dots).$$

Definition

- The Fourier sine series of $f(x)$ on $[0, L]$ is

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx, \quad (n = 1, 2, \dots).$$

Example (1) Compute the Fourier sine series for the function

$$f(x) = \cos\frac{\pi x}{3},\ 0 < x < 3.$$

We extend $f(x)$ as an odd function on $[-3, 3]$

$$f_o(x) = \begin{cases} \cos\frac{\pi x}{3}, & 0 \le x < 3, \\ -\cos\frac{\pi x}{3} & -3 \le x < 0. \end{cases}$$

The Fourier sine series representation of

$$f(x) = \cos\frac{\pi x}{3}$$

is

$$f(x) = \cos\frac{\pi x}{3} = \sum_{n=1}^{\infty} b_n \sin\frac{nx\pi}{3}, \quad 0 < x < 3,$$

where

$$\begin{aligned} b_n &= \frac{2}{3}\int_0^3 \cos\frac{\pi x}{3}\sin\frac{n\pi x}{3}dx \\ &= \frac{1}{3}\int_0^3 \left(\sin\frac{(n+1)\pi x}{3} - \sin\frac{(n-1)\pi x}{3}\right)dx \\ &= \begin{cases} 0, & n \text{ odd} \\ \frac{4n}{\pi(n^2-1)}, & n \text{ even} \end{cases} \end{aligned}$$

According to Fourier theorem, equality holds for $0 < x < 3$, but not at $x = 0$ and $x = 3$:

$$\cos\frac{\pi x}{3} = \frac{8}{\pi}\int_{n=1}^{\infty}\frac{n}{(4n^2-1)}\sin\frac{2nx\pi}{3}, \quad 0 < x < 3.$$

At $x = 0$ and $x = 3$, the Fourier series converges to

$$\frac{f(0^+) + f(0^-)}{2} = 0$$

and

$$\frac{f(3^+) + f(3^-)}{2} = 0,$$

respectively.

Example (2) Compute the Fourier cosine series for the function

$$f(x) = e^{2x},\ 0 \leq x \leq 1.$$

and deduce that

$$\frac{3-e^2}{2} = \sum_{n=1}^{\infty} \frac{4}{4+n^2\pi^2}\left[e^2(-1)^n - 1\right]$$

We extend $f(x)$ as an even function on $[-1,1]$

$$f_e(x) = \begin{cases} e^{2x}, & 0 < x < 1, \\ e^{-2x} & -1 < x < 0. \end{cases}$$

The Fourier cosine series representation of

$$f(x) = e^{2x},$$

is

$$f(x) = e^{2x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x, \quad 0 \leq x \leq 1,$$

where

$$a_0 = 2\int_0^1 e^{2x}dx = e^2 - 1,$$

$$\begin{aligned} a_n &= 2\int_0^1 e^{2x}\cos n\pi x dx \\ &= 2\left[\frac{1}{2}\, e^{2x}\cos n\pi x\Big|_0^1 + \frac{1}{2}n\pi\int_0^1 e^{2x}\sin n\pi x dx\right] \\ &= e^2(-1)^n - 1 + n\pi\left[\frac{1}{2}\, n\pi e^{2x}\sin n\pi x\Big|_0^1 - \frac{1}{2}n\pi\int_0^1 e^{2x}\cos n\pi x dx\right] \\ &= e^2(-1)^n - 1 - \frac{1}{2}n^2\pi^2\int_0^1 e^{2x}\cos n\pi x dx. \end{aligned}$$

Hence

$$a_n = \frac{4}{4+n^2\pi^2}\left[e^2(-1)^n - 1\right]$$

The Fourier series is then

$$e^{2x} = \frac{e^2-1}{2} + \sum_{n=1}^{\infty}\frac{4}{4+n^2\pi^2}\left[e^2(-1)^n - 1\right]\cos n\pi x, \quad 0 \le x \le 1.$$

At $x = 0$, we have

$$\frac{3 - e^2}{2} = \sum_{n=1}^{\infty} \frac{4}{4 + n^2\pi^2} \left[e^2(-1)^n - 1\right].$$

Complex form of a Fourier Series

We have seen that Fourier Series in the interval $(-T, T)$ of a functon $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{T} + b_n \sin \frac{n\pi x}{T} \right).$$

Thus, from The Euler's formula we have the complex form of Fourier Series of f is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{T}},$$

where

$$c_n = \frac{1}{2T} \int_{-T}^{T} f(x) e^{\frac{in\pi x}{T}} \, dx.$$

Example Obtain the complex form of the Fourier series for the function $f(x) = e^{\lambda x} \quad -\pi < x < \pi$ in the form

$$e^{\lambda x} = \frac{\sinh \lambda \pi}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{\lambda + in}{\lambda^2 + n^2} e^{inx},$$

and deduce that

$$\frac{\pi}{\lambda \sinh \lambda \pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\lambda^2 + n^2}.$$

We look for the coefficients c_n in the series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$,

$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\lambda x}e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(\lambda-in)x}dx \\
&= \frac{1}{2\pi}\left[\frac{e^{(\lambda-in)\pi}-e^{-(\lambda-in)\pi}}{\lambda-in}\right] \\
&= \frac{1}{2\pi}\left[\frac{e^{\lambda\pi}(\cos n\pi - i\sin n\pi)-e^{-\lambda\pi}(\cos n\pi + i\sin n\pi)}{\lambda-in}\right] \\
&= \frac{1}{2\pi(\lambda-in)}\left(e^{\lambda\pi}-e^{-\lambda\pi}\right)\cos n\pi \\
&= \frac{1}{2\pi(\lambda-in)}(e^{\lambda\pi}-e^{-\lambda\pi})\cos n\pi \\
&= \frac{1}{2\pi(\lambda-in)}(2\sinh\lambda\pi)\cos n\pi \\
&= \frac{(-1)^n\sinh\lambda\pi}{\pi(\lambda-in)} = \frac{(-1)^n(\lambda+in)\sinh\lambda\pi}{\pi(\lambda^2+n^2)}.
\end{aligned}
$$

Substituting this found c_n in the series to get

$$f(x) = e^{\lambda x} = \frac{\sinh \lambda\pi}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n(\lambda + in)}{\lambda^2 + n^2} e^{inx}. \tag{3}$$

Now by setting $x = 0$ in (3), we obtain

$$\frac{\pi}{\sinh \lambda\pi} = \sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{\lambda}{\lambda^2 + n^2} + i \frac{n}{\lambda^2 + n^2} \right).$$

By equating the real part, we have

$$\frac{\pi}{\lambda \sinh \lambda\pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\lambda^2 + n^2}.$$