VECTORS and MATRICES

Rotations

Let Rot_φ denote the rotation of vectors in the plane $\mathbb{R}^2$ by the angle φ counter-clockwise. The rotation Rot_φ is given by the matrix

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix},$$

i.e. if a vector v is given by $v = (x, y)$, then rotating v by the angle φ counter-clockwise we obtain

$$\mathrm{Rot}_\varphi(v) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\varphi - y\sin\varphi \\ x\sin\varphi + y\cos\varphi \end{pmatrix}.$$

In particular, the rotation Rot_{90° of vectors in the plane $\mathbb{R}^2$ by the angle 90° counter-clockwise is given by the matrix

$$\begin{pmatrix} \cos(90^\circ) & -\sin(90^\circ) \\ \sin(90^\circ) & \cos(90^\circ) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

i.e. if a vector v is given by $v = (x, y)$, then rotating v by the angle 90° counter-clockwise we obtain

$$\mathrm{Rot}_{90^\circ}(v) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}.$$

Remember: If a vector v has coordinates $v = (x, y)$, then rotating v by the angle 90° counter-clockwise we obtain $\mathrm{Rot}_{90^\circ}(v) = (-y, x)$.

Dot-Product

Definition: The **dot product** of two vectors $v = (v_1, \dots, v_n), w = (w_1, \dots, w_n) \in \mathbb{R}^n$ is given by

$$v \bullet w = v_1 w_1 + \cdots + v_n w_n.$$

The geometrical meaning of the dot-product:

$$v \bullet w = |v| \cdot |w| \cdot \cos\angle(v, w),$$

where $|v|$ resp. $|w|$ is the length of the vector v resp. w and $\angle(v, w)$ is the angle between the vectors v and w.

Properties of the dot-product: For vectors a, b, a_1, a_2, b_1, b_2 and a real number λ:

1) $a \bullet b = b \bullet a$.
2) $(a_1 + a_2) \bullet b = a_1 \bullet b + a_2 \bullet b$, $a \bullet (b_1 + b_2) = a \bullet b_1 + a \bullet b_2$.
3) $(\lambda a) \bullet b = \lambda(a \bullet b)$, $a \bullet (\lambda b) = \lambda(a \bullet b)$.
4) $|a \bullet b| = |a| \cdot |b| \cdot \cos\theta$, where θ is the angle between a and b.
5) $a \bullet b = 0 \iff \angle(a, b) = \pi/2$ or $a = 0$ or $b = 0$.
6) $a \bullet b > 0$ for $\angle(a, b) \in [0, \pi/2)$.
7) $a \bullet b < 0$ for $\angle(a, b) \in (\pi/2, \pi]$.
8) $a \bullet a = |a|^2$.

Remember: For vectors $a, b \neq 0$ we have $a \bullet b = 0 \iff a \perp b$.

Remember: $a \bullet a = |a|^2$.

Differentiation of the dot-product: Let $a, b : I \to \mathbb{R}^2$ or $a, b : I \to \mathbb{R}^3$, where I is an interval in $\mathbb{R}$, then

$$(a \bullet b)' = a \bullet b' + a' \bullet b,$$

i.e. for any $t \in I$

$$(a \bullet b)'(t) = a(t) \bullet b'(t) + a'(t) \bullet b(t).$$

Determinant

Notation: For vectors $v = (v_1, v_2)$ and $w = (w_1, w_2)$ in $\mathbb{R}^2$ we write

$$|v\ w| = \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} = \det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} = v_1 w_2 - v_2 w_1.$$

The geometrical meaning of the determinant:

$$|v\ w| = |v| \cdot |w| \cdot \sin \angle(v, w),$$

where $|v|$ resp. $|w|$ is the length of the vector v resp. w and $\angle(v, w)$ is the angle between the vectors v and w with sign of the angle depending on the orientation of the vectors.

$|v\ w|$ measures the area of the parallelogram spanned by the vectors v and w.

Properties of the determinant: For vectors a, b, a_1, a_2, b_1, b_2 and a real number λ:

1) $|a\ b| = -|b\ a|$.
2) $|a_1 + a_2\ b| = |a_1\ b| + |a_2\ b|$, $|a\ b_1 + b_2| = |a\ b_1| + |a\ b_2|$.
3) $|\lambda \cdot a\ b| = \lambda \cdot |a\ b|$, $|a\ \lambda \cdot b| = \lambda \cdot |a\ b|$.

Remark: Connection with the vector product: For $v = (v_1, v_2)$, $w = (w_1, w_2)$:

$$(v_1, v_2, 0) \times (w_1, w_2, 0) = (0, 0, |v\ w|)\,.$$

Vector Cross Product

Definition: The **vector cross product** of vectors $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ in $\mathbb{R}^3$ is given by

$$v \times w = \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, -\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right) = (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1).$$

Remark: If you have good skills in working with determinants, you might find the following method for working out vector products useful: If $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$ denote the three standard vectors, we can write the vector product as

$$\begin{aligned} v \times w &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= \mathbf{i} \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \\ &= \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right) \\ &= (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1). \end{aligned}$$

Remark: The cross product is also known as the **wedge product** and written $v \wedge w$.

Properties of the vector product: For vectors $a, b, c, a_1, a_2, b_1, b_2$ and a real number λ:

1) The vector product $a \times b$ is perpendicular to the vectors a and b.
2) $a \times b = -b \times a$.
3) $(a_1 + a_2) \times b = a_1 \times b + a_2 \times b$, $a \times (b_1 + b_2) = a \times b_1 + a \times b_2$.
4) $(\lambda a) \times b = \lambda(a \times b)$, $a \times (\lambda b) = \lambda(a \times b)$.
5) $|a \times b|^2 = |a|^2 |b|^2 - (a \bullet b)^2$.
6) $(a \times b) \times c = (a \bullet c)b - (b \bullet c)a$.
7) $|a \times b| = |a| \cdot |b| \cdot \sin\theta$, where θ is the angle $(0 \leqslant \theta < \pi)$ between a and b.

Remember: $a \times b \perp a$, $a \times b \perp b$.

Remark: When working out the vector product of two vectors, it is always a good idea to check (using the dot-product) that your answer is perpendicular to each of the vectors you started with. That is, having worked out $a \times b$, check that $(a \times b) \bullet a = 0$ and $(a \times b) \bullet b = 0$.

Differentiation of the vector product: Let $a, b : I \to \mathbb{R}^3$, where I is an interval in $\mathbb{R}$, then

$$(a \times b)' = a \times b' + a' \times b,$$

i.e. for any $t \in I$

$$(a \times b)'(t) = a(t) \times b'(t) + a'(t) \times b(t).$$