

Vectors in Three-Space

A **vector in three-space** is an ordered triple of real numbers:

$$\mathbf{v} = (v_1, v_2, v_3).$$

Vectors represent quantities that have both *magnitude* and *direction* in 3D space.

Vector Equality

Two vectors are equal if their corresponding components are equal:

$$(u_1, u_2, u_3) = (v_1, v_2, v_3) \iff u_1 = v_1, u_2 = v_2, u_3 = v_3.$$

Vector Addition and Subtraction

If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, then

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3),$$

$$\mathbf{u} - \mathbf{v} = (u_1 - v_1, u_2 - v_2, u_3 - v_3).$$

Scalar Multiplication

If $c \in \mathbb{R}$ and $\mathbf{v} = (v_1, v_2, v_3)$, then

$$c\mathbf{v} = (cv_1, cv_2, cv_3).$$

Zero Vector

The **zero vector** in space is

$$\mathbf{0} = (0, 0, 0).$$

It has no direction and zero magnitude.

Magnitude of a Vector

The **magnitude** (or length) of $\mathbf{v} = (v_1, v_2, v_3)$ is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

Examples

- $\mathbf{u} = (2, -1, 3)$, $\mathbf{v} = (1, 4, 0)$

$$\mathbf{u} + \mathbf{v} = (3, 3, 3), \quad \mathbf{u} - \mathbf{v} = (1, -5, 3).$$

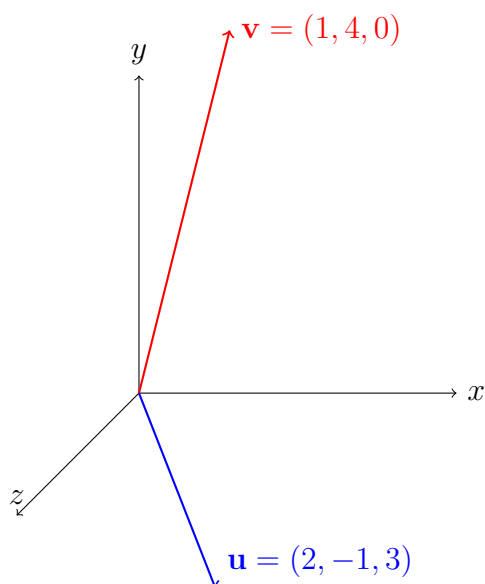
- If $c = 2$, then

$$2\mathbf{u} = (4, -2, 6).$$

- Magnitude:

$$\|\mathbf{u}\| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}.$$

Figure



Position Vector. If $P = (x, y, z)$ is a point in space, then the vector from the origin $O(0, 0, 0)$ to P is called the *position vector* of P :

$$\vec{OP} = (x, y, z).$$

Standard Unit Vectors. In \mathbb{R}^3 , the standard unit vectors along the coordinate axes are

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1).$$

Any vector $a = (a_1, a_2, a_3)$ can be written as a linear combination of these unit vectors:

$$a = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

Unit Vector. A vector of length 1 in the direction of $a \neq 0$ is called the *unit vector* in the direction of a , and is defined as

$$\frac{a}{\|a\|}.$$

Condition for Parallel Vectors. Two nonzero vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ are *parallel* if one is a scalar multiple of the other:

$$\vec{u} \parallel \vec{v} \iff \exists c \in \mathbb{R} \setminus \{0\} \text{ such that } \vec{u} = c\vec{v}.$$

Example:

$$u = (2, 4, -6), \quad v = (1, 2, -3).$$

Here $u = 2v$, so indeed $u \parallel v$.

Exercise

Let $\mathbf{a} = (-6, -3, 6)$. Find the vector that has:

1. the same direction as \mathbf{a} and twice the magnitude of \mathbf{a} ,
2. the opposite direction of \mathbf{a} and one-third of the magnitude of \mathbf{a} ,
3. the same direction as \mathbf{a} and magnitude 2.

Magnitude of \mathbf{a} .

$$\|\mathbf{a}\| = \sqrt{(-6)^2 + (-3)^2 + 6^2} = \sqrt{36 + 9 + 36} = \sqrt{81} = 9.$$

Unit vector in the direction of \mathbf{a} .

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{1}{9}(-6, -3, 6) = \left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right).$$

1.

$$\mathbf{v}_1 = 18\hat{\mathbf{a}} = 18\left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right) = (-12, -6, 12).$$

2. Opposite direction \Rightarrow multiply by -1 .

$$\mathbf{v}_2 = -3\hat{\mathbf{a}} = -3\left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right) = (2, 1, -2).$$

3.

$$\mathbf{v}_3 = 2\hat{\mathbf{a}} = 2\left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right) = \left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right).$$

Exercise. Determine whether the points $A(1, -1, 5)$, $B(0, -1, 6)$, and $C(3, -1, 3)$ lie on the same straight line.

To check if A, B, C lie on the same straight line, we form the vectors

$$\overrightarrow{AB} = B - A = (0 - 1, -1 - (-1), 6 - 5) = (-1, 0, 1),$$

$$\overrightarrow{AC} = C - A = (3 - 1, -1 - (-1), 3 - 5) = (2, 0, -2).$$

If A, B, C are collinear, then \overrightarrow{AB} and \overrightarrow{AC} must be parallel, i.e.,

$$\overrightarrow{AC} = \lambda \overrightarrow{AB}, \quad \text{for some scalar } \lambda \in \mathbb{R}.$$

We compare:

$$\overrightarrow{AB} = (-1, 0, 1), \quad \overrightarrow{AC} = (2, 0, -2).$$

Clearly,

$$\overrightarrow{AC} = -2 \cdot \overrightarrow{AB}$$

Thus, \overrightarrow{AB} and \overrightarrow{AC} are parallel, which implies that the points A, B, C are on the same straight line.

Vector Dot Product

The **dot product** (or scalar product) of two vectors is a real number obtained in two equivalent ways.

Analytic Definition

If $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, then the dot product is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Geometric Definition

If θ is the angle between \mathbf{a} and \mathbf{b} (where $0 \leq \theta \leq \pi$), then

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta.$$

From this, we can solve for the cosine of the angle:

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|}, \quad 0 \leq \theta \leq \pi.$$

Properties of the Dot Product

1. **Commutativity:**

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}.$$

2. **Distributivity:**

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}.$$

3. **Scalar multiplication:**

$$(k\mathbf{a}) \cdot \mathbf{b} = k(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (k\mathbf{b}).$$

4. **Self dot product (magnitude squared):**

$$\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2.$$

5. **Orthogonality:**

$$\mathbf{a} \cdot \mathbf{b} = 0 \quad \Leftrightarrow \quad \mathbf{a} \perp \mathbf{b}.$$

Let $\mathbf{a} = (1, 2, 3)$, $\mathbf{b} = (2, 0, 1)$, and $\mathbf{c} = (1, 1, 1)$. Compute $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$.

First,

$$\mathbf{b} + \mathbf{c} = (2 + 1, 0 + 1, 1 + 1) = (3, 1, 2).$$

Then

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = (1, 2, 3) \cdot (3, 1, 2) = 1 \cdot 3 + 2 \cdot 1 + 3 \cdot 2 = 3 + 2 + 6 = 11.$$

Alternatively,

$$\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} = (1 \cdot 2 + 2 \cdot 0 + 3 \cdot 1) + (1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1) = (2 + 0 + 3) + (1 + 2 + 3) = 5 + 6 = 11.$$

Finding the angle between two vectors. Let $\mathbf{a} = (1, 0, 0)$ and $\mathbf{b} = (1, 1, 0)$.

First, compute the dot product:

$$\mathbf{a} \cdot \mathbf{b} = 1 \cdot 1 + 0 \cdot 1 + 0 \cdot 0 = 1.$$

Magnitudes:

$$\|\mathbf{a}\| = \sqrt{1^2 + 0^2 + 0^2} = 1, \quad \|\mathbf{b}\| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}.$$

Thus,

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} = \frac{1}{1 \cdot \sqrt{2}} = \frac{1}{\sqrt{2}}.$$

So,

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^\circ \left(\frac{\pi}{4} \text{ radians}\right).$$

Orthogonal vectors. Let $\mathbf{a} = (2, 1, -1)$ and $\mathbf{b} = (1, 1, 3)$.

$$\mathbf{a} \cdot \mathbf{b} = (2)(1) + (1)(1) + (-1)(3) = 2 + 1 - 3 = 0.$$

Since the dot product is zero, \mathbf{a} and \mathbf{b} are orthogonal (the angle between them is 90°).

Non-orthogonal vectors. Let $\mathbf{c} = (1, 2, 3)$ and $\mathbf{d} = (2, -1, 1)$.

$$\mathbf{c} \cdot \mathbf{d} = (1)(2) + (2)(-1) + (3)(1) = 2 - 2 + 3 = 3.$$

So $\mathbf{c} = (1, 2, 3)$ and $\mathbf{d} = (2, -1, 1)$ are non-orthogonal.

Component of a Vector Along Another Vector

Given two vectors \mathbf{a} and \mathbf{b} , the **component of \mathbf{a} along \mathbf{b}** is the scalar that measures how much of \mathbf{a} points in the direction of \mathbf{b} .

Definition

The component of \mathbf{a} along \mathbf{b} is

$$\text{comp}_{\mathbf{b}}(\mathbf{a}) = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|}.$$

This scalar can be positive (same direction) or negative (opposite direction).

Projection Vector

The actual vector projection of \mathbf{a} onto \mathbf{b} is

$$\text{proj}_{\mathbf{b}}(\mathbf{a}) = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\right) \mathbf{b}.$$

Example

Let $\mathbf{a} = (3, 4, 0)$ and $\mathbf{b} = (4, 0, 0)$.

- Compute the dot product:

$$\mathbf{a} \cdot \mathbf{b} = 3 \cdot 4 + 4 \cdot 0 + 0 \cdot 0 = 12.$$

- Compute the magnitude of \mathbf{b} :

$$\|\mathbf{b}\| = \sqrt{4^2 + 0^2 + 0^2} = 4.$$

- Component of \mathbf{a} along \mathbf{b} :

$$\text{comp}_{\mathbf{b}}(\mathbf{a}) = \frac{12}{4} = 3.$$

- Projection vector:

$$\text{proj}_{\mathbf{b}}(\mathbf{a}) = \left(\frac{12}{16}\right)(4, 0, 0) = (3, 0, 0).$$

The component of \mathbf{a} along \mathbf{b} is 3, and the projection vector is $(3, 0, 0)$.

Exercise

Let $\mathbf{a} = (2, -1, 2)$ and $\mathbf{b} = (1, 2, 2)$. Find (i) the component of \mathbf{a} along \mathbf{b} , and (ii) the projection of \mathbf{a} onto \mathbf{b} .

First compute the dot product:

$$\mathbf{a} \cdot \mathbf{b} = (2)(1) + (-1)(2) + (2)(2) = 2 - 2 + 4 = 4.$$

Next, compute $\|\mathbf{b}\|$:

$$\|\mathbf{b}\| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{1 + 4 + 4} = 3.$$

(i) Component:

$$\text{comp}_{\mathbf{b}}(\mathbf{a}) = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|} = \frac{4}{3}.$$

(ii) Projection:

$$\text{proj}_{\mathbf{b}}(\mathbf{a}) = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\right) \mathbf{b} = \left(\frac{4}{9}\right)(1, 2, 2) = \left(\frac{4}{9}, \frac{8}{9}, \frac{8}{9}\right).$$

Exercise

Find the component of $\mathbf{a} = (1, 2, 3)$ along $\mathbf{b} = (2, -1, 2)$. Then find the angle between \mathbf{a} and \mathbf{b} .

Dot product:

$$\mathbf{a} \cdot \mathbf{b} = (1)(2) + (2)(-1) + (3)(2) = 2 - 2 + 6 = 6.$$

Magnitudes:

$$\|\mathbf{a}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}, \quad \|\mathbf{b}\| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{9} = 3.$$

Component:

$$\text{comp}_{\mathbf{b}}(\mathbf{a}) = \frac{6}{3} = 2.$$

Angle:

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} = \frac{6}{\sqrt{14} \cdot 3} = \frac{2}{\sqrt{14}}.$$

So,

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right).$$

Exercise

Let $\mathbf{a} = (4, -2, 1)$ and $\mathbf{b} = (2, 1, -2)$. Decompose \mathbf{a} into two parts: one parallel to \mathbf{b} and one perpendicular to \mathbf{b} .

The parallel part is the projection:

$$\mathbf{a}_{\parallel} = \text{proj}_{\mathbf{b}}(\mathbf{a}) = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2} \right) \mathbf{b}.$$

Compute dot product:

$$\mathbf{a} \cdot \mathbf{b} = (4)(2) + (-2)(1) + (1)(-2) = 8 - 2 - 2 = 4.$$

Magnitude squared:

$$\|\mathbf{b}\|^2 = 2^2 + 1^2 + (-2)^2 = 4 + 1 + 4 = 9.$$

Thus,

$$\mathbf{a}_{\parallel} = \left(\frac{4}{9}\right) (2, 1, -2) = \left(\frac{8}{9}, \frac{4}{9}, -\frac{8}{9}\right).$$

The perpendicular part is

$$\mathbf{a}_{\perp} = \mathbf{a} - \mathbf{a}_{\parallel} = (4, -2, 1) - \left(\frac{8}{9}, \frac{4}{9}, -\frac{8}{9}\right) = \left(\frac{28}{9}, -\frac{22}{9}, \frac{17}{9}\right).$$

Check: $\mathbf{a}_{\parallel} + \mathbf{a}_{\perp} = \mathbf{a}$ and $\mathbf{a}_{\perp} \cdot \mathbf{b} = 0$.

Work Done by a Force

In physics, the **work** done by a constant force \mathbf{F} in moving an object through a displacement \mathbf{d} is defined as the dot product:

$$W = \mathbf{F} \cdot \mathbf{d}.$$

Geometric Form

If θ is the angle between \mathbf{F} and \mathbf{d} , then

$$W = \|\mathbf{F}\| \|\mathbf{d}\| \cos \theta.$$

- If $\theta = 0^\circ$, then $\cos \theta = 1$ and all of the force contributes to the work.
- If $\theta = 90^\circ$, then $\cos \theta = 0$ and no work is done (force is perpendicular to motion).
- If $\theta = 180^\circ$, then $\cos \theta = -1$ and the work is negative (force opposes motion).

Example

A force $\mathbf{F} = (10, 5, 0)$ newtons moves a particle through a displacement $\mathbf{d} = (3, 4, 0)$ meters.

Step 1. Compute the dot product.

$$\mathbf{F} \cdot \mathbf{d} = (10)(3) + (5)(4) + (0)(0) = 30 + 20 + 0 = 50.$$

Step 2. Interpret.

$$W = 50 \text{ joules.}$$

Thus the work done by the force along the displacement is 50 joules.

Direction Angles and Direction Cosines

Let a vector in three-dimensional space be

$$\mathbf{a} = (a_1, a_2, a_3).$$

Direction Angles

The *direction angles* of a vector are the angles α, β, γ that the vector makes with the positive x -, y -, and z -axes, respectively.

$$\alpha = \angle(\mathbf{a}, x\text{-axis}), \quad \beta = \angle(\mathbf{a}, y\text{-axis}), \quad \gamma = \angle(\mathbf{a}, z\text{-axis}).$$

Each lies in the range

$$0 \leq \alpha, \beta, \gamma \leq \pi.$$

Direction Cosines

The *direction cosines* are the cosines of these angles:

$$\cos \alpha = \frac{a_1}{\|\mathbf{a}\|}, \quad \cos \beta = \frac{a_2}{\|\mathbf{a}\|}, \quad \cos \gamma = \frac{a_3}{\|\mathbf{a}\|},$$

where $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

They satisfy the identity

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

Example

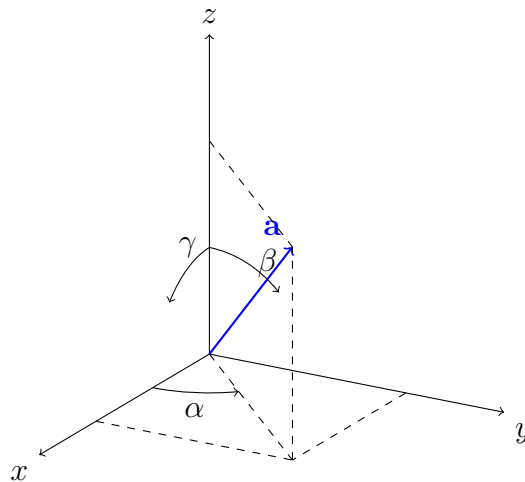
For $\mathbf{a} = (2, -2, 1)$:

$$\begin{aligned} \|\mathbf{a}\| &= \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{9} = 3, \\ \cos \alpha &= \frac{2}{3}, \quad \cos \beta = \frac{-2}{3}, \quad \cos \gamma = \frac{1}{3}. \end{aligned}$$

Thus,

$$\alpha = \cos^{-1}\left(\frac{2}{3}\right), \quad \beta = \cos^{-1}\left(\frac{-2}{3}\right), \quad \gamma = \cos^{-1}\left(\frac{1}{3}\right).$$

Figure: Direction Angles of a Vector



Vector Product (Cross Product)

Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be two vectors in \mathbb{R}^3 .

Analytic Definition

The *cross product* (or vector product) of \mathbf{a} and \mathbf{b} is the vector

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix},$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the standard unit vectors along the x -, y -, and z -axes.

Expanding the determinant gives

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

Geometric Definition

Geometrically, the cross product $\mathbf{a} \times \mathbf{b}$ is defined as the unique vector that satisfies the following properties:

- It is *perpendicular to both* \mathbf{a} and \mathbf{b} .
- Consequently, it serves as a *normal vector* to the plane determined by \mathbf{a} and \mathbf{b} .
- Its magnitude equals the area of the parallelogram spanned by \mathbf{a} and \mathbf{b} :

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta,$$

where θ is the angle between \mathbf{a} and \mathbf{b} , with $0 \leq \theta \leq \pi$.

- Its orientation is determined by the *right-hand rule*: if the fingers of the right hand curl from \mathbf{a} to \mathbf{b} , then the extended thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.

If we denote by \mathbf{n} the *unit normal vector* to the plane containing \mathbf{a} and \mathbf{b} , then the cross product can be expressed as

$$\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta \mathbf{n},$$

with \mathbf{n} oriented according to the right-hand rule.

Properties

The cross product has several important properties:

- **Anti-commutative:**

$$\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a}).$$

Swapping the order reverses the direction.

- **Self-product is zero:**

$$\mathbf{a} \times \mathbf{a} = \mathbf{0}.$$

- **Orthogonality:**

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0, \quad \mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0.$$

That is, $\mathbf{a} \times \mathbf{b}$ is orthogonal to both \mathbf{a} and \mathbf{b} .

- **Distributive over addition:**

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}.$$

- The cross product produces a new vector, not just a number.
- This new vector points straight “out of the plane” made by \mathbf{a} and \mathbf{b} .
- Its length tells you how large the parallelogram spanned by \mathbf{a} and \mathbf{b} is.
- Its direction is fixed by the right-hand rule.
- Because it is normal to the plane, $\mathbf{a} \times \mathbf{b}$ is automatically perpendicular to both \mathbf{a} and \mathbf{b} .

Applications of the Cross Product

1. Area of a Parallelogram. The magnitude of the cross product gives the area of the parallelogram spanned by \mathbf{a} and \mathbf{b} :

$$\text{Area}_{\text{parallelogram}} = \|\mathbf{a} \times \mathbf{b}\|.$$

2. Area of a Triangle. Since the area of a triangle is half the area of the corresponding parallelogram, we have

$$\text{Area}_{\text{triangle}} = \frac{1}{2} \|\mathbf{a} \times \mathbf{b}\|.$$

3. Test for Parallelism. Two vectors \mathbf{a} and \mathbf{b} are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}.$$

Numerical Examples

Example 1. Cross Product. Let $\mathbf{a} = (1, 2, 3)$ and $\mathbf{b} = (4, 5, 6)$.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = (-3)\mathbf{i} + (6)\mathbf{j} - (3)\mathbf{k}.$$

Thus,

$$\mathbf{a} \times \mathbf{b} = (-3, 6, -3).$$

Example 2. Area of a Parallelogram. With the same vectors:

$$\|\mathbf{a} \times \mathbf{b}\| = \sqrt{(-3)^2 + 6^2 + (-3)^2} = \sqrt{9 + 36 + 9} = \sqrt{54}.$$

Hence, the area of the parallelogram is $\sqrt{54}$, and the area of the triangle spanned by \mathbf{a} and \mathbf{b} is $\frac{1}{2}\sqrt{54}$.

Example 3. Test of Parallelism. Take $\mathbf{a} = (2, 4, 6)$ and $\mathbf{b} = (1, 2, 3)$. Since $\mathbf{a} = 2\mathbf{b}$, they are parallel. Indeed:

$$\mathbf{a} \times \mathbf{b} = (0, 0, 0).$$

Exercises

Exercise 1. Compute $\mathbf{u} \times \mathbf{v}$ for $\mathbf{u} = (2, -1, 3)$ and $\mathbf{v} = (4, 0, -2)$, and determine the area of the parallelogram spanned by these vectors.

Solution.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 4 & 0 & -2 \end{vmatrix} = (2, 16, 4).$$

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{2^2 + 16^2 + 4^2} = \sqrt{276} = 2\sqrt{69}.$$

So the area of the parallelogram is $2\sqrt{69}$.

Exercise 2. Find the cross product of $\mathbf{a} = (1, 1, 0)$ and $\mathbf{b} = (0, 1, 1)$, then verify explicitly that it is orthogonal to both \mathbf{a} and \mathbf{b} .

Solution.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = (1, -1, 1).$$

Check orthogonality:

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 1(1) + 1(-1) + 0(1) = 0,$$

$$\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0(1) + 1(-1) + 1(1) = 0.$$

Thus, $(1, -1, 1)$ is orthogonal to both.

Exercise 3. Given three points $P(1, 2, 3)$, $Q(2, 3, 4)$, and $R(4, 0, 5)$, compute the area of triangle PQR using the cross product.

Solution.

$$\overrightarrow{PQ} = (1, 1, 1), \quad \overrightarrow{PR} = (3, -2, 2).$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 3 & -2 & 2 \end{vmatrix} = (4, 1, -5).$$

$$\|(4, 1, -5)\| = \sqrt{42}, \quad \text{Area of } \triangle PQR = \frac{1}{2}\sqrt{42}.$$

Exercise 4. Prove that if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, then \mathbf{a} and \mathbf{b} are parallel.

Solution. If $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, then

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta = 0.$$

If $\mathbf{a}, \mathbf{b} \neq \mathbf{0}$, then $\sin \theta = 0$, so $\theta = 0$ or π . Thus, they are parallel.

Exercise 5. Let $\mathbf{a} = (3, -1, 2)$ and $\mathbf{b} = (1, 4, 2)$. Compute $\mathbf{a} \times \mathbf{b}$ and use it to find the equation of the plane passing through the origin and spanned by \mathbf{a} and \mathbf{b} .

Solution.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ 1 & 4 & 2 \end{vmatrix} = (-10, -4, 13).$$

So the normal vector is $(-10, -4, 13)$. Equation of plane through the origin:

$$-10x - 4y + 13z = 0.$$

Exercise 6. (Hard) For vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, prove that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}),$$

and interpret geometrically.

Solution. Each equals

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Cyclic permutations of rows preserve determinant, so all three expressions are equal.

Geometric meaning: This is the (signed) volume of the parallelepiped spanned by $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

Scalar Triple Product

For three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, the *scalar triple product* is defined as

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

It is a scalar (real number), unlike the cross product.

Properties

- **Determinant form:**

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

- **Cyclic symmetry:**

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}).$$

- **Anticommutativity:** Swapping two vectors changes the sign:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}).$$

- **Linear dependence:** If $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$, then the three vectors are coplanar.

Applications

1. Volume of a Parallelepiped. The absolute value of the scalar triple product gives the volume of the parallelepiped spanned by \mathbf{a} , \mathbf{b} , \mathbf{c} :

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|.$$

2. Distance from a Point to a Line. Let a line pass through point A with direction vector \mathbf{a} , and let P be another point. Then the distance from P to the line is

$$d = \frac{\|\vec{AP} \times \mathbf{a}\|}{\|\mathbf{a}\|}.$$

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We want the distance from $P(3, 1, -2)$ to the line through $Q(2, 5, 1)$ and $R(-1, 4, 2)$. Here $\vec{RQ} = (-3, -1, 1)$ and $\vec{QP} = (1, -4, -3)$. The formula gives

$$\text{dist}(P, QR) = \frac{\|\vec{QP} \times \vec{RQ}\|}{\|\vec{RQ}\|}.$$

Computing,

$$\vec{QP} \times \vec{RQ} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & -3 \\ -3 & -1 & 1 \end{vmatrix} = (-7, 8, -13).$$

Thus,

$$\|\vec{QP} \times \vec{RQ}\| = \sqrt{282}, \quad \|\vec{RQ}\| = \sqrt{11}.$$

Hence,

$$\text{dist}(P, QR) = \frac{\sqrt{282}}{\sqrt{11}}.$$

Exercises with Solutions

Exercise 1. Compute $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ for $\mathbf{a} = (1, 2, 3)$, $\mathbf{b} = (0, 1, 2)$, $\mathbf{c} = (2, 1, 1)$, and find the volume of the parallelepiped.

Answer.

$$\begin{aligned}\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \end{vmatrix} = 1(1 \cdot 1 - 2 \cdot 1) - 2(0 \cdot 1 - 2 \cdot 2) + 3(0 \cdot 1 - 1 \cdot 2). \\ &= 1(-1) - 2(-4) + 3(-2) = -1 + 8 - 6 = 1.\end{aligned}$$

Thus, the volume of the parallelepiped is $|1| = 1$.

Exercise 3. Find the volume of the tetrahedron with vertices $O(0, 0, 0)$, $A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 1)$ using the scalar triple product.

Answer.

$$\mathbf{OA} = (1, 0, 0), \quad \mathbf{OB} = (0, 1, 0), \quad \mathbf{OC} = (0, 0, 1).$$

$$\mathbf{OA} \cdot (\mathbf{OB} \times \mathbf{OC}) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

Volume of tetrahedron:

$$V = \frac{1}{6} |\mathbf{OA} \cdot (\mathbf{OB} \times \mathbf{OC})| = \frac{1}{6}.$$

Equation of a Line in 3D

Parametric Form

Let a line pass through a point $A(x_0, y_0, z_0)$ with direction vector $\mathbf{a} = (a_1, a_2, a_3)$. A general point $P(x, y, z)$ lies on the line if

$$\overrightarrow{AP} = t\mathbf{a}, \quad t \in \mathbb{R}.$$

Equivalently, the parametric equations are

$$x = x_0 + a_1t, \quad y = y_0 + a_2t, \quad z = z_0 + a_3t, \quad t \in \mathbb{R}.$$

Symmetric Form

If all $a_1, a_2, a_3 \neq 0$, the parametric equations can be written as

$$\frac{x - x_0}{a_1} = \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3}.$$

Parallel and Orthogonal Lines

- Two lines with direction vectors \mathbf{a} and \mathbf{b} are **parallel** if $\mathbf{a} = k\mathbf{b}$ for some $k \neq 0$.
- They are **orthogonal** if $\mathbf{a} \cdot \mathbf{b} = 0$.

Exercises

Exercise 1. Find the equation of the line passing through $A(1, 2, 3)$ and parallel to $\mathbf{a} = (2, -1, 4)$.

Solution. Parametric form:

$$x = 1 + 2t, \quad y = 2 - t, \quad z = 3 + 4t.$$

Symmetric form:

$$\frac{x - 1}{2} = \frac{y - 2}{-1} = \frac{z - 3}{4}.$$

Exercise 2. Find the line through $P(1, -1, 2)$ and $Q(3, 0, 5)$ in both parametric and symmetric forms.

Solution. Direction vector: $\mathbf{a} = \overrightarrow{PQ} = (2, 1, 3)$. Parametric:

$$x = 1 + 2t, \quad y = -1 + t, \quad z = 2 + 3t.$$

Symmetric:

$$\frac{x - 1}{2} = \frac{y + 1}{1} = \frac{z - 2}{3}.$$

Exercise 3. Show that the lines

$$\ell_1 : \frac{x - 1}{2} = \frac{y + 1}{-1} = \frac{z}{1}, \quad \ell_2 : \frac{x}{4} = \frac{y - 2}{-2} = \frac{z + 1}{2},$$

are parallel.

Solution. Directions: $\mathbf{a} = (2, -1, 1)$ and $\mathbf{b} = (4, -2, 2)$. Since $\mathbf{b} = 2\mathbf{a}$, the lines are parallel.

Exercise 4. Find the equation of the line through $A(1, 0, 0)$ that is orthogonal to the line

$$\ell : \frac{x - 2}{1} = \frac{y + 1}{2} = \frac{z - 3}{-1}.$$

Solution. Direction of ℓ is $\mathbf{a} = (1, 2, -1)$. We need a direction vector \mathbf{b} with $\mathbf{a} \cdot \mathbf{b} = 0$. Take $\mathbf{b} = (2, -1, 0)$ (since $1 \cdot 2 + 2 \cdot (-1) + (-1) \cdot 0 = 0$). Equation of required line:

$$x = 1 + 2t, \quad y = -t, \quad z = 0.$$

Equation of Planes in 3D

1. General Equation

A plane in 3D can be described by

$$ax + by + cz + d = 0,$$

where $\mathbf{n} = (a, b, c)$ is the *normal vector* to the plane.

If the plane passes through a point $P_0(x_0, y_0, z_0)$ and has normal vector $\mathbf{n} = (a, b, c)$, then the equation is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

2. Parallelism and Orthogonality

- Two planes are **parallel** if their normal vectors are parallel.
- Two planes are **orthogonal** if their normal vectors are orthogonal:

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = 0.$$

3. Angle Between Two Planes

If θ is the angle between two planes with normals $\mathbf{n}_1, \mathbf{n}_2$, then

$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|}.$$

Exercises

Exercise 1. Find the equation of the plane through $P(1, 2, 3)$ with normal vector $\mathbf{n} = (2, -1, 1)$.

Solution. Equation:

$$2(x - 1) - 1(y - 2) + 1(z - 3) = 0 \quad \Rightarrow \quad 2x - y + z - 3 = 0.$$

Exercise 2. Find the equation of the plane through three points $A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 1)$.

Solution. Two direction vectors:

$$\overrightarrow{AB} = (-1, 1, 0), \quad \overrightarrow{AC} = (-1, 0, 1).$$

Normal vector:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = (1, 1, 1).$$

Equation through $A(1, 0, 0)$:

$$(x - 1) + y + z = 0 \quad \Rightarrow \quad x + y + z = 1.$$

Exercise 3. Find the angle between the planes

$$\pi_1 : 2x - y + 2z - 5 = 0, \quad \pi_2 : x + z + 3 = 0.$$

Solution. Normals: $\mathbf{n}_1 = (2, -1, 2)$, $\mathbf{n}_2 = (1, 0, 1)$.

$$\begin{aligned} \cos \theta &= \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} = \frac{|2 \cdot 1 + (-1) \cdot 0 + 2 \cdot 1|}{\sqrt{2^2 + (-1)^2 + 2^2} \sqrt{1^2 + 0^2 + 1^2}} \\ &= \frac{4}{\sqrt{9} \cdot \sqrt{2}} = \frac{4}{3\sqrt{2}}. \end{aligned}$$

Exercise 4. (Hard) Find the line of intersection of the planes

$$\pi_1 : x + 2y + z - 3 = 0, \quad \pi_2 : 2x - y + z - 1 = 0.$$

Solution. Normals: $\mathbf{n}_1 = (1, 2, 1)$, $\mathbf{n}_2 = (2, -1, 1)$. Direction of line: $\mathbf{a} = \mathbf{n}_1 \times \mathbf{n}_2 = (3, 1, -5)$. To find a point: solve system

$$\begin{cases} x + 2y + z = 3 \\ 2x - y + z = 1 \end{cases}$$

Take $z = 0$: $x + 2y = 3$, $2x - y = 1$. Solving gives $x = 1, y = 1$. Point: $(1, 1, 0)$. Equation of line of intersection:

$$x = 1 + 3t, \quad y = 1 + t, \quad z = -5t.$$

Distance from a Point to a Plane

If $\pi : ax + by + cz + d = 0$ and $P(x_0, y_0, z_0)$, the distance is

$$d(P, \pi) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Example. Distance from $P(1, 2, 3)$ to plane $\pi : 2x - y + 2z - 5 = 0$:

$$d = \frac{|2(1) - 1(2) + 2(3) - 5|}{\sqrt{2^2 + (-1)^2 + 2^2}} = \frac{|2 - 2 + 6 - 5|}{\sqrt{9}} = \frac{1}{3}.$$

Exercises with Solutions

Exercise 1. Find the distance from $P(3, 1, -2)$ to the plane $2x - 2y + z - 3 = 0$.

Solution.

$$d = \frac{|2(3) - 2(1) + (-2) - 3|}{\sqrt{2^2 + (-2)^2 + 1^2}} = \frac{|6 - 2 - 2 - 3|}{3} = \frac{-1}{3} \Rightarrow \frac{1}{3}.$$

Exercise 2. Find the line of intersection of planes $\pi_1 : x + y + z = 6$ and $\pi_2 : 2x - y + z = 3$.

Solution. Normals: $\mathbf{n}_1 = (1, 1, 1)$, $\mathbf{n}_2 = (2, -1, 1)$. Direction of line: $\mathbf{a} = \mathbf{n}_1 \times \mathbf{n}_2 = (2, 1, -3)$. Solve system with $z = 0$: $x + y = 6$, $2x - y = 3 \rightarrow x = 3, y = 3$. Point $(3, 3, 0)$. Equation of line: $x = 3 + 2t$, $y = 3 + t$, $z = -3t$.

Exercise 3. Find the distance between the parallel planes $3x - 2y + 6z - 4 = 0$ and $3x - 2y + 6z + 5 = 0$.

Solution. Normals: $\mathbf{n} = (3, -2, 6)$, $\|\mathbf{n}\| = 7$. Distance:

$$d = \frac{|(-4) - (+5)|}{7} = \frac{9}{7}.$$