Uniform Continuity

$f(x)$ is uniformly continuous on the set D if:

$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$, st if $|x-t| < \delta \Rightarrow |f(x) - f(t)| < \varepsilon$

P_1 Theorem (Cantor)

if $f(x)$ is Contiuous on a closed interval $[a,b]$, then $f(x)$ is UC on $[a,b]$.

P_2

P_2: if $f(x)$ is Contiuous on the interval (a,b) and $\exists\, g(x)$ st $g(x)$ is UC on $[a,b]$ and $g(x) \equiv f(x)$ on (a,b), then $f(x)$ is UC on (a,b).

ex show that $f(x) = \frac{\sin x}{x}$ is UC on $(0,1)$

P_3 if $f(x)$ is UC on $[a,c]$ and $[c,b]$, then $f(x)$ is UC on $[a,b]$, $a<c<b$

P_4 if $f(x)$ is UC on $[a,c]$ and $[c,\infty)$, then $f(x)$ is UC on $[a,\infty)$.

P_5 if $f(x)$ is UC on $(-\infty,c]$ and $[c,\infty)$, then $f(x)$ is UC on $\mathbb{R}$.

P_6 if $f(x)$ is Continuos on $[a,\infty)$ and $\exists \lim\limits_{x\to\infty} f(x)=A\in\mathbb{R}$, then $f(x)$ is UC on $[a,\infty)$.

P_7 if $f(x)$ is Continuos on $(-\infty,\infty)$ and $\exists \lim\limits_{x\to\infty} f(x)=A\in\mathbb{R}$, $\exists \lim\limits_{x\to-\infty} f(x)=B\in\mathbb{R}$ then $f(x)$ is UC on $\mathbb{R}$.

P_8 if $f'(x)$ is bounded on I $\left(|f'(x)|\leq M\right)$ then $f(x)$ is UC on I.

ex $f(x)=\sin^2 x$ on $\mathbb{R}$.

Sol $f'(x)=2\sin x\cos x$ $\quad |f'(x)|\leq 2$ on $\mathbb{R}$

$\xRightarrow{P}$ $f(x)$ is UC on $\mathbb{R}$.

ex. $f(x)=\sqrt{x^2+1}$ on $(-\infty,\infty)$.

Sol $f'(x)=\dfrac{x}{\sqrt{x^2+1}}$ $\quad |f'(x)|=\left|\dfrac{x}{\sqrt{x^2+1}}\right|=\dfrac{|x|}{\sqrt{x^2+1}}\leq 1$

So $\xRightarrow{P}$ f is UC on $\mathbb{R}$.

ex $f(x)=x^2$ on $[0,4]$.

$f'(x)=2x$ $\quad |f'(x)|=|2x|\leq 8$

$\xRightarrow{P}$ $f(x)=x^2$ is UC on $[0,4]$.