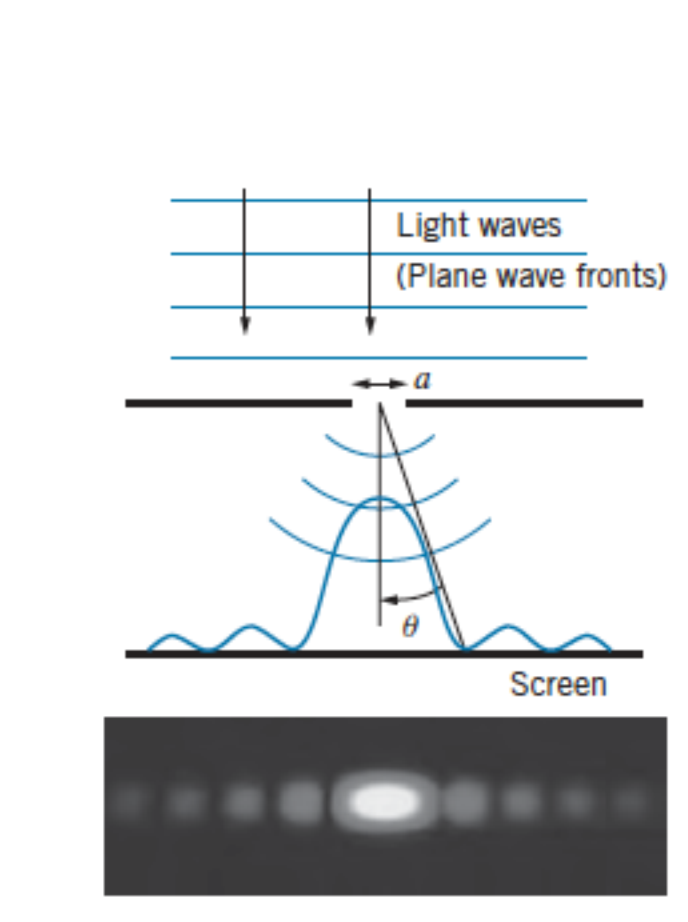
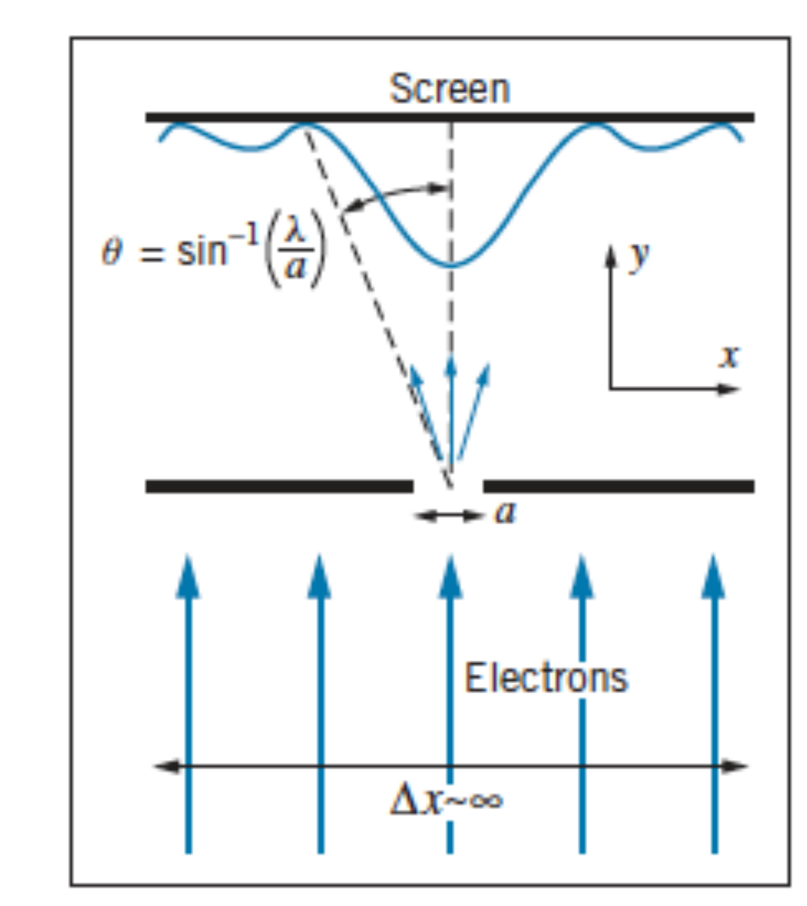
HEISENBERG UNCERTAINTY RELATIONSHIPS

Uncertainty Principle



For light of wavelength λ incident on a slit of width a, the diffraction minima are located at angles given by

$$a \sin\theta = n\lambda \qquad n = 1, 2, 3, \ldots$$



- Consider a beam of electrons incident on a single slit, single-slit diffraction experiment.
- We'll assume that the particles are initially moving in the y direction and that we know their momentum in that direction as precisely as possible.
- Δp_x=0,no movement in the x-direction, thus we know nothing about the x coordinates of the electrons (Δ x = ∞).
- At the instant that some of the electrons pass through the slit, we know quite a bit more about their x location.
- To pass through the slit, the uncertainty in their x location is no larger than a, the width of the slit; thus Δ x = a.
- This improvement in our knowledge of the electron's location comes at the expense of our knowledge of its momentum.
- In passing through the slit, a particle acquires on the average an x component of momentum of roughly $\hbar/a$, according to the uncertainty principle.

$$\sin\theta \approx \tan\theta = \frac{p_x}{p_y} = \frac{\hbar/a}{p_y} = \frac{\lambda}{2\pi a}$$

- It is not possible to make a simultaneous determination of the position and the momentum of a particle with unlimited precision,

and

- It is not possible to make a simultaneous determination of the energy and the time coordinate of a particle with unlimited precision.

Example

An electron moves in the x direction with a speed of 3.6×10^6 m/s. We can measure its speed to a precision of 1%. With what precision can we simultaneously measure its x coordinate?

The electron's momentum is

$$p_x = mv_x = (9.11 \times 10^{-31}\ \text{kg})(3.6 \times 10^6\ \text{m/s})$$
$$= 3.3 \times 10^{-24}\ \text{kg} \cdot \text{m/s}$$

The uncertainty Δp_x is 1% of this value, or 3.3×10^{-26} kg · m/s. The uncertainty in position is then

$$\Delta x \sim \frac{\hbar}{\Delta p_x} = \frac{1.05 \times 10^{-34}\ \text{J} \cdot \text{s}}{3.3 \times 10^{-26}\ \text{kg} \cdot \text{m/s}}$$
$$= 3.2\ \text{nm}$$

which is roughly 10 atomic diameters.

Example

Repeat the calculations of the previous example in the case of a pitched baseball ($m = 0.145\,\text{kg}$) moving at a speed of 95 mi/h (42.5 m/s). Again assume that its speed can be measured to a precision of 1%.

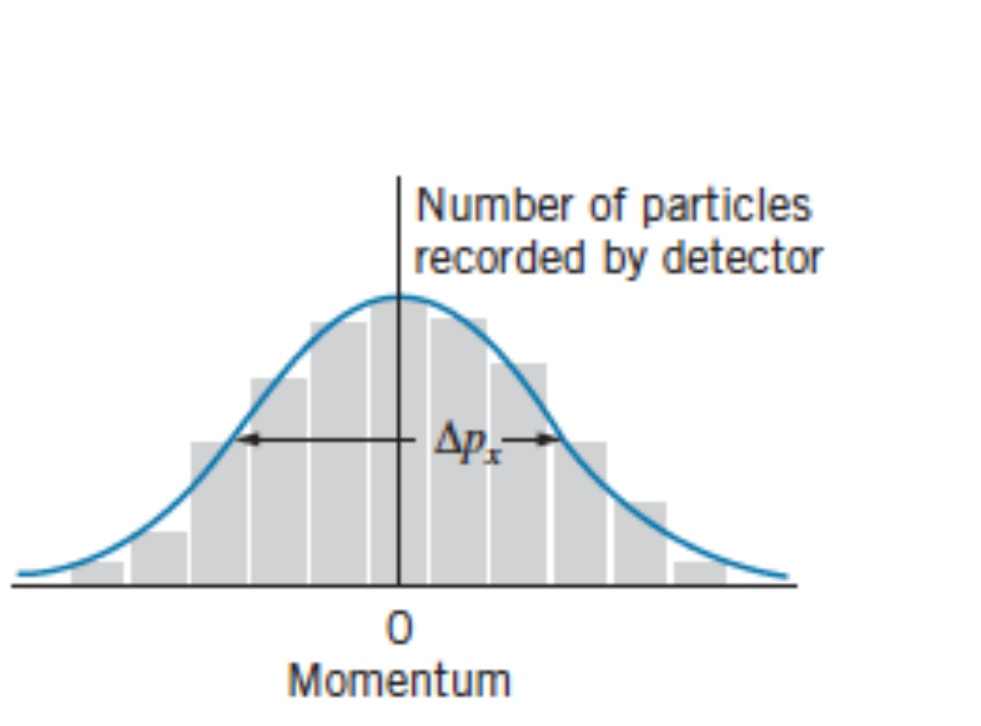
$$p_x = mv_x = (0.145\,\text{kg})(42.5\,\text{m/s}) = 6.16\,\text{kg}\cdot\text{m/s}$$

The uncertainty in momentum is $6.16 \times 10^{-2}\,\text{kg}\cdot\text{m/s}$, and the corresponding uncertainty in position is

$$\Delta x \sim \frac{\hbar}{\Delta p_x} = \frac{1.05 \times 10^{-34}\,\text{J}\cdot\text{s}}{6.16 \times 10^{-2}\,\text{kg}\cdot\text{m/s}} = 1.7 \times 10^{-33}\,\text{m}$$

This uncertainty is 19 orders of magnitude smaller than the size of an atomic nucleus

A Statistical Interpretation of Uncertainty



This resembles a statistical distribution.
the precise definition
of Δp_x is similar to that of the standard deviation

$$\sigma_A = \sqrt{(A^2)_{av} - (A_{av})^2}$$

$$\Delta p_x = \sqrt{(p_x^2)_{av} - (p_{x,av})^2}$$

$$\Delta p_x = \sqrt{(p_x^2)_{av}}$$

This gives in effect a root-mean-square value of p_x.

- Estimate the minimum velocity that would be measured for a billiard ball ($m \approx 100$ g) confined to a billiard table of dimension 1 m.

For $\Delta x \approx 1$ m, we have

$$\Delta p_x \sim \frac{\hbar}{\Delta x} = \frac{1.05 \times 10^{-34}\ \mathrm{J \cdot s}}{1\ \mathrm{m}} = 1 \times 10^{-34}\ \mathrm{kg \cdot m/s}$$

so

$$\Delta v_x = \frac{\Delta p_x}{m} = \frac{1 \times 10^{-34}\ \mathrm{kg \cdot m/s}}{0.1\ \mathrm{kg}} = 1 \times 10^{-33}\ \mathrm{m/s}$$

Quantum effects are not observable with macroscopic objects.

- Likewise, if the time duration, Δt*, of the pulse is small, we require a wide* spread of frequencies, $\Delta\omega$, to form the group. That is,

$$\Delta t \, \Delta\omega \approx 1$$

Some Consequences of the Uncertainty Principle

- Minimum Energy of a Particle in a Box

$$(\Delta p)^2 = \overline{p^2} \geq \left(\frac{\hbar}{L}\right)^2$$

$$\overline{E} = \frac{\overline{p^2}}{2m} \geq \frac{\hbar^2}{2mL^2}$$

zero point energy.

A Macroscopic Particle in a Box

- Consider a small but macroscopic particle of mass m =5 10^{-6} g confined to a one-dimensional box with L =10^{-6} m, for example, a tiny bead on a very short wire. Compute the bead's minimum kinetic energy and the corresponding speed.

1. The minimum kinetic energy is given by Equation 5-28:

$$\overline{E} = \frac{\hbar^2}{2mL^2} = \frac{(1.055 \times 10^{-34}\ \mathrm{J \cdot s})^2}{(2)(10^{-9}\ \mathrm{kg})(10^{-6}\ \mathrm{m})^2}$$

$$= 5.57 \times 10^{-48}\ \mathrm{J}$$

$$= 3.47 \times 10^{-29}\ \mathrm{eV}$$

2. The speed corresponding to this kinetic energy is

$$v = \sqrt{\frac{2\overline{E}}{m}} = \sqrt{\frac{2(5.57 \times 10^{-48}\ \mathrm{J})}{10^{-9}\ \mathrm{kg}}}$$

$$= 1.06 \times 10^{-19}\ \mathrm{m/s}$$

An Electron in an Atomic Box

- If the particle in a one dimensional box of length L= 0.1 nm (about the diameter of an atom) is an electron, what will be its zero-point energy?

$$E = \frac{(\hbar c)^2}{2mc^2L^2} = \frac{(197.3 \text{ eV} \cdot \text{nm})^2}{2(0.511 \times 10^6 \text{ eV})(0.1 \text{ nm})^2} = 3.81 \text{ eV}$$

This is the correct order of magnitude for the kinetic energy of an electron in an atom.

Size of the Hydrogen Atom

The energy of an electron of momentum p a distance r from a proton is

$$E = \frac{p^2}{2m} - \frac{ke^2}{r}$$

If we take for the order of magnitude of the position uncertainty $\Delta x = r$, we have

$$(\Delta p)^2 = \overline{p^2} \geq \frac{\hbar^2}{r^2}$$

The energy is then

$$E = \frac{\hbar^2}{2mr^2} - \frac{ke^2}{r}$$

There is a radius r_m at which E is a minimum. Setting $dE/dr = 0$ yields r_m and E_m:

$$r_m = \frac{\hbar^2}{ke^2 m} = a_0 = 0.0529 \text{ nm}$$

and

$$E_m = -\frac{k^2 e^4 m}{2\hbar^2} = -13.6 \text{ eV}$$

Widths of Spectral Lines

- If an atom is in an excited state, it does not remain in that state indefinitely but makes transitions to lower energy states until it reaches the ground state.
- lifetime τ : the mean time for decay is a measure of the time available to determine the energy of the state.
- For atomic transitions, τ is of the order of 10^{-8} s.

$$\Delta E \geq \frac{\hbar}{\tau} = \frac{6.58 \times 10^{-16}\,\text{eV}\cdot\text{s}}{10^{-8}\,\text{s}} \approx 10^{-7}\,\text{eV}$$

- This uncertainty in energy causes a spread $\Delta\lambda$ in the wavelength of the light emitted.
- For transitions to the ground state Find $\Delta\lambda / \lambda$.

WAVE GROUPS AND DISPERSION

- The matter wave representing a moving particle must reflect the fact that the particle has a large probability of being found in a small region of space only at a specific time.

- This means that a traveling sinusoidal matter wave of infinite extent and constant amplitude cannot properly represent a localized moving particle.

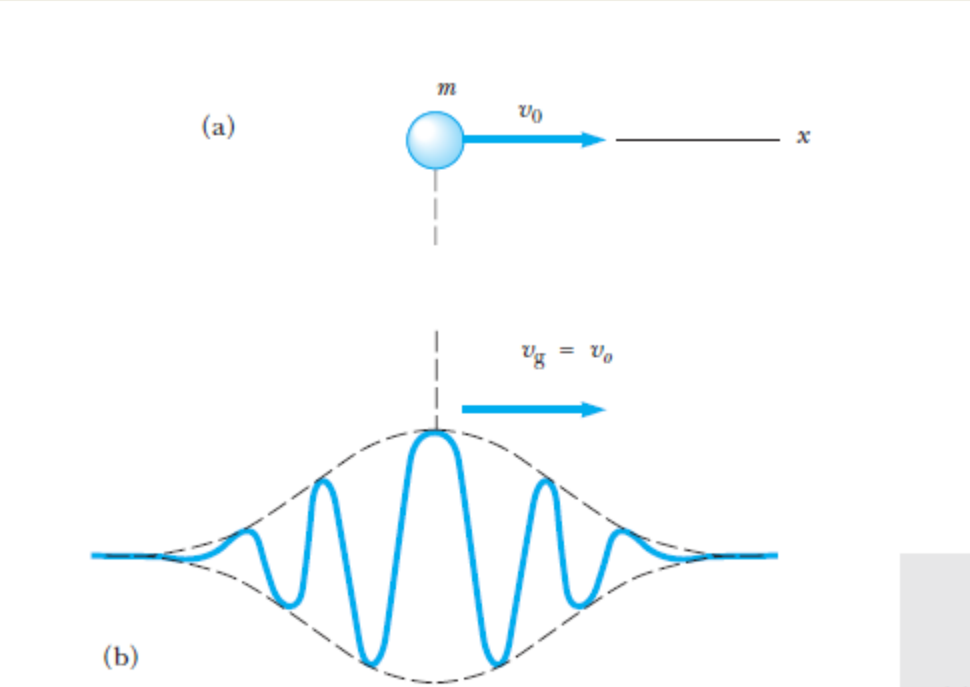
- We need a wave group or wave packet.

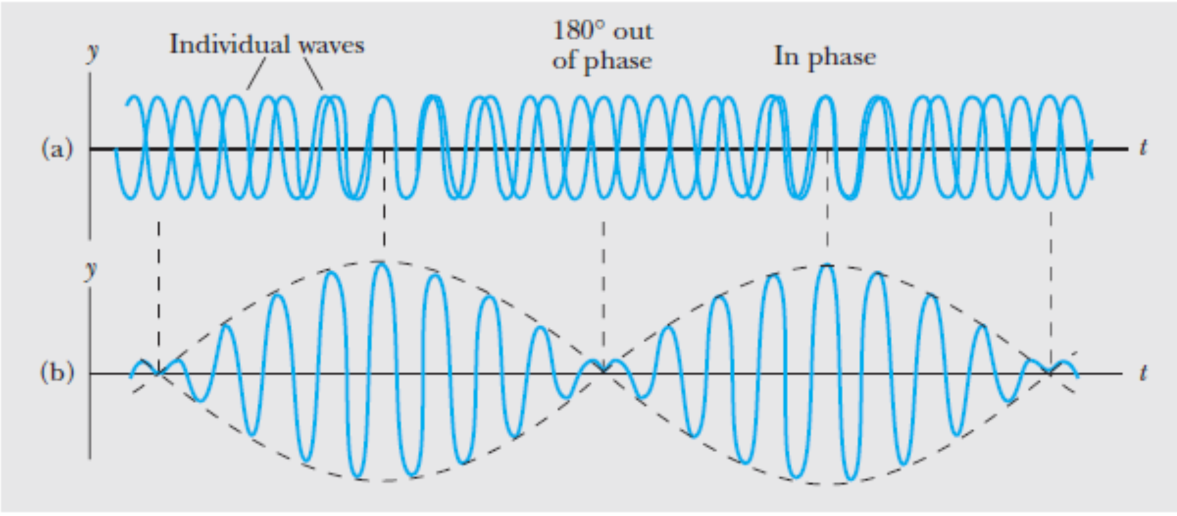
- The resulting wave group can then be shown to move with a speed v_g (the group speed) identical to the classical particle speed.

Question

- What happens to the zero-point energy of a particle in a one-dimensional box as the length of the box $L = \infty$?

WAVE PACKETS





the group velocity $$v_g = \left.\frac{d\omega}{dk}\right|_{k_0}$$

<u>Group Velocity in a Dispersive Medium:</u>

In a particular substance the phase velocity of waves doubles when the wavelength is halved. Show that wave groups in this system move at twice the *central phase* velocity.

Solution From the given information, the dependence of phase velocity on wavelength must be

$$v_p = \frac{A'}{\lambda} = Ak$$

for some constants A' and A. From Equation 5.21 we obtain

$$v_g = v_p\Big|_{k_0} + k\frac{dv_p}{dk}\Big|_{k_0} = Ak_0 + Ak_0 = 2Ak_0$$

Thus,

$$v_g = 2v_p\Big|_{k_0}$$