NOTES ON ULTRAFILTERS

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Let X be a space.

Definition 0.0.1. A set $\mathcal{F}$ of subsets of X has the finite intersection property iff for each finite list $S_1, \dots S_n$ with each $S_k \in X$ we have $S_1 \cap \dots S_n \neq \emptyset$. Such a set $\mathcal{F}$ will be called a family with the finite intersection property or an FFIP or a filter subbase.

Annoying Remark 0.0.2. There are no FFIP's on the empty set. This is because $S_1 \cap \dots S_n$ has to be interpreted as X when $n = 0$. You can take it to be a separate convention if you prefer.

Examples:

(0) $\{(a, \infty) \mid a \in \mathbb{R}\}$ is an FFIP on $\mathbb{R}$.
(1) $\{S \subseteq \mathbb{N} \mid \mathbb{N} \setminus S \text{ is finite}\}$ is an FFIP on $\mathbb{N}$.
(2) $\{S \subseteq \mathbb{C} \mid \mathbb{C} \setminus S \text{ is compact}\}$ is an FFIP on $\mathbb{C}$.
(3) For any space X and $x \in X$, the set
$$\mathcal{N}_x = \{ \text{ neighbourhoods of } x \text{ in } X\}$$
has FIP.
(4) If $(x_n)_{n \in \mathbb{N}}$ is a sequence in X then the set
$$\mathcal{F} = \{S \subseteq X \mid \exists N \quad n \geq N \Rightarrow x_n \in S\} = \{S \mid \exists N \quad \{x_N, x_{N+1}, \dots\} \subseteq S\}$$
has FIP.

Definition 0.0.3. An ultrafilter (or UF) on X is a set $\mathcal{W}$ of subsets of X which is a maximal FFIP. In other words:

U0 $\mathcal{W}$ has the FIP.
U1 If $\mathcal{W} \subseteq \mathcal{W}'$ and $\mathcal{W}'$ has FIP then $\mathcal{W}' = \mathcal{W}$.

For example:
$$\mathcal{W}_x = \{S \subseteq X \mid x \in S\}$$
is an ultrafilter. Indeed, suppose $\mathcal{W}' \supseteq \mathcal{W}_x$ has FIP. Suppose $S_1 \in \mathcal{W}'$. Considering $S_2 = \{x\} \in \mathcal{W}_x \subseteq \mathcal{W}'$ we see that $S_1 \cap S_2 \neq \emptyset$ and so $x \in S_1$. However, this means that $S_1 \in \mathcal{W}_x$ by the definition of $\mathcal{W}_x$. As this holds for all $S_1 \in \mathcal{W}'$, we see that $\mathcal{W}' = \mathcal{W}_x$ as required.

Definition 0.0.4. An ultrafilter of the form $\mathcal{W}_x$ is called a fixed ultrafilter. Ultrafilters which are not fixed are called free.

It is essentially impossible to give an explicit example of a free ultrafilter, although in a moment we will have a theorem guaranteeing that very many of them exist.

We shall use the following properties repeatedly, so you should make yourself familiar with them. Note that they are all immediate in the case $\mathcal{W} = \mathcal{W}_x$.

Proposition 0.0.5. *Let $\mathcal{W}$ be an ultrafilter.*

UP0 If $S \in \mathcal{W}$ and $T \supseteq S$ then $T \in \mathcal{W}$.
UP1 If $S_k \in \mathcal{W}$ for each k then $S_1 \cap \dots S_n \in \mathcal{W}$.
UP2 If $S \subseteq X$ then either $S \in \mathcal{W}$ or $S^c \in \mathcal{W}$ (but not both).
UP3 If $T \subseteq X$ and $T \cap S \neq \emptyset$ for every $S \in \mathcal{W}$ then $T \in \mathcal{W}$.
UP4 If $S_1 \cup \dots S_n \in \mathcal{W}$ then $S_k \in \mathcal{W}$ for some k.
UP5 $X \in \mathcal{W}$

Proof. (0) Write

$$\mathcal{W}' = \{T \subseteq X \mid \exists S \in \mathcal{W}\ S \subseteq T\}$$

Clearly $\mathcal{W} \subseteq \mathcal{W}'$. Moreover, $\mathcal{W}'$ has FIP. Indeed, if $T_1, \dots T_n \in \mathcal{W}'$ then there are sets $S_1, \dots S_n \in \mathcal{W}$ with $T_k \supseteq S_k$ and so

$$T_1 \cap \dots T_n \supseteq S_1 \cap \dots S_n \neq \emptyset$$

It follows by maximality of $\mathcal{W}$ that $\mathcal{W}' = \mathcal{W}$, hence the claim.

(1) Similarly, write

$$\mathcal{W}' = \{S_1 \cap \dots S_n \mid n \in \mathbb{N}, S_k \in \mathcal{W}\}$$

This contains $\mathcal{W}$ and has FIP so equals $\mathcal{W}$ as required.

(2) Suppose $S \notin \mathcal{W}$, so $\mathcal{W}' = \mathcal{W} \cup \{S\} \neq \mathcal{W}$. Thus, as $\mathcal{W}$ is maximal among families with FIP, we see that $\mathcal{W}'$ cannot have FIP. This means that there are sets $T_1, \dots T_n$ in $\mathcal{W}$ such that

$$T_1 \cap \dots T_n \cap S = \emptyset$$

It follows from UP1 that $T = T_1 \cap \dots T_n \in \mathcal{W}$. As $T \cap S = \emptyset$, we have $T \subseteq S^c$. Using UP0, we find that $S^c \in \mathcal{W}$, as required. We cannot have both $S \in \mathcal{W}$ and $S^c \in \mathcal{W}$ as $S \cap S^c = \emptyset$ and $\mathcal{W}$ is supposed to have FIP.

(3) Using UP1 we see that $\mathcal{W} \cup \{T\}$ has FIP and so equals $\mathcal{W}$ by maximality; thus $T \in \mathcal{W}$.

(4) Suppose $S = \bigcup_{k=1}^n S_k \in \mathcal{W}$, so $S^c = \bigcap_k S_k^c \notin \mathcal{W}$. By UP1, $S_k^c \notin \mathcal{W}$ for some k. By UP2, we have $S_k \in \mathcal{W}$ as required.

(5) By the annoying remark, $X \neq \emptyset$. Now use UP3, for example.

□

Proposition 0.0.6. *Suppose $\mathcal{W}$ has FIP. Then $\mathcal{W}$ is an ultrafilter iff for each $S \subseteq X$ we have $S \in \mathcal{W}$ or $S^c \in \mathcal{W}$.*

Proof. One half of this is UP2. Conversely, suppose $\mathcal{W}$ has FIP and contains S or S^c for each $S \subseteq X$. Suppose $\mathcal{W}' \supseteq \mathcal{W}$ has FIP. Consider $S \in \mathcal{W}'$. By assumption $S \in \mathcal{W}$ or $S^c \in \mathcal{W} \subseteq \mathcal{W}'$. The latter would contradict the FIP for $\mathcal{W}'$, so $S \in \mathcal{W}$. This holds for all $S \in \mathcal{W}'$, so $\mathcal{W}' = \mathcal{W}$. Thus $\mathcal{W}$ is an ultrafilter. □

We next turn to the problem of proving that ultrafilters exist.

Definition 0.0.7. A chain of FFIP's is a set $\mathcal{L}$ of FFIP's on X such that whenever $\mathcal{F}, \mathcal{G} \in \mathcal{L}$ we have either $\mathcal{F} \subseteq \mathcal{G}$ or $\mathcal{G} \subseteq \mathcal{F}$. In other words, $\mathcal{L}$ is linearly ordered by inclusion.

Proposition 0.0.8. *If $\mathcal{L}$ is a chain of FFIP's on X then the set*

$$\mathcal{F} = \bigcup_{\mathcal{G} \in \mathcal{L}} \mathcal{G} = \{S \subseteq X \mid \exists \mathcal{G} \in \mathcal{L} \quad S \in \mathcal{G}\}$$

has FIP.

Proof. Suppose $S_1, \dots S_n \in \mathcal{F}$. Then there are sets $\mathcal{G}_1, \dots \mathcal{G}_n$ in $\mathcal{L}$ with $S_k \in \mathcal{G}_k$ for each k. As $\mathcal{L}$ is a chain, for each k and l we have $\mathcal{G}_k \subseteq \mathcal{G}_l$ or $\mathcal{G}_l \subseteq \mathcal{G}_k$. By changing the indexing if neccessary, we may assume that

$$\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \dots \mathcal{G}_n$$

Thus $S_k \in \mathcal{G}_n$ for each k. Moreover, $\mathcal{L}$ is supposed to be a set of FFIP's so $\mathcal{G}_n \in \mathcal{L}$ has FIP. Thus $S_1 \cap \dots S_n \neq \emptyset$, as required. □

Theorem 0.0.9. *If $\mathcal{F}$ is an FFIP on X then there exists an ultrafilter $\mathcal{W}$ on X with $\mathcal{F} \subseteq \mathcal{W}$.*

Outline. For each non-maximal FFIP $\mathcal{G}$ choose a strictly larger FFIP $\mathcal{G}' = l(\mathcal{G})$. If $\mathcal{G}$ is a maximal FFIP (= ultrafilter) write $l(\mathcal{G}) = \mathcal{G}$.

Define $\mathcal{F}_0 = \mathcal{F}$. For integers $n > 0$ define $\mathcal{F}_n = l(\mathcal{F}_{n-1})$. It may well be that none of the FFIP's $\mathcal{F}_n$ is maximal. Never mind. For $n \leq m$ we have $\mathcal{F}_n \subseteq \mathcal{F}_m$ by construction, so

$$\mathcal{L}_\omega = \{\mathcal{F}_n \mid n \in \mathbb{N}\}$$

is a chain. Thus

$$\mathcal{F}_\omega = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$$

has FIP. We then define $\mathcal{F}_{\omega+1} = l(\mathcal{F}_\omega)$ and so on, and then

$$\mathcal{F}_{2\omega} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_{\omega+n} = \bigcup_{\alpha < 2\omega} \mathcal{F}_\alpha$$

continuing in this way, we eventually get $3\omega, \omega^2, \omega^\omega, \dots$ and so on. In general, for every ordinal α (whatever that means) we have an FFIP $\mathcal{F}_\alpha$. If $\alpha = \beta + 1$ for some other ordinal β, then $\mathcal{F}_\alpha = l(\mathcal{F}_\beta)$. If α is a limit ordinal like ω which has no immediate predecessor, then we have

$$\mathcal{F}_\alpha = \bigcup_{\beta < \alpha} \mathcal{F}_\beta$$

This still has FIP by the proposition above. There is only a fixed collection of FFIP's which $\mathcal{F}_\alpha$ could possibly be, and there are too many ordinals for all the $\mathcal{F}_\alpha$'s to be different. Thus, we are eventually forced (for extremely large α) to have $\mathcal{F}_\alpha = \mathcal{F}_{\alpha+1} = l(\mathcal{F}_\alpha)$, which means (see the definition of l) that $\mathcal{F}_\alpha$ is an ultrafilter. □

Remark 0.0.10. This can be reorganised to use Zorn's lemma instead of transfinite recursion. The above proof essentially contains the proof of Zorn's lemma. Of course, one has to develop the theory of ordinals to make it rigorous.

Recall that

$$\mathcal{N}_x = \{\text{ neighbourhoods of } x\} = \{\text{ open sets } U \mid x \in U\}$$

Suppose that σ is a subbasis for the topology on X. Write

$$\mathcal{N}'_x = \{\text{ subbasic neighbourhoods of } x\} = \{U \in \sigma \mid x \in U\}$$

Definition 0.0.11. An ultrafilter $\mathcal{W}$ converges to $x \in X$ if any of the following equivalent conditions hold:

UC0 $x \in \bigcap_{S \in \mathcal{W}} \overline{S}$
UC1 $\mathcal{N}_x \subseteq \mathcal{W}$
UC2 $\mathcal{N}'_x \subseteq \mathcal{W}$
UC3 For all $S \in \mathcal{W}$ and $U \in \mathcal{N}'_x$ we have $S \cap U \neq \emptyset$.

If so, we write $\mathcal{W} \to x$ and say that x is a limit of $\mathcal{W}$.

of equivalence. Clearly UC1 implies UC2. By the definition of a subbasis, every neighbourhood contains a finite intersection of subbasic neighbourhoods. Using UP1 and UP0, this shows that UC2 implies UC1. Similarly, UP3 shows that UC2 is equivalent to UC3.

Finally,

$$\begin{aligned} x \in \bigcap_{S \in \mathcal{W}} \overline{S} &\Leftrightarrow \forall S \in \mathcal{W}\, \forall U \in \mathcal{N}_x \;\; S \cap U \neq \emptyset \\ &\Leftrightarrow \mathcal{W} \cup \mathcal{N}_x \text{ has FIP} \\ &\Leftrightarrow \mathcal{N}_x \subseteq \mathcal{W} \end{aligned}$$

This shows that UC0 is equivalent to UC1. The last step uses the maximality of $\mathcal{W}$. The one before that is valid because finite intersections of neighbourhoods are neighbourhoods, and finite intersections of sets in $\mathcal{W}$ lie in $\mathcal{W}$. □

Examples:

(0) The fixed ultrafilter $\mathcal{W}_x$ converges to x.
(1) If X is discrete and $\mathcal{W} \to x$ then $\mathcal{W} = \mathcal{W}_x$. Indeed, $\{x\}$ is a neighbourhood of x so $\{x\} \in \mathcal{W}$ so $x \in S \Rightarrow \{x\} \subseteq S \Rightarrow S \in \mathcal{W}$. This shows that $\mathcal{W}_x \subseteq \mathcal{W}$ and thus they are equal by maximality of $\mathcal{W}_x$.

(2) If
$$\mathcal{F} = \{S \subseteq \mathbb{C} \mid \mathbb{C} \setminus S \text{ is compact}\}$$
and $\mathcal{W}$ is an ultrafilter on $\mathbb{C}$ containing $\mathcal{F}$ then $\mathcal{W}$ does not converge.

(3) If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X then the set
$$\mathcal{F} = \{S \subseteq X \mid \exists N \in \mathbb{N} \quad n \geq N \Rightarrow x_n \in S\}$$
has FIP. The sequence converges to $y \in X$ iff every ultrafilter $\mathcal{W} \supseteq \mathcal{F}$ converges to y.

Theorem 0.0.12. *The space X is Hausdorff iff every ultrafilter converges to at most one point.*

Proof. Suppose that X is Hausdorff, that $\mathcal{W} \to x$, and that $y \neq x$. Then there are disjoint open sets U, V with $x \in U$ and $y \in V$. As $\mathcal{W} \to x$, condition UC1 tells us that $U \in \mathcal{W}$. As $U \cap V = \emptyset$ and $\mathcal{W}$ has FIP, this means that $V \notin \mathcal{W}$. Thus (UC1 again) $\mathcal{W} \not\longrightarrow y$, as required.

Conversely, suppose that ultrafilter limits are unique. Suppose that x and y do not have disjoint neighbourhoods. Then $\mathcal{N}_x \cup \mathcal{N}_y$ has FIP, so there is an ultrafilter $\mathcal{W} \supseteq \mathcal{N}_x \cup \mathcal{N}_y$. This means that $\mathcal{W} \to x$ and $\mathcal{W} \to y$, so by hypothesis $x = y$. This shows that X is Hausdorff. □

Theorem 0.0.13. *The following are equivalent:*

(0) *X is compact.*
(1) *Every covering of X by subbasic open sets has a finite subcover.*
(2) *Every ultrafilter on X has a limit.*

Proof. It is immediate that (0) implies (1). Suppose that (1) holds, and that $\mathcal{W}$ is an ultrafilter on X. Recall (condition UC2) that $\mathcal{N}'_x \subseteq \mathcal{W} \Rightarrow \mathcal{W} \to x$. Suppose that $\mathcal{W}$ has no limit, so each point x has a subbasic neighbourhood $x \in U_x \in \mathcal{N}'_x$ with $U_x \notin \mathcal{W}$. Choose a finite subcover $U_{x_1} \cup \ldots U_{x_n} = X$. As $U_{x_k} \notin \mathcal{W}$, properties UP4 and UP5 give a contradiction. This shows that $\mathcal{W}$ must have a limit after all. Thus (1) implies (2).

Finally, suppose (2) holds. Let $\mathcal{F}$ be a family of closed sets with FIP, so we need to show that $\bigcap_{\mathcal{F}} S \neq \emptyset$ (this is the closed-set characterisation of compactness).

Choose an ultrafilter $\mathcal{W} \supseteq \mathcal{F}$. By hypothesis, this converges, so (condition UC0):
$$\bigcap_{S\in\mathcal{W}} \overline{S} \neq \emptyset$$
However,
$$\bigcap_{S\in\mathcal{F}} S = \bigcap_{S\in\mathcal{F}} \overline{S} \supseteq \bigcap_{S\in\mathcal{W}} \overline{S} \neq \emptyset$$
as required. The first equality holds simply because the sets $S \in \mathcal{F}$ are assumed to be closed, and the inequality ($\supseteq$) is obvious if you think about it. □

The equivalence of (0) and (1) is called Alexander's subbasis theorem.

Theorem 0.0.14 (Tychonov)**.** *Suppose $(X_i)_{i\in I}$ is a family of compact spaces. Then $X = \prod_I X_i$ is compact.*

Proof. Suppose $\mathcal{W}$ is an ultrafilter on X. Then
$$\mathcal{F}_i = \{\overline{\pi_i(S)} \mid S \in \mathcal{W}\}$$
is a family of closed sets with FIP (because $f(A) \cap f(B) \supseteq f(A \cap B)$). As X_i is compact, we can choose a point
$$x_i \in \bigcap_{S\in\mathcal{F}_i} S = \bigcap_{S\in\mathcal{W}} \overline{\pi_i(S)} \neq \emptyset$$
Putting these together, we get a point $x = (x_i)_{i\in I}$ in X. The claim is of course that $\mathcal{W} \to x$. It is enough (condition UC3) to show that every subbasic neighbourhood V of x meets every set $S \in \mathcal{W}$. The subbasic neighbourhoods have the form
$$V = \pi_i^{-1}(U) \qquad U \text{ open in } X_i \qquad x \in V$$
The condition $x \in V = \pi_i^{-1}(U)$ is equivalent to $x_i = \pi_i(x) \in U$. The point x_i was chosen so that $x_i \in \overline{\pi_i(S)}$, so $U \cap \pi_i(S) \neq \emptyset$. Thus there is a point $y \in S$ with $\pi_i(y) \in U$. In other words,

$y \in \pi_i^{-1}(U) \cap S$ so $V \cap S = \pi_i^{-1}(U) \cap S \neq \emptyset$. This holds for every subbasic neighbourhood V of X and every $S \in \mathcal{W}$. As previously mentioned, this implies that $\mathcal{W} \to x$. We have shown that every ultrafilter $\mathcal{W}$ on X has a limit, so X is compact. $\square$