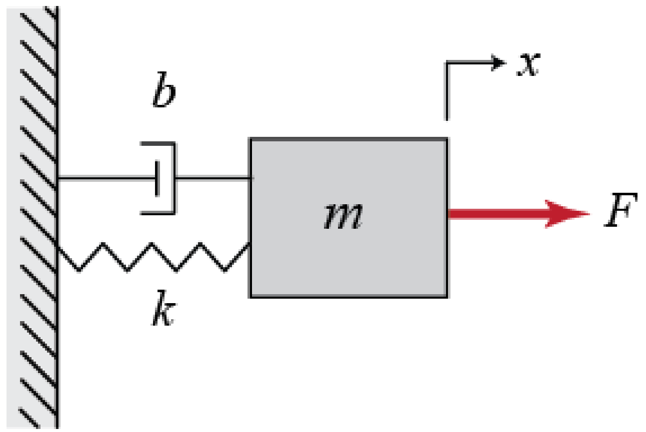
Exercise 1



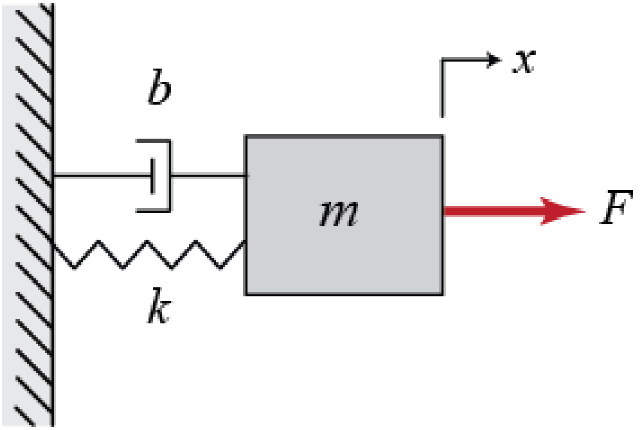
Suppose we have a simple mass, spring, and damper problem. Find

1. The modeling equation of this system (F input, x output).
2. The transfer function.

Let M = 1 kg b = 10 N s/m k = 20 N/m F = 1 N

3. Find Open-Loop Step Response.
4. Design a proportional controller to improve the output (ω_n=17.89).
5. Design a proportional derivative controller to improve the output (ζ=0.56 and ω_n=17.89).

Solution 1



1. The modeling equation of this system

$$\sum f = M\frac{d^2x}{dt^2}$$

$$M\ddot{x} = -b\dot{x} - k\,x + F \implies M\ddot{x} + b\dot{x} + k\,x = F$$

2. The transfer function.

Taking the Laplace transform of the modeling equation, we get

$$Ms^2X(s) + b\,sX(s) + k\,X(s) = F(s) \implies G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2+b\,s+k}$$

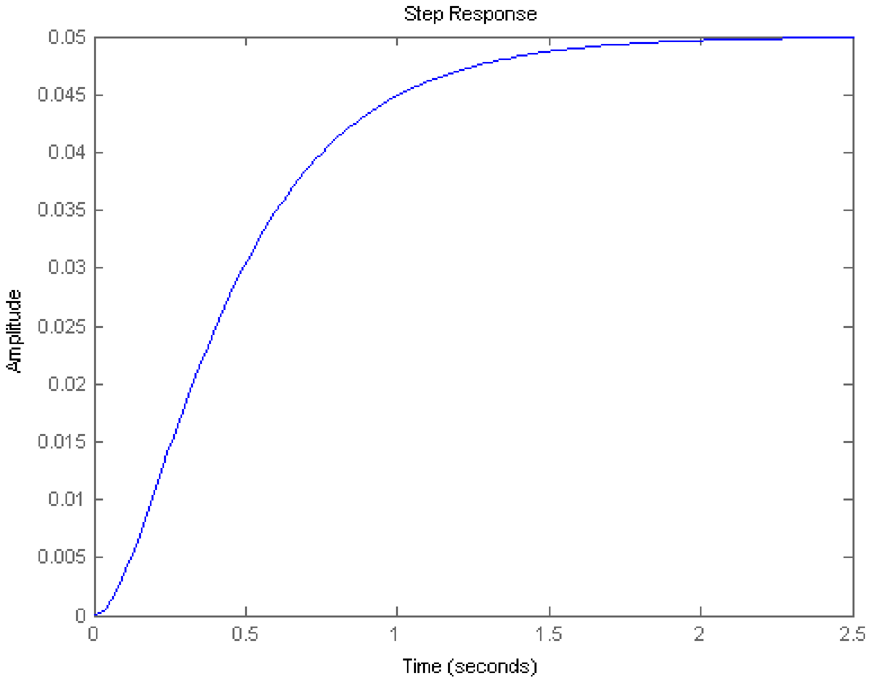
Let M = 1 kg b = 10 N s/m k = 20 N/m F = 1 N

Plug these values into the above transfer function $G(s) = \frac{X(s)}{F(s)} = \frac{1}{s^2 + 10\,s + 20}$

3. Find Open-Loop Step Response.

Create a new m-file (Matlab) and run the following code:

```
s = tf('s');
P = 1/(s^2 + 10*s + 20);
step(P)
```



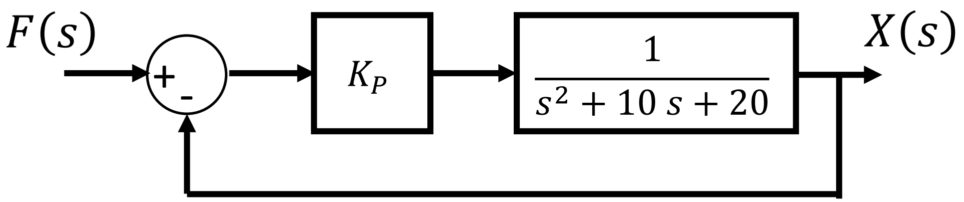
- the DC gain of the plant transfer function is 1/20, so 0.05 is the final value of the output to an unit step input.
- this corresponds to the steady-state error of 0.95, quite large indeed.
- the rise time is about one second,
- the settling time is about 1.5 seconds

4. **Proportional Control**

The proportional controller (Kp) reduces the rise time, increases the overshoot, and reduces the steady-state error.

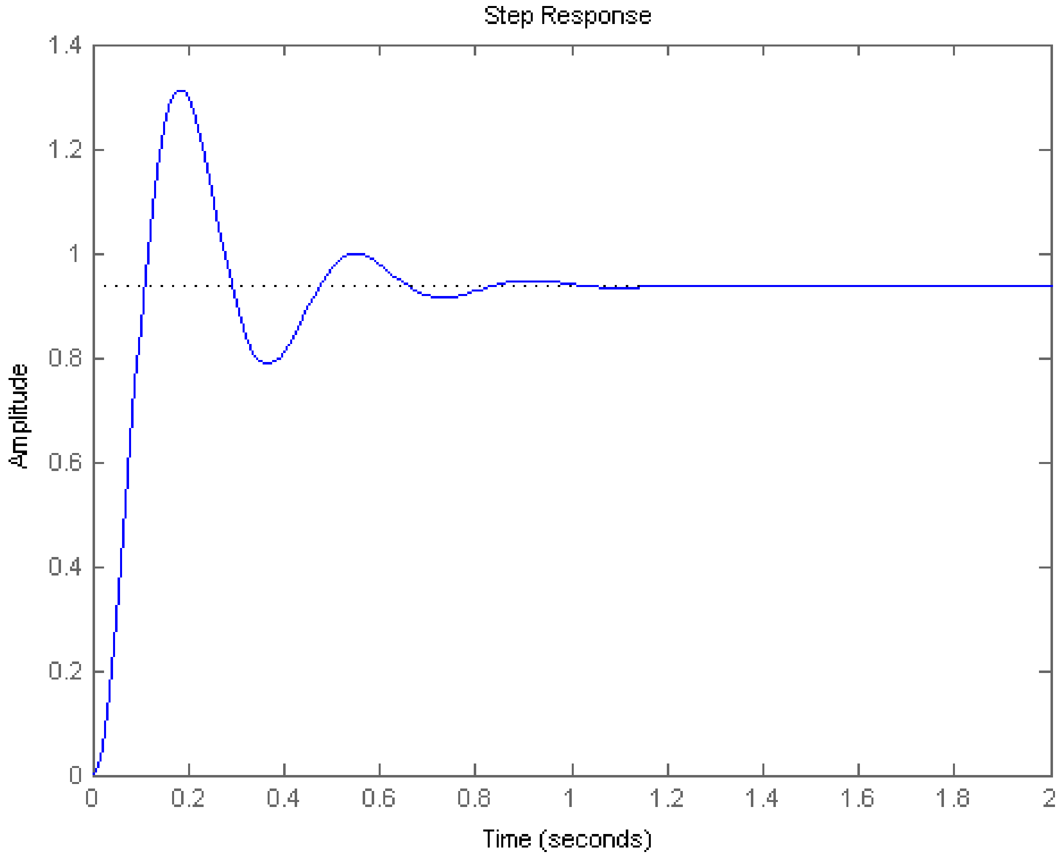
$$G(s) = \frac{X(s)}{F(s)} = \frac{K_P}{s^2 + 10\,s + (20 + K_P)}$$

$$\omega_n{}^2 = 20 + K_P \Longrightarrow K_P = \omega_n{}^2 - 20 = 300$$



Let the proportional gain equal 300 and change the m-file to the following

```
Kp = 300;
C = pid(Kp)
T = feedback(C*P,1)


t = 0:0.01:2;
step(T,t)
```



The above plot shows that the proportional controller reduced both the rise time and the steady-state error, increased the overshoot, and decreased the settling time by small amount.

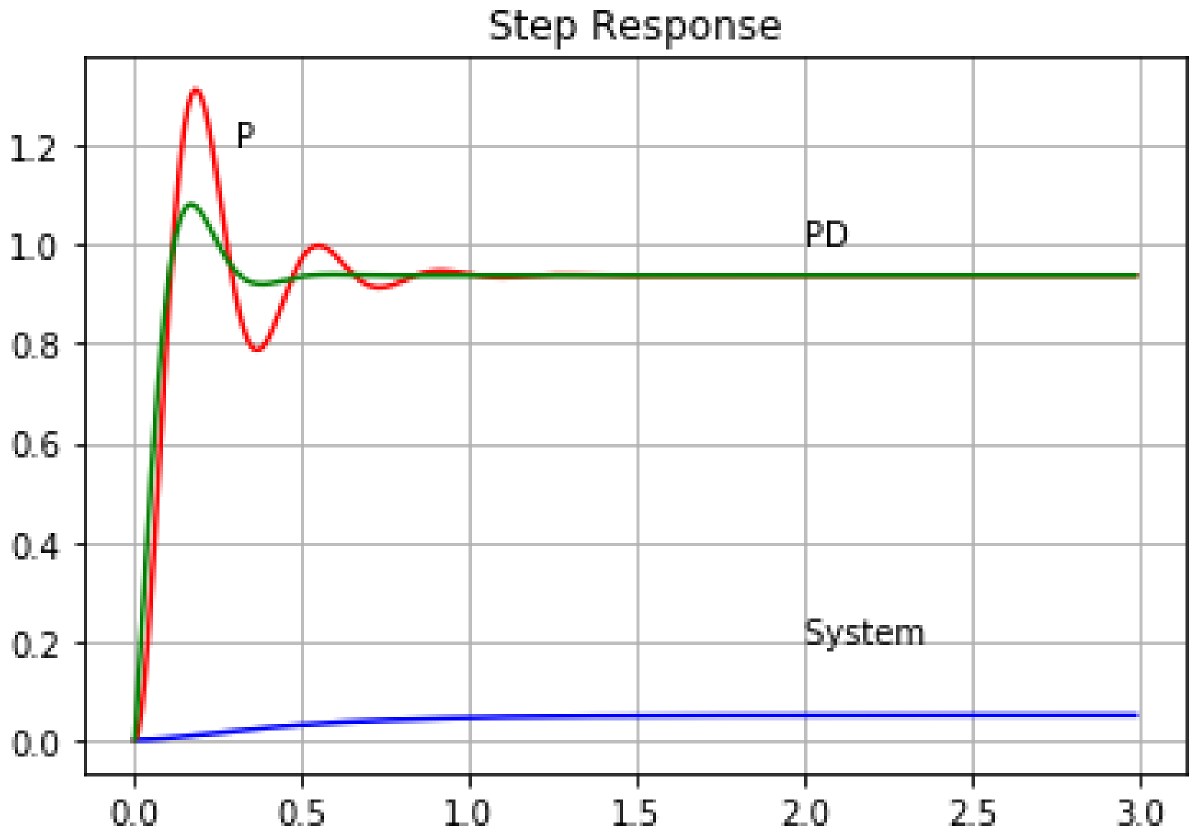
Proportional-Derivative Control

Now, let's take a look at a PD control. From the table shown above, we see that the derivative controller (Kd) reduces both the overshoot and the settling time. The closed-loop transfer function of the given system with a PD controller is:

$$\frac{X(s)}{F(s)} = \frac{K_d s + K_p}{s^2 + (10 + K_d)s + (20 + K_p)}$$

$$20 + K_P = {\omega_n}^2 \Longrightarrow K_P = {\omega_n}^2 - 20 = 300$$
$$10 + K_D = 2\zeta\omega_n \Longrightarrow K_D = 2\zeta\omega_n - 20 = 2\,(0.56)(17.89) - 10 \Longrightarrow K_D = 10$$

```
Kp = 300;

Kd = 10;

C = pid(Kp,0,Kd)

T = feedback(C*P,1)


t = 0:0.01:2;

step(T,t)
```

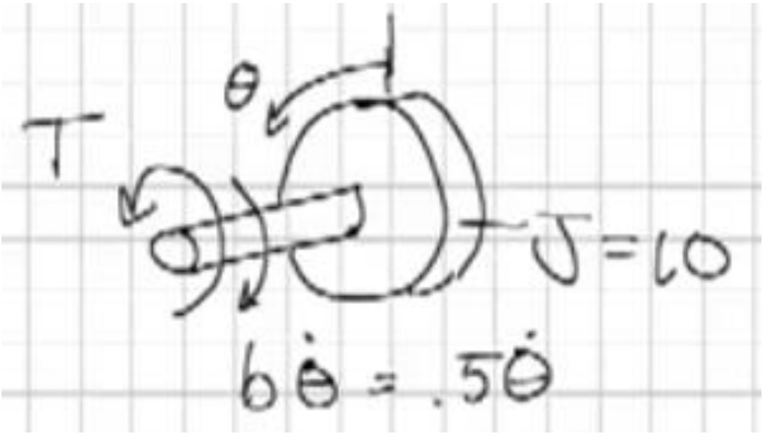


This plot shows that the derivative controller reduced both the overshoot and the settling time, and had a small effect on the rise time and the steady-state error.

Exercise 2

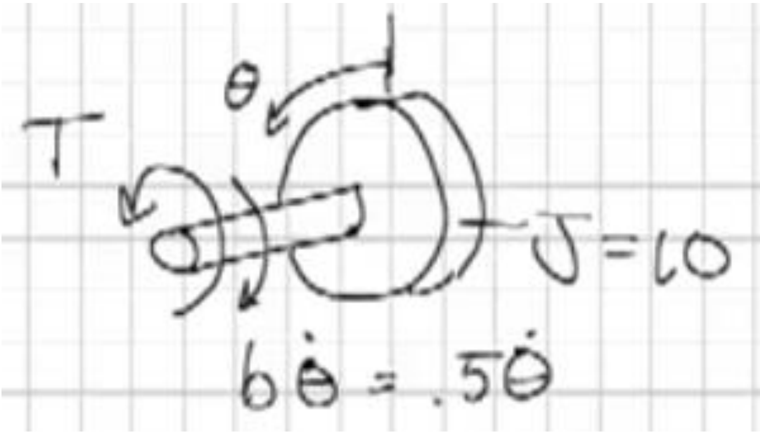
The plant consists of rotating mass with inertia J and a viscous friction b, a torque T is applied to control the position of the mass.

1. Find the system model.
2. Find the transfer function.
3. Control the position θ using T to have ζ = 0.7 (Proportional controller)



Solution 2

1. The modeling equation of this system



$$\sum Torques = J\frac{d^2\theta}{dt^2}$$

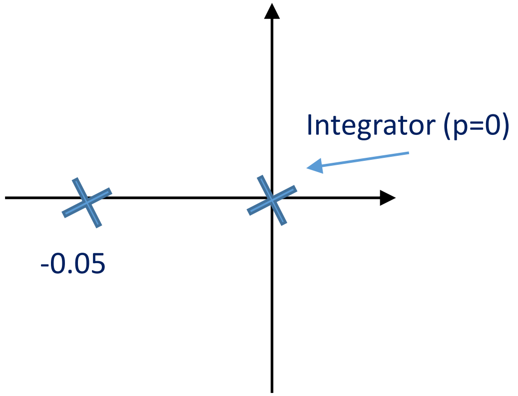
$$J\ddot{\theta} = T - b\dot{\theta} \Longrightarrow J\ddot{\theta} + b\dot{\theta} = T \Longrightarrow \ \ \ddot{\theta} + 0.05\,\dot{\theta} = 0.1\,T$$

2. The transfer function

$$s^2\theta(s) + 0.05\,S\,\theta(s) = 0.1\,T(s) \Longrightarrow \frac{\theta(s)}{T(s)} = \frac{0.1}{s^2 + 0.05\,s}$$

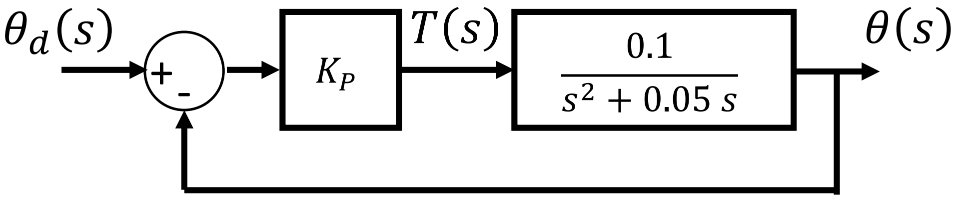
3. System control

The open Loop Analysis



$t_s = \frac{4}{\sigma} = \frac{4}{0.05} = 80\ sec$ (very slow moving system)

The close-Loop with Proportional controller Analysis



$$\frac{\frac{0.1K_P}{s^2 + 0.05\,s}}{1 + \frac{0.1K_P}{s^2 + 0.05\,s}} = \frac{0.1K_P}{s^2 + 0.05\,s + 0.1K_P}$$

Given the characteristic equation we calculate the value of K_P to have the desired transient response

$$CLCE: s^2 + 0.05\,s + 0.1K_P$$

$$CLCE\ Desired: s^2 + 2\zeta\omega_n\,s + {\omega_n}^2$$

The desired specifications: $\zeta = 0.7$

Matching $\Rightarrow$

$$2\zeta\omega_n = 0.05$$

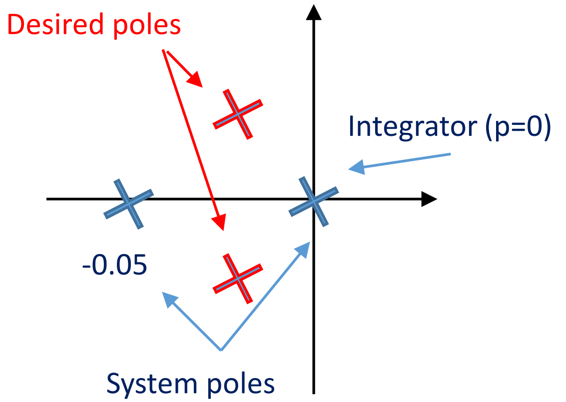
$$0.1K_P = {\omega_n}^2$$

$\Rightarrow$

$$\omega_n = \frac{0.05}{2\,(0.7)} = 0.0357\ \ rad/sec$$

$$K_P = 10\,(0.0357)^2 = 0.0127$$

The desired poles: $P_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -0.025 \pm j0.025$



Exercise 2

The plant G(s) is given $G(s) = \frac{10}{s^2-3\,s+2}$, design a PD controller (using pole placement) to have the desired system response: $\boldsymbol{\xi = 0.5}$, and $\boldsymbol{T_s = 1s}$

1. Find the desired poles and give the PD controller transfer function $G_c(s)$ (general form).
2. Find the closed loop transfer function CLTF (with PD controller).
3. Find the closed loop characteristic equation CLCE.
4. Find the gains K_D and K_P of the PD controller.

Solution

1. desired poles at : $\boldsymbol{T_s} = \frac{4}{\xi\boldsymbol{\omega_n}} \rightarrow \boldsymbol{\omega_n} = \frac{4}{\xi\boldsymbol{T_s}} = 8\, rad/sec$

$$s_{1,2} = -\boldsymbol{\xi}\boldsymbol{\omega_n} \pm \boldsymbol{\omega_n}\sqrt{1-\boldsymbol{\xi}^2} = -4 \pm j\ 6.93$$

PD controller transfer function: $G_c(s) = K_P + K_D\, s$

2. CLTF : $T(s) = \frac{G_C(s)G(s)}{1+G_C(s)G(s)} = \frac{10\,(K_P+K_D\,s)}{s^2+(10\,K_D-3)s+10\,K_P+2}$

3. CLCE: $s^2+(10\,K_D-3)s+10\,K_P+2=0$

4. Desired CLCE: $s^2+2\,\boldsymbol{\xi}\,\boldsymbol{\omega_n}\,s+\omega_n^2=0$

PD controller Gains: $K_P = \frac{\omega_n^2-2}{10}$ and $K_D = \frac{2\,\boldsymbol{\xi}\,\boldsymbol{\omega_n}+3}{10}$

$K_P = 6.2$ and $K_D = 1.1$

Exercise 3

Use a PI controller to control the system $\mathrm{G}(s) = \frac{2}{s+4}$ to meet the specifications $\zeta = 0.7 \; and \; t_s < 0.5 \; sec$.

Solution

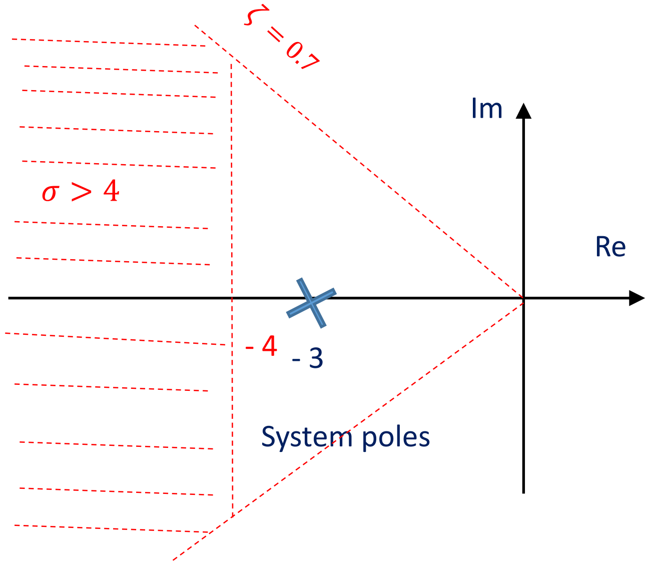
The system is type zero, it has no integrator. In order to have no steady-state error we need to add an integrator to make the system type one.

The open Loop Analysis: the system pole at p= - 3.

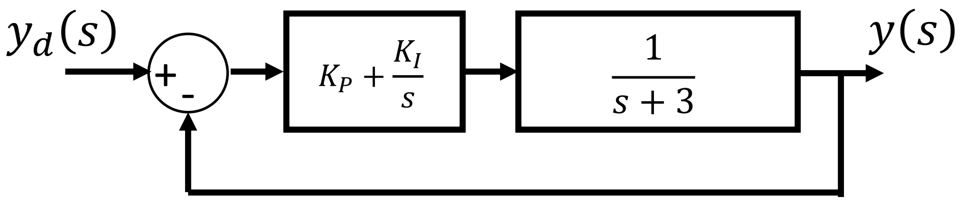
$\zeta = 0.7.$

$t_s = \frac{4}{\sigma} < 1$ ⇒ $\sigma = \zeta\omega_n > 4$ ⇒ $\omega_n > \frac{4}{\zeta} = \frac{4}{0.7}$ ⇒ $\omega_n > 5.71$ ⇒ $\omega_n = 6$

satisfy the settling time



The controller design: PI controller



The PI controller can be written as:

$$K_P + \frac{K_I}{s} = K_P \frac{(s + \frac{K_I}{K_P})}{s}$$

Zero $=K_{IP}$

Gain

Integrator

$$CLCE : s(s+3) + K_P\left(s + \frac{K_I}{K_P}\right) = 0 \Rightarrow s^2 + (2 + K_P)\,s + K_I = 0$$

$$Desired\ CLCE : s^2 + 2\zeta\omega_n\, s + {\omega_n}^2 = s^2 + 2(0.7)(6)s + (6)^2 = s^2 + 8.4\,s + 36 = 0$$

$$\Rightarrow \begin{matrix} 2 + K_P = 8.4 \\ K_I = 36 \end{matrix} \Rightarrow \begin{cases} K_P = 7.4 \\ K_I = 36 \end{cases}$$