STAT 436

Tutorial set #5

Question 1:

Suppose that the process $\{y_t\}$ follows an AR(1) model, with $|\phi_1| < 1$, find the autocovariance function for the process $W_t = \nabla y_t = (y_t - y_{t-1})$ in terms of ϕ_1 and σ_ε^2, (where σ_ε^2 is the white noise variance).

Since $\{y_t\}$ follows an AR(1) model, then we know that:

$$\gamma_k = \mathrm{Cov}(y_t, y_{t-k}) = \phi^k \frac{\sigma_\varepsilon^2}{1-\phi^2} \quad , k \geq 0$$

and thus:

$$\mathrm{Cov}(W_t, W_{t-k}) = \mathrm{Cov}[(y_t - y_{t-1}, y_{t-k} - y_{t-k-1})]$$

$$= \mathrm{cov}(y_t, y_{t-k}) + \mathrm{cov}(y_t, -y_{t-k-1}) + \mathrm{cov}(-y_{t-1}, y_{t-k}) + \mathrm{cov}(-y_{t-1}, -y_{t-k-1})$$

$$= \frac{\sigma_\varepsilon^2}{1-\phi^2}[\phi^k - \phi^{k+1} - \phi^{k-1} + \phi^k]$$

$$= \frac{\sigma_\varepsilon^2}{1-\phi^2}[2\phi^k - \phi^{k+1} - \phi^{k-1}]$$

Question 2:

Let the process $\{y_t\}$ follows an AR(2) model, with the following special form: $y_t = \phi_2 y_{t-2} + \varepsilon_t$, use the general method to find the values of ϕ_2 that make the process stationary.

The variance of the process $\{y_t\}$ is:

$$\mathrm{Var}(y_t) = \mathrm{Var}(\phi_2 y_{t-2} + \varepsilon_t) = {\phi_2}^2 \mathrm{Var}(y_{t-2}) + \sigma_\varepsilon^2$$

and for the process $\{y_t\}$ to be stationary, then the following must be satisfied:

$$Var(y_t) = Var(y_{t-2})$$

thus,

$$Var(y_t) = {\phi_2}^2 \mathrm{Var}(y_t) + \sigma_\varepsilon^2$$

or,

$$Var(y_t) = \frac{\sigma_\varepsilon^2}{1 - {\phi_2}^2}$$

for $Var(y_t)$ to be finite then we have to put some conditions on the parameter ϕ_2, which is $|\phi_2| < 1$, and hence this is the stationarity condition for this process.

Question 3:

Let the process $\{y_t\}$ follows an AR(2) model, with the following parameter values: $\phi_1 = 0.5$, $\phi_2 = -0.5$:

1- is the process $\{y_t\}$ stationary?

The stationarity conditions for the AR(2) process are:

$$|\phi_2| < 1 \implies |-0.5| \; (< 1)$$

$$\phi_1 + \phi_2 < 1 \implies 0.5 - 0.5 = 0 \; (< 1)$$

$$\phi_2 - \phi_1 < 1 \implies -0.5 - 0.5 = -1 \; (< 1)$$

and since all stationarity conditions are satisfied then the process $\{y_t\}$ is stationary.

2- find the ψ_j weights in the general linear process.

To find the ψ_j weights in the general linear process, we rewrite the model in the following form:

$$y_t = (1 - 0.5\,B + 0.5B^2)^{-1}\varepsilon_t$$

and from the form of general linear process:

$$y_t = (1 - 0.5\,B + 0.5B^2)^{-1}\varepsilon_t$$

we thus can find the ψ_j weights by equating the coefficients of the terms B in the following equation:

$$(1 - 0.5\,B + 0.5B^2)(\psi_0 + \psi_1 B + \psi_2 B^2 + \psi_3 B^3 + \cdots) = 1$$

so, the coefficient of:

B^0: $1 \times \psi_0 = 1 \implies \psi_0 = 1$

B^1: $-0.5 + \psi_1 = 0 \implies \psi_1 = 0.5$

B^2: $\psi_2 - 0.5\psi_1 + 0.5 = 0 \implies \psi_2 = 0.5\psi_1 - 0.5$

B^3: $\psi_3 - 0.5\psi_2 + 0.5\psi_1 = 0 \implies \psi_3 = 0.5\psi_2 - 0.5\psi_1$

and continuing in the same manner, we can see that the general form of these weights takes the form:

$$\boxed{\psi_j = 0.5\psi_{j-1} - 0.5\psi_{j-2} \;, j \geq 2}$$

Question 4:

Let the process $\{y_t\}$ follows an AR(2) model, for the following cases find the roots of the characteristic equation, and check if the process is stationary:

1- with parameters: $\phi_1 = 0.6,\ \phi_2 = -0.8$

the process $\{y_t\}$ takes the form:

$$y_t = 0.6y_{t-1} - 0.8\,y_{t-2} + \varepsilon_t$$
$$\Longrightarrow (1 - 0.6\,B + 0.8B^2)y_t = \varepsilon_t$$

thus, the characteristic equation has the form:

$$(1 - 0.6\,B + 0.8B^2) = 0$$

and to find its roots we equate it to zero:

$$\Longrightarrow (1 - 0.2\,B)\,(1 - 0.4B) = 0$$
$$(1 - 0.2\,B\,) = 0 \Longrightarrow G_1^{-1} = \frac{1}{0.2} = 5 > 1$$
$$(1 - 0.4\,B\,) = 0 \Longrightarrow G_2^{-1} = \frac{1}{0.4} = 2.5 > 1$$

thus the process satisfies the stationarity conditions.

2- with parameters: $\phi_1 = 2.4,\ \phi_2 = -0.8$

the process $\{y_t\}$ takes the form:

$$y_t = 2.4y_{t-1} - 0.8\,y_{t-2} + \varepsilon_t$$
$$\Longrightarrow (1 - 2.4\,B + 0.8B^2)y_t = \varepsilon_t$$

thus, the characteristic equation has the form:

$$\phi(B) = (1 - 2.4\,B + 0.8B^2)$$

and to find its roots we equate it to zero:

$$(1 - 2.4\,B + 0.8B^2) = 0$$
$$\Longrightarrow (1 - 2B)\,(1 - 0.4B) = 0$$
$$(1 - 2B\,) = 0 \Longrightarrow G_1^{-1} = \frac{1}{2} < 1$$
$$(1 - 0.4\,B\,) = 0 \Longrightarrow G_2^{-1} = \frac{1}{0.4} = 2.5 > 1$$

and since one of the roots is not greater than 1, the process is not stationary.

Question 5:

Find the Yule-Walker equations for the following models and solve these equations to get values for ρ_1 and ρ_2.

1- $y_t - 0.8y_{t-1} = \varepsilon_t$

Multiplying both sides by y_{t-k}, and taking the mathematical expectation:

$$E(y_t\, y_{t-k}) = 0.8E(y_{t-1}\, y_{t-k}) + E(\varepsilon_t\, y_{t-k})$$

$$\Rightarrow \gamma_k = 0.8\gamma_{k-1} + E(\varepsilon_t\, y_{t-k})$$

Now notice that Y_{t-k} depends only on $\varepsilon_{t-k}, \varepsilon_{t-k-1}, \ldots$, then:

$$E[\varepsilon_t y_{t-k}] = \begin{cases} \sigma_\varepsilon^2 & ,k = 0 \\ 0 & ,k = 1,2,3 \end{cases}$$

so,

$$\gamma_0 = 0.8\gamma_{-1} + \sigma_\varepsilon^2$$
$$= 0.8\gamma_1 + \sigma_\varepsilon^2,$$
$$\gamma_k = 0.8\gamma_{k-1}, \qquad k > 0$$

from which we can calculate the ACF ρ_k:

$$\boxed{\rho_k = 0.8\rho_{k-1} \quad ,k = 1,2,\ldots}$$

these are the Yule-Walker equations for this model.

For example, for $k = 1$:

$$\rho_1 = 0.8\rho_0 = 0.8$$

and for $k = 2$:

$$\rho_2 = 0.8\rho_1 \Longrightarrow \rho_2 = 0.8^2 = 0.64$$

2- $y_t - 0.9y_{t-1} + 0.4y_{t-2} = \varepsilon_t$

Multiplying both sides by y_{t-k}, and taking the mathematical expectation:

$$\gamma_k = E[y_t y_{t-k}] = 0.9E[y_{t-1}y_{t-k}] - 0.4\,E[y_{t-2}y_{t-k}] + E[\varepsilon_t y_{t-k}]$$

and since Y_{t-k} depends only on $\varepsilon_{t-k}, \varepsilon_{t-k-1}, \ldots$, then:

$$E[\varepsilon_t y_{t-k}] = \begin{cases} \sigma_\varepsilon^2 & ,k = 0 \\ 0 & ,k = 1,2,3 \end{cases}$$

so,

$$\gamma_0 = 0.9\gamma_{-1} - 0.4\gamma_{-2} + \sigma_\varepsilon^2$$
$$= 0.9\gamma_1 - 0.4\gamma_2 + \sigma_\varepsilon^2,$$
$$\gamma_k = 0.9\gamma_{k-1} - 0.4\gamma_{k-2}, \qquad k > 0$$

from which we can calculate the ACF ρ_k:

$$\boxed{\rho_k = 0.9\rho_{k-1} - 0.4\rho_{k-2} \quad ,k = 1,2,\ldots}$$

these are the Yule-Walker equations for this model.

For example, for $k = 1$:

$$\rho_1 = 0.9\rho_0 - 0.4\rho_1$$

$$\rho_1(1 + 0.4) = 0.9 \;\Longrightarrow\; \rho_1 = \frac{0.9}{(1 + 0.4)} = 0.643$$

and for $k = 2$:

$$\rho_2 = 0.9\rho_1 - 0.4\rho_0 \implies \rho_2 = 0.9(0.643) - 0.4 = 0.1787$$