قسم الاحصاء وبحوث العمليات

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Tutorial set #3

Question 1:

Write the Yule-Walker equations for every model of the following, where $\varepsilon_{t} \sim i.i.d(0, \sigma_{\varepsilon}^{2})$:

1- $y_t = 0.5y_{t-1} + \varepsilon_t$

To find the Yule-Walker equations for the model, we multiply both sides of the equation by y_{t-k} and take expectations:

$E(y_t y_{t-k}) = 0.5\, E(y_{t-1} y_{t-k}) + E(\varepsilon_t y_{t-k})$

$\gamma_k = 0.5\, \gamma_{k-1} + 0 \qquad (\text{For } k \neq 0)$

dividing both sides by γ_0, we get:

$$\rho_k = 0.5\, \rho_{k-1}, \quad k = 1, 2, 3, \dots$$

These equations are called Yule-Walker equations, we can use them in finding autocorrelation and partial autocorrelation coefficients of the model.

2- $y_t = 1.2\, y_{t-1} - 0.7\, y_{t-2} + \varepsilon_t$

multiply both sides of the equation by y_{t-k} and take expectations:

$E(y_t y_{t-k}) = 1.2\, E(y_{t-1} y_{t-k}) - 0.7\, E(y_{t-2} y_{t-k}) + E(\varepsilon_t y_{t-k})$

$\gamma_k = 1.2\, \gamma_{k-1} - 0.7\, \gamma_{k-2} \qquad (\text{For } k \neq 0)$

dividing both sides by γ_0, we get:

$$\rho_k = 1.2\, \rho_{k-1} - 0.7 \rho_{k-2}, \quad k = 1, 2, 3, \dots$$

3- Find ρ_1, ρ_2 , ϕ_{kk} for the models in (1) and (2).

For model (1):

we found the following ACF:

$$\rho_k = 0.5\, \rho_{k-1}, \quad k = 1, 2, 3, \dots$$

Thus:

$$\rho_1 = 0.5\rho_0 = 0.5 \qquad (\rho_0 = 1)$$

$$\rho_2 = 0.5\rho_1 = (0.5)(0.5) = 0.25$$

Applying the recurrence relation for finding the PACF:

$$\phi_{00} = 1 \quad ; \quad \phi_{11} = \rho_1$$

$$\phi_{kk} = \frac{\rho_k - \sum_{j=1}^{k-1} \phi_{k-1,j}\ \rho_{k-j}}{1 - \sum_{j=1}^{k-1} \phi_{k-1,j}\ \rho_j}$$

$$\emptyset_{kj} = \emptyset_{k-1,j} - \emptyset_{k,k}\emptyset_{k-1,k-j} \quad ; \quad j = 1,2,\dots,k-1$$

$$\phi_{00} = 1$$

$$\phi_{11} = \rho_1 = 0.5$$

$$\phi_{22} = \frac{\rho_2 - \sum_{j=1}^{1} \phi_{1,j}\ \rho_{2-j}}{1 - \sum_{j=1}^{1} \phi_{1,j}\ \rho_j} = \frac{\rho_2 - \phi_{11}\rho_1}{1 - \phi_{11}\rho_1}$$

$$= \frac{0.25 - (0.5)(0.5)}{1 - (0.5)(0.5)} = 0$$

we can show that:

$$\phi_{33} = \phi_{44} = \cdots = 0$$

For model 2: we found the following ACF:

$$\boldsymbol{\rho_k = 1.2\ \rho_{k-1} - 0.7\rho_{k-2} \quad ; \quad k = 1, 2, 3, \dots}$$

Thus:

$$\rho_1 = 1.2\ \rho_0 - 0.7\rho_{1-2}$$

$$= 1.2 - 0.7\rho_1 \qquad (\rho_{-1} = \rho_1)$$

$$\rho_1(1 + 0.7) = 1.2 \implies \rho_1 = \frac{1.2}{1.7} = 0.7059$$

$$\rho_2 = 1.2\ \rho_1 - 0.7\rho_0 \qquad (\rho_0 = 1)$$

$$= 1.2\ \rho_1 - 0.7 = 1.2(0.7059) - 0.7 = 0.1471$$

$$\rho_3 = 1.2\ \rho_2 - 0.7\rho_1$$

$$= 1.2\ (0.1471) - 0.7(0.7059) = -0.3176$$

Applying the recurrence relation for finding the PACF:

$$\phi_{00} = 1 \quad ; \quad \phi_{11} = \rho_1$$

$$\phi_{kk} = \frac{\rho_k - \sum_{j=1}^{k-1} \phi_{k-1,j}\ \rho_{k-j}}{1 - \sum_{j=1}^{k-1} \phi_{k-1,j}\ \rho_j}$$

$$\emptyset_{kj} = \emptyset_{k-1,j} - \emptyset_{k,k}\emptyset_{k-1,k-j} \quad ; \quad j = 1,2,\dots,k-1$$

$$\phi_{00} = 1$$

$$\phi_{11} = \rho_1 = 0.7059$$

$$\phi_{22} = \frac{\rho_2 - \sum_{j=1}^{1} \phi_{1,j}\ \rho_{2-j}}{1 - \sum_{j=1}^{1} \phi_{1,j}\ \rho_j} = \frac{\rho_2 - \phi_{11}\rho_1}{1 - \phi_{11}\rho_1} = \frac{0.1471 - (0.7059)(0.7059)}{1 - (0.7059)(0.7059)} = -0.7$$

$$\phi_{33} = \frac{\rho_3 - \sum_{j=1}^{2} \phi_{2,j}\ \rho_{3-j}}{1 - \sum_{j=1}^{2} \phi_{2,j}\ \rho_j} = \frac{\rho_3 - [\phi_{21}\rho_2 + \phi_{22}\rho_1]}{1 - [\phi_{21}\rho_1 + \phi_{22}\rho_2]}$$

Thus, we notice that we need to find ϕ_{21} :

$$\phi_{21} = \phi_{11} - \phi_{22}\phi_{11} = 0.7059 - (-0.7)(0.7059) = 1.2$$

Hence:

$$\phi_{33} = \frac{-0.1376 - [1.2\ (0.1471) + (-0.7)(0.7059)]}{1 - [1.2(0.7059) + (-0.7)(0.1471)]} = 0$$

we can show that:

$$\phi_{44} = \phi_{55} = \cdots = 0$$

Question 2:

Find the Yule-Walker equations for the following models and solve these equations to get values for ρ_1 and ρ_2.

1- $\mathbf{y_t - 0.8y_{t-1} = \varepsilon_t}$

$$\mathbf{y_t = 0.8y_{t-1} + \varepsilon_t}$$

Multiplying both sides by y_{t-k}, and taking the mathematical expectation:

$$E(y_t\, y_{t-k}) = 0.8E(y_{t-1}\, y_{t-k}) + E(\varepsilon_t\, y_{t-k})$$

$$\Rightarrow \gamma_k = 0.8\gamma_{k-1} + E(\varepsilon_t\, y_{t-k})$$

Now notice that Y_{t-k} depends only on $\varepsilon_{t-k}, \varepsilon_{t-k-1}, \ldots$, then:

$$E[\varepsilon_t y_{t-k}] = \begin{cases} \sigma_\varepsilon^2 & ,k = 0 \\ 0 & ,k = 1,2,3 \end{cases}$$

so,

$$\gamma_0 = 0.8\gamma_{-1} + \sigma_\varepsilon^2$$
$$= 0.8\gamma_1 + \sigma_\varepsilon^2 \quad ; \quad k = 0$$
$$\gamma_k = 0.8\gamma_{k-1} \quad ; \quad k > 0$$

dividing both sides by $\boldsymbol{\gamma_0}$, from which we can calculate the ACF ρ_k:

$$\boxed{\rho_k = 0.8\rho_{k-1} \quad ,k = 1,2,\ldots}$$

these are the Yule-Walker equations for this model.

For example, for $k = 1$:

$$\rho_1 = 0.8\rho_0 = 0.8$$

and for $k = 2$:

$$\rho_2 = 0.8\rho_1 \Longrightarrow \rho_2 = 0.8^2 = 0.64$$

2- $\mathbf{y_t - 0.9y_{t-1} + 0.4y_{t-2} = \varepsilon_t}$

$$\mathbf{y_t = 0.9y_{t-1} - 0.4y_{t-2} + \varepsilon_t}$$

Multiplying both sides by y_{t-k}, and taking the mathematical expectation:

$$\gamma_k = E[y_t y_{t-k}] = 0.9E[y_{t-1}y_{t-k}] - 0.4\,E[y_{t-2}y_{t-k}] + E[\varepsilon_t y_{t-k}]$$

and since Y_{t-k} depends only on $\varepsilon_{t-k}, \varepsilon_{t-k-1}, \ldots$, then:

$$E[\varepsilon_t y_{t-k}] = \begin{cases} \sigma_\varepsilon^2 & ,k = 0 \\ 0 & ,k = 1,2,3 \end{cases}$$

so,

$$\gamma_0 = 0.9\gamma_{-1} - 0.4\gamma_{-2} + \sigma_\varepsilon^2$$
$$= 0.9\gamma_1 - 0.4\gamma_2 + \sigma_\varepsilon^2 \quad ; k = 0$$
$$\gamma_k = 0.9\gamma_{k-1} - 0.4\gamma_{k-2} \quad ; \quad k > 0$$

from which we can calculate the ACF ρ_k:

$$\boxed{\rho_k = 0.9\rho_{k-1} - 0.4\rho_{k-2} \quad ,k = 1,2,\ldots}$$

these are the Yule-Walker equations for this model.

For example, for $\boldsymbol{k} = \mathbf{1}$:

$$\rho_1 = 0.9\rho_0 - 0.4\rho_1$$

$$\rho_1(1 + 0.4) = 0.9 \implies \rho_1 = \frac{0.9}{(1 + 0.4)} = 0.643$$

and for $\boldsymbol{k} = \mathbf{2}$:

$$\rho_2 = 0.9\rho_1 - 0.4\rho_0 \implies \rho_2 = 0.9(0.643) - 0.4 = 0.1787$$

Question 3:

Assume $\varepsilon_{t} \sim i.i.d(0, \sigma_\varepsilon^2)$, and let the observed series be defined as $y_t = \varepsilon_t - \theta\varepsilon_{t-1}$

Where the parameter θ can take either the value $\theta = 3$ or $\theta = \frac{1}{3}$.

1- Find the autocorrelation function of the series $\{Y_t\}$ for both cases, compare them.

For the model where $\theta = 3$:

$$E(y_t) = E(\varepsilon_t - 3\varepsilon_{t-1}) = 0$$

$$V(y_t) = \gamma_0 = V(\varepsilon_t - 3\varepsilon_{t-1}) = 10\,\sigma_\varepsilon^2 \qquad ; [cov(\varepsilon_t, \varepsilon_{t-1}) = 0]$$

$$\gamma_k = Cov[y_t, y_{t-k}]$$

$$= Cov[(\varepsilon_t - 3\varepsilon_{t-1}), (\varepsilon_{t-k} - 3\varepsilon_{t-k-1})]$$

now for k =1:

$$\gamma_1 = Cov[(\varepsilon_t - 3\varepsilon_{t-1}), (\varepsilon_{t-1} - 3\varepsilon_{t-2})] = -3\ \sigma_\varepsilon^2$$

for k =2:

$$\gamma_2 = Cov[(\varepsilon_t - 3\varepsilon_{t-1}), (\varepsilon_{t-2} - 3\varepsilon_{t-3})] = 0$$

Hence, the ACF has the form:

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \begin{cases} 1, & k = 0 \\ -0.3, & k = 1 \\ 0, & k \geq 2 \end{cases}$$

For the model where $\theta = {}^{1}/_{3}$:

$$E(y_t) = E\left(\varepsilon_t - \frac{1}{3}\varepsilon_{t-1}\right) = 0$$

$$V(y_t) = V\left(\varepsilon_t - \frac{1}{3}\varepsilon_{t-1}\right) = \frac{10}{9}\,\sigma_\varepsilon^2$$

$$\gamma_k = Cov\left[\left(\varepsilon_t - \frac{1}{3}\varepsilon_{t-1}\right), \left(\varepsilon_{t-k} - \frac{1}{3}\varepsilon_{t-k-1}\right)\right]$$

for k =1:

$$\gamma_1 = Cov\left[\left(\varepsilon_t - \frac{1}{3}\varepsilon_{t-1}\right), \left(\varepsilon_{t-1} - \frac{1}{3}\varepsilon_{t-2}\right)\right] = -\frac{1}{3}\,\sigma_\varepsilon^2$$

for k =2:

$$\gamma_2 = Cov\left[\left(\varepsilon_t - \frac{1}{3}\varepsilon_{t-1}\right), \left(\varepsilon_{t-2} - \frac{1}{3}\varepsilon_{t-3}\right)\right] = 0$$

Hence, the ACF has the form:

$$\rho_k = \begin{Bmatrix} 1, & k = 0 \\ -0.3, & k = 1 \\ 0, & k \geq 2 \end{Bmatrix}$$

Thus, we notice that both process has the same ACF!

Question 4: Write the following models using the Backshift operator B:

1- $y_t - 0.5\, y_{t-1} = \varepsilon_t$:

The backshift operator B is $\boldsymbol{B^r y_t = y_{t-r}}$; $\boldsymbol{r = 1, 2, \ldots}$

$(1 - 0.5B)\, y_t = \varepsilon_t$

2- $y_t = \varepsilon_t - 1.3\,\varepsilon_{t-1} + 0.4\,\varepsilon_{t-2}$

$y_t = (1 - 1.3\,B + 0.4B^2)\,\varepsilon_t$

3- $y_t - 0.5\,y_{t-1} = \varepsilon_t - 1.3\,\varepsilon_{t-1} + 0.4\,\varepsilon_{t-2}$

$(1 - 0.5B)\,y_t = (1 - 1.3\,B + 0.4B^2)\,\varepsilon_t$

4- $y_t - 0.2\,y_{t-2} = \varepsilon_t + 1.8\,\varepsilon_{t-1}$

$(1 - 0.2B^2)y_t = (1 + 1.8\,B)\,\varepsilon_t$

5- $y_t - 0.2\,y_{t-1} + y_{t-2} = \varepsilon_t - 1.7\,\varepsilon_{t-1} + 0.3\,\varepsilon_{t-3}$

$(1 - 0.2B + B^2)\,y_t = (1 - 1.7\,B + 0.3B^3)\,\varepsilon_t$

Question 4:

Express the following models in terms of the process $\{y_t\}$ and $\{\varepsilon_t\}$:

1- $\nabla^3 y_t = \nabla\varepsilon_t$

Difference operator is defined as: $\boldsymbol{\nabla y_t = y_t - y_{t-1}}$

$\nabla\varepsilon_t = \varepsilon_t - \varepsilon_{t-1}$ ،

$$\nabla^3 y_t = \nabla\nabla\nabla(y_t) = \nabla\nabla(y_t - y_{t-1}) = \nabla[(y_t - y_{t-1}) - (y_{t-1} - y_{t-2})]$$

$$= \nabla(y_t - 2y_{t-1} + y_{t-2})$$

$$= (y_t - 2y_{t-1} + y_{t-2}) - (y_{t-1} - 2y_{t-2} + y_{t-3})$$

$$= (y_t - 3y_{t-1} + 3y_{t-2} - y_{t-3})$$

$$\therefore\ (y_t - 3y_{t-1} + 3y_{t-2} - y_{t-3}) = \varepsilon_t - \varepsilon_{t-1}$$

2- $\nabla^2 y_t = \nabla^3\varepsilon_t$

Same as above

3- $\nabla\, y_t = \nabla^2\varepsilon_t$

$$y_t - y_{t-1} = \nabla(\varepsilon_t - \varepsilon_{t-1})$$

$$y_t - y_{t-1} = \nabla\varepsilon_t - \nabla\varepsilon_{t-1}$$

$$y_t - y_{t-1} = \varepsilon_t - \varepsilon_{t-1} - \varepsilon_{t-1} + \varepsilon_{t-2}$$

$$y_t - y_{t-1} = \varepsilon_t - 2\varepsilon_{t-1} + \varepsilon_{t-2}$$