Ex 1:

Let X_1 and X_2 have the covariance matrix

$$A = \Sigma = Cov(X) = \begin{bmatrix} Var(X_1) & Cov(X_1, X_2) \\ Cov(X_2, X_1) & Var(X_2) \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 4 & 100 \end{bmatrix}$$

where $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$.

1) Find the eigen values and the eigen vectors.

Solution 1:

```
> model0 <- read.csv(file="A.csv",header=F, sep=";")
> model0
  V1  V2
1  1   4
2  4 100
> attributes(model0)
$names
[1] "V1" "V2"

$class
[1] "data.frame"

$row.names
[1] 1 2

> e <- eigen(model0)
> e
eigen() decomposition
$values
[1] 100.1613532   0.8386468

$vectors
           [,1]        [,2]
[1,] 0.04030552 -0.99918740
[2,] 0.99918740  0.04030552

> values <- e$values
> values
[1] 100.1613532   0.8386468
> values[1]
[1] 100.1614
> vectors <- e$vectors
> vectors
           [,1]        [,2]
[1,] 0.04030552 -0.99918740
[2,] 0.99918740  0.04030552

> vectors[ ,1]
[1] 0.04030552 0.99918740
```

So,

$$\lambda_1 = 100.1613532 \rightarrow e_1 = \begin{bmatrix} 0.04030552 \\ 0.99918740 \end{bmatrix}$$

$$\lambda_2 = 0.8386468 \rightarrow e_2 = \begin{bmatrix} -0.99918740 \\ 0.04030552 \end{bmatrix}$$

Note that always $\lambda_1 \geq \lambda_2 \geq 0$.

2) Find the Principles Components $\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{Y_1} \\ \boldsymbol{Y_2} \end{bmatrix}$.

Solution 2:

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = e'X = e' \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \text{ where } e = \begin{bmatrix} \overset{e_1}{0.04030552} & \overset{e_2}{-0.99918740} \\ 0.99918740 & 0.04030552 \end{bmatrix}_{2\times 2}$$

So,

$$Y_1 = e_1'X = e_1' \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = [0.04030552 \quad 0.99918740] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$= 0.04030552X_1 + 0.99918740X_2$$

$$Y_2 = e_2'X = e_2' \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = [-0.99918740 \quad 0.04030552] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$= -0.99918740X_1 + 0.04030552X_2$$

3) Show that $\boldsymbol{Trace\ (A)} = \sum_{i=1}^{2} \boldsymbol{Var(X_i)} = \sum_{i=1}^{2} \boldsymbol{\lambda_i}$.

```
> A <- data.matrix(model0)
> A
     V1  V2
[1,]  1   4
[2,]  4 100
> attributes(A)
$dim
[1] 2 2

$dimnames
$dimnames[[1]]
NULL

$dimnames[[2]]
[1] "V1" "V2"


> class(A)
[1] "matrix"
> install.packages("matlib")

> library(matlib)
> tr(A)
[1] 101
> sum(values)
[1] 101
```

4) Show that $\boldsymbol{Var(Y_i) = \lambda_i}$ for $\boldsymbol{i = 1, 2}$ and $\boldsymbol{Cov(Y_i, Y_j) = 0, \forall\ i \neq j}$ for $\boldsymbol{i, j = 1, 2}$.

Solution 4:

$$* \ Var(Y_1) = Var(0.04030552X_1 + 0.99918740X_2)$$

$$= Cov(0.04030552X_1 + 0.99918740X_2, 0.04030552X_1 + 0.99918740X_2)$$

$$= Var(0.04030552X_1)$$
$$+Cov(0.04030552X_1, 0.99918740X_2) + Cov(0.99918740X_2, 0.04030552X_1)$$
$$+Var(0.99918740X_2)$$
$$= (0.04030552)^2 Var(X_1)$$
$$+2(0.04030552)(0.99918740)Cov(X_1, X_2)$$
$$+(0.99918740)^2 Var(X_2)$$
$$= 100.1613532$$
$$= \lambda_1.$$

And the same way for showing that $Var(Y_2) = \lambda_2$.

* $Cov(Y_1, Y_2) = Cov(0.04030552X_1 + 0.99918740X_2, -0.99918740X_1 + 0.04030552X_2)$

$$= Cov(0.04030552X_1, -0.99918740X_1)$$
$$+Cov(0.04030552X_1, 0.04030552X_2)$$
$$+Cov(0.99918740X_2, -0.99918740X_1)$$
$$+Cov(0.99918740X_2, 0.04030552X_2)$$
$$= (0.04030552)(-0.99918740)Var(X_1)$$
$$+(0.04030552)^2 Cov(X_1, X_2)$$
$$-(0.99918740)^2 Cov(X_2, X_1)$$
$$+(0.99918740)(0.04030552)Var(X_2)$$
$$= -0.040272767734448 + 0.0064981397698816$$
$$-3.99350184127504 + 4.0272767734448$$
$$\approx 0.$$

By using R where $Cov(Y) = Cov(e'X) = e'Cov(X)e$, we get the same results as the following:

```
> covY <- t(vectors)%*% A %*% vectors
> covY
             [,1]         [,2]
[1,] 1.001614e+02 8.881784e-16
[2,] 9.714451e-17 8.386468e-01
```

5) Find the ratio of the total variance produced by Y_1, Y_2 and for both Y_1 and Y_2.

Solution 5:

```
> ratio <- vector (mode="numeric")
> for (i in 1:nrow(A) ){
+ ratio[i] <- values[i]/sum(values)
+  }
> ratio
[1] 0.991696566 0.008303434
> cumsum(ratio)
[1] 0.9916966 1.0000000
```

Note that if we have $Y_1, Y_2, Y_3, \dots\dots\ , Y_n$ then:

*The ratio of the total variance produced by Y_j is $= \frac{\lambda_j}{\sum_{i=1}^{n} \lambda_i}$.

*The ratio of the total variance produced by $Y_1, Y_2, Y_3, \dots\dots\ , Y_j$ is $= \frac{\sum_{i \le j} \lambda_i}{\sum_{i=1}^{n} \lambda_i}$.

6) Find the correlation between X and Y.

Solution 6:

By using R where $\rho_{Y_i, X_k} = \frac{e_{ki}\sqrt{\lambda_i}}{\sqrt{Var(X_i)}}$ for $i, k = 1,2$, we get the following:

```
> corr.y.x <- matrix ( , nrow=nrow(A) , ncol= nrow(A))
> for (i in 1:nrow(A) ){
+  for (j in 1:nrow(A) ){
+ corr.y.x[i,j] <- vectors[j,i]*sqrt(values[i])/sqrt(A[j,j])
+    }
+  }
> corr.y.x
           [,1]        [,2]
[1,]  0.4033802 0.999993188
[2,] -0.9150325 0.003691085
```

ρ_{Y_1,X_1} ρ_{Y_1,X_2} ρ_{Y_2,X_1} ρ_{Y_2,X_2}

7) If the units of X_1 and X_2 are different <u>or</u> the variation between X_1 and X_2 are too high, then

i) find the correlation of X ($\rho_{2\times 2}$), which is better than Σ as

$$\rho = V^{-1}\Sigma V^{-1} \text{ where } V = \begin{bmatrix} \sqrt{Var(X_1)} & 0 \\ 0 & \sqrt{Var(X_2)} \end{bmatrix}$$ **is the matrix standard deviation.**

ii) use ρ to:

a) Find the eigen values and the eigen vectors.

b) Find the Principles Components $Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$.

c) Show that $Trace\ (\rho) = \sum_{i=1}^{2} 1 = \sum_{i=1}^{2} \lambda_i$. (up to the student)

d) Show that $Var(Y_i) = \lambda_i$ for $i = 1, 2$ and $Cov(Y_i, Y_j) = 0, \forall\ i \neq j$ for $i, j = 1, 2$.

e) Find the ratio of the total variance produced by Y_1, Y_2 and for both Y_1 and Y_2.

(up to the student)

Solution7:

i)

```
> sqrt.diag.A <- vector (mode="numeric")
> for (i in 1:nrow(A) ){
+ sqrt.diag.A[i] <- sqrt(A[i,i])
+  }
> sqrt.diag.A
[1]  1 10
> sd.matrix <- diag(sqrt.diag.A, nrow=nrow(A), ncol=nrow(A))
> p <- solve(sd.matrix)%*% A %*% solve(sd.matrix)
> p
     [,1] [,2]
[1,]  1.0  0.4
[2,]  0.4  1.0
```

ii)

a)

```
> e <- eigen(p)
> e
eigen() decomposition
$values
[1] 1.4 0.6

$vectors
          [,1]       [,2]
[1,] 0.7071068 -0.7071068
[2,] 0.7071068  0.7071068
```

b) We know that $Z = V_{2\times2}^{-1}(X_{2\times1} - \mu_{2\times1}) = \begin{bmatrix} \frac{1}{\sqrt{Var(X_1)}} & 0 \\ 0 & \frac{1}{\sqrt{Var(X_2)}} \end{bmatrix}\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} - \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}\right)$, then

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = e'Z = e'\begin{bmatrix} Z_1 = \frac{X_1-\mu_1}{\sqrt{Var(X_1)}} \\ Z_2 = \frac{X_2-\mu_2}{\sqrt{Var(X_2)}} \end{bmatrix} \text{ where } e = \begin{bmatrix} e_1 & e_2 \\ 0.7071068 & -0.7071068 \\ 0.7071068 & 0.7071068 \end{bmatrix}_{2\times2}$$

So,

$$Y_1 = e_1'Z = e_1'\begin{bmatrix} \frac{X_1-\mu_1}{\sqrt{Var(X_1)}} \\ \frac{X_2-\mu_2}{\sqrt{Var(X_2)}} \end{bmatrix} = [0.7071068 \quad 0.7071068]\begin{bmatrix} \frac{X_1-\mu_1}{\sqrt{1}} \\ \frac{X_2-\mu_2}{\sqrt{100}} \end{bmatrix}$$

$$= 0.7071068(X_1-\mu_1) + \frac{0.7071068}{10}(X_2-\mu_2)$$

$$Y_2 = e_2'Z = e_2'\begin{bmatrix} \frac{X_1-\mu_1}{\sqrt{Var(X_1)}} \\ \frac{X_2-\mu_2}{\sqrt{Var(X_2)}} \end{bmatrix} = [-0.7071068 \quad 0.7071068]\begin{bmatrix} \frac{X_1-\mu_1}{\sqrt{1}} \\ \frac{X_2-\mu_2}{\sqrt{100}} \end{bmatrix}$$

$$= -0.7071068(X_1-\mu_1) + \frac{0.7071068}{10}(X_2-\mu_2)$$

d) By using R where $Cov\ (Y) = Cov\ \ (e'Z) = e'Cov\ (Z)e = e'\boldsymbol{\rho}\ e$, we get the results as the following:

```
> values <- e$values
> vectors <- e$vectors
> covY <- t(vectors)%*% p %*% vectors
> covY
     [,1] [,2]
[1,]  1.4  0.0
[2,]  0.0  0.6
```

<u>Ex2:</u>

Let X_1, X_2 and X_3 have the covariance matrix

$$B = \Sigma = Cov(X) = \begin{bmatrix} Var(X_1) & Cov(X_1,X_2) & Cov(X_1,X_3) \\ Cov(X_2,X_1) & Var(X_2) & Cov(X_2,X_3) \\ Cov(X_3,X_1) & Cov(X_3,X_2) & Var(X_3) \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

where $X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$.

1) Find the eigen values and the eigen vectors.

Solution 1:

```
> B <- matrix( c(1,-2,0,-2,5,0,0,0,2) , nr=3 )
> B
     [,1] [,2] [,3]
[1,]    1   -2    0
[2,]   -2    5    0
[3,]    0    0    2
> e <- eigen(B)
> e
eigen() decomposition
$values
[1] 5.8284271 2.0000000 0.1715729

$vectors
           [,1] [,2]      [,3]
[1,] -0.3826834    0 0.9238795
[2,]  0.9238795    0 0.3826834
[3,]  0.0000000    1 0.0000000

> values <- e$values
> vectors <- e$vectors
```

2) Find the Principles Components $Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$. (up to the student)

3) Show that $Trace\ (B) = \sum_{i=1}^{3} Var(X_i) = \sum_{i=1}^{3} \lambda_i$.

Solution 3:

```
> values <- e$values
> vectors <- e$vectors
> library(matlib)
> tr(B)
[1] 8
> sum(values)
[1] 8
```

4) Show that $Var(Y_i) = \lambda_i$ for $i = 1,2,3$ and $Cov(Y_i, Y_j) = 0, \forall\, i \neq j$ for $i, j = 1,2,3$.

Solution 4:

By using R where $Cov(Y) = Cov(e'X) = e'Cov(X)e$, we get the same results as the following:

```
> covY <- t(vectors)%*% B %*% vectors
> covY
                [,1] [,2]          [,3]
[1,]  5.828427e+00    0 -8.881784e-16
[2,]  0.000000e+00    2  0.000000e+00
[3,] -7.979728e-16    0  1.715729e-01
```

5) Find the ratio of the total variance produced by Y_1, Y_2, Y_3, both Y_1 and Y_2 and Y_1, Y_2 and Y_3 together.

Solution 5:

```
> ratio <- vector (mode="numeric")
> for (i in 1:nrow(B) ){
+ ratio[i] <- values[i]/sum(values)
+  }
> ratio
[1] 0.72855339 0.25000000 0.02144661
> cumsum(ratio)
[1] 0.7285534 0.9785534 1.0000000
```

6) Find the correlation between X and Y.

Solution 6:

By using R where $\rho_{Y_i,X_k} = \frac{e_{ki}\sqrt{\lambda_i}}{\sqrt{Var(X_i)}}$ for $i, k = 1,2,3$, we get the following:

```
> corr.y.x <- matrix ( , nrow=nrow(B) , ncol= nrow(B))
> for (i in 1:nrow(B) ){
+  for (j in 1:nrow(B) ){
+ corr.y.x[i,j] <- vectors[j,i]*sqrt(values[i])/sqrt(B[j,j])
+   }
+  }
> corr.y.x
           [,1]       [,2] [,3]
[1,] -0.9238795 0.99748421    0
[2,]  0.0000000 0.00000000    1
[3,]  0.3826834 0.07088902    0
```

7) If the units of X_1, $X_2$and X_3 are different <u>or</u> the variation between X_1, $X_2$and X_3 are too high, then

i) find the correlation of X ($\rho_{3\times3}$).

ii) use ρ to:

a) Find the eigen values and the eigen vectors.

b) Find the Principles Components $Y = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$.

c) Show that $Trace\ (\rho) = \sum_{i=1}^{3} 1 = \sum_{i=1}^{3} \lambda_i$.

d) Show that $Var(Y_i) = \lambda_i$ for $i = 1, 2, 3$ and $Cov(Y_i, Y_j) = 0, \forall\ i \neq j$ for $i, j = 1, 2, 3$.

e) Find the ratio of the total variance produced by all different ways in Principles Components.

Solution7:

(up to the student)