★ Linear Ordinary differential equation:

General form:

$$\frac{d^n}{dt^n} y(t) + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} y(t) + \cdots + a_1 \frac{d}{dt} y(t) + a_0 y(t) = f(t) \quad \cdots\cdots (1)$$

order of this D.E. is $\underline{\underline{n}}$

★ Characteristic polynomial and characteristic equation:

Let $\frac{d}{dt} = D$, $\frac{d^n}{dt^n} = D^n$

then we can write the Characteristic polynomial of (1) as follows

$$D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$$

From the above we can write the characteristic equation as follows

$$D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0 = 0$$

Example: write the C.E. of this O.D.E.

$$\frac{d^2}{dt^2} y(t) - 4 \frac{d}{dt} y(t) + 5 y(t) = 0$$

ans.

$$\Rightarrow \boxed{D^2 - 4D + 5 = 0}$$

★ Linear ordinary Differential equation solution:

Free response: input is zero $\Rightarrow f(t) = 0$

Forced response: initial conditions are zero $\Rightarrow f(t) \neq 0$

total response is the sum of free & Frorced response

$\Rightarrow$ Free response : $y_1 = A_1 e^{D_1 t} + A_2 e^{D_2 t} + \cdots$

$\Rightarrow$ Forced response depends on $f(t)$

Example: Solve the following D.E. assuming zero initial conditions

① $\frac{dy}{dt} + 7y = 5\cos(2t)$ ---- ①

First: we start by finding the forced response

$f(t) = 5\cos(2t) \Rightarrow y_2 = A_1\cos(2t) + A_2\sin(2t)$

substituting in ①

$\frac{d}{dt}y = -2A_1\sin(2t) + 2A_2\cos(2t)$

$\Rightarrow -2A_1\sin(2t) + 2A_2\cos(2t) + 7(A_1\cos(2t) + A_2\sin(2t)) = 5\cos(2t)$

$\Rightarrow 2A_2\cos(2t) + 7A_1\cos(2t) + 7A_2\sin(2t) - 2A_1\sin(2t) = 5\cos(2t)$

$\rightarrow (7A_1 + 2A_2)\cos(2t) + (-2A_1 + 7A_2)\sin(2t) = 5\cos(2t)$

Since the left side is equal to the right side, we can find the values of A_1, A_2.

$\Rightarrow$ $7A_1 + 2A_2 = 5$

$-2A_1 + 7A_2 = 0$

$\Rightarrow$ Solving for A_1, A_2

$\Rightarrow A_1 = \frac{35}{53}$, $A_2 = 10/53$

Second: we find the free response using the C.E.

$\frac{d}{dt}y + 7y = 5\cos(2t)$

$\Rightarrow$ C.E. $D + 7 = 0 \Rightarrow D = -7$

$y_1 = A_3 e^{-7t}$; $y(t) = y_1(t) + y_2(t)$

$\Rightarrow y(t) = A_3 e^{-7t} + \frac{35}{53}\cos(2t) + \frac{10}{53}\sin(2t)$

to find A_3, we use the initial conditions we have, $y(0) = 0$

$y(0) = A_3 e^{-7(0)} + \frac{35}{53}\cos(2(0)) + \frac{10}{53}\underbrace{\sin(2(0))}_{\to 0} = 0$

$\Rightarrow A_3 + \frac{35}{53} = 0$ $A_3 = -35/53$

$y(t) = -\frac{35}{53}e^{-7t} + \frac{35}{53}\cos(2t) + \frac{10}{53}\sin(2t)$

Example 2: Solve the following differential equations:

$$\frac{d^2}{dt^2}y + 2\frac{d}{dt}y + y = 5e^{-2t} + t \quad ; \quad y(0) = 2 \quad , \quad \frac{dy}{dt}(0) = 1$$

First: Forced response

$$f(t) = 5e^{-2t} + t \quad \Rightarrow \quad y_2 = A_1 e^{-2t} + A_2 t + A_3$$

$$\frac{d}{dt}y_2 = -2A_1 e^{-2t} + A_2$$

$$\frac{d^2}{dt^2}y_2 = 4A_1 e^{-2t}$$

Substituting in origiqral D.E.

$$\Rightarrow 4A_1 e^{-2t} - 4A_1 e^{-2t} + 2A_2 + A_1 e^{-2t} + A_2 t + A_3 = 5e^{-2t} + t$$

$$\Rightarrow 2A_2 + A_1 e^{-2t} + A_2 t + A_3 = 5e^{-2t} + t$$

rearranging the equation

$$A_1 e^{-2t} + A_2 t + (2A_2 + A_3) = 5e^{-2t} + t$$

comparing both sides, we find

$$\boxed{A_1 = 5} \quad , \quad \boxed{A_2 = 1} \quad , \quad 2A_2 + A_3 = 0 \quad \Rightarrow \quad \boxed{A_3 = -2A_2 = -2}$$

$$\Rightarrow y_2 = 5e^{-2t} + t - 2$$

Second: Free response, using original D.E. we can find the C.E.

$$D^2 + 2D + 1 = 0 \quad \Rightarrow \quad D_1 = -1, \quad D_2 = -1$$

Since $D_1 = D_2 \Rightarrow y_1 = A_4 e^{-t} + A_5 t e^{-t}$

the total response is

$$y(t) = y_1 + y_2 = A_4 e^{-t} + A_5 t e^{-t} + 5e^{-2t} + t - 2$$

Using the initial conditions

- $y(0) = 2 \Rightarrow A_4 e^{-(0)} + A_5 (0) e^{-(0)} + 5e^{-2(0)} + (0) - 2 = A_4 + 5 - 2 = 2 \Rightarrow \boxed{A_4 = -1}$
- $\frac{dy}{dt}(0) = 1 \; ; \; \frac{d}{dt}y = -(-1)e^{-t} + A_5 e^{-t} - A_5 t e^{-t} - 10e^{-2t} + 1$

$$\Rightarrow \frac{dy}{dt}(0) = +1e^{-(0)} + A_5 e^{-(0)} - A_5 (0) e^{-(0)} - 10e^{-2(0)} + 1 = 1$$

$$= +1 + A_5 - 10 + 1 = 1 \quad \Rightarrow \quad \boxed{A_5 = 9}$$

$$\Rightarrow y(t) = -e^{-t} + 9te^{-t} + 5e^{-2t} + t - 2$$

Laplace Transform

Inverse Laplace transform

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\, e^{st}\, ds$$

However, this is very ~~difficult~~ and time-consuming, so we usually use the tables to ~~find~~ the inverse Laplace transform.

Example: Find the inverse ~~Laplace~~ transform ~~of~~ this transfer function

$$H(s) = \frac{1}{(s+1)(s+2)}$$

Using Partial fraction, we can express $H(s)$ as

$$H(s) = \frac{A}{s+1} + \frac{B}{s+2}$$

$$A = \left[(s+1)\, H(s)\right]\Big|_{s=-1} = 1 \quad ; \quad B = \left[(s+2)\, H(s)\right]\Big|_{s=-2} = -1$$

$$H(s) = \frac{1}{s+1} - \frac{1}{s+2} \Rightarrow h(t) = \mathcal{L}^{-1}\left[H(s)\right] = \mathcal{L}^{-1}\left[\frac{1}{s+1} - \frac{1}{s+2}\right]$$

From the table, we can find $\mathcal{L}^{-1}[H(s)]$: $e^{-at} \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}$

$$\Rightarrow h(t) = e^{-t} + e^{-2t} \quad \text{for } t \geq 0$$

Example: Find $h(t)$ of this ~~transform~~ function

$$H(s) = \frac{s+2}{s^3 + 4s^2 + 3s} \quad ; \quad \text{expressing } H(s) \text{ using Partial fraction}$$

$$H(s) = \frac{2/3}{s} + \frac{-1/2}{s+1} + \frac{-1/6}{s+3} = \frac{2}{3}\frac{1}{s} - \frac{1}{2}\frac{1}{s+1} + \frac{1}{6}\frac{1}{s+3}$$

$$h(t) = \frac{2}{3} - \frac{1}{2} e^{-t} - \frac{1}{6} e^{-3t} \quad \text{for } t \geq 0$$

$$1 \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}$$

$$e^{-at} \longleftrightarrow \frac{1}{s+a}$$

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initial value theorem:

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s) \quad \text{if the limit exists}$$

final value theorem:

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) \quad \text{if } F(s) \text{ is stable only}$$

example: $F(s) = \frac{2}{s^2(s+2)(s+3)} \Rightarrow$ Unstable, $F(s) = \frac{s+2}{s^3 + 4s^2 + 3s} \Rightarrow$ I.V. = 0, F.V. = $2/3$