

## Chapter 2: Introduction to Mathematical Tools

Consider an operator  $H$  that produces an output image,  $g(x, y)$ , for a given input image,  $f(x, y)$

$$H[f(x, y)] = g(x, y)$$

$H$  is said to be a linear operator if

$$\begin{aligned} H[a_i f_i(x, y) + a_j f_j(x, y)] &= a_i H[f_i(x, y)] + a_j H[f_j(x, y)] \\ &= a_i g_i(x, y) + a_j g_j(x, y) \end{aligned}$$

$a_i, a_j, f_i(x, y)$  and  $f_j(x, y)$  - arbitrary constants and images of the same size

# MAX Operation: Linear or Non-linear?

consider the max operation,  $f_i = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$ ,  $f_j = \begin{bmatrix} 6 & 5 \\ 4 & 7 \end{bmatrix}$ ,  $a_i = 1$  and  $a_j = -1$

Testing for linearity with the left side of the following equation

$$\begin{aligned} H[a_i f_i(x, y) + a_j f_j(x, y)] &= a_i H[f_i(x, y)] + a_j H[f_j(x, y)] \\ &= a_i g_i(x, y) + a_j g_j(x, y) \end{aligned}$$

$$\begin{aligned} \max \left\{ (1) \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix} + (-1) \begin{bmatrix} 6 & 5 \\ 4 & 7 \end{bmatrix} \right\} &= \max \left\{ \begin{bmatrix} -6 & -3 \\ -2 & -4 \end{bmatrix} \right\} \\ &= -2 \end{aligned}$$

and for the right side we obtain

$$\begin{aligned} (1) \max \left\{ \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix} \right\} + (-1) \max \left\{ \begin{bmatrix} 6 & 5 \\ 4 & 7 \end{bmatrix} \right\} &= 3 - 7 \\ &= -4 \end{aligned}$$

# Arithmetic Operation: Example

$$g(x, y) = f(x, y) + \eta(x, y)$$

where  $g(x, y)$  is a corrupted image, formed by addition of noise  $\eta(x, y)$  into  $f(x, y)$

$\bar{g}(x, y)$  can be formed by averaging  $K$  different noise images

$$\bar{g}(x, y) = \frac{1}{K} \sum_{i=1}^K g_i(x, y)$$

and we can say that

$$E\{\bar{g}(x, y)\} = f(x, y)$$

# Arithmetic Operation: Example

and

$$\sigma_{\bar{g}(x,y)}^2 = \frac{1}{K} \sigma_{\eta(x,y)}^2$$

where  $E\{\bar{g}(x,y)\}$  - expected value of  $\bar{g}$ .  $\sigma_{\bar{g}(x,y)}^2$  and  $\sigma_{\eta(x,y)}^2$  are variances of  $\bar{g}$  and  $\eta$ .

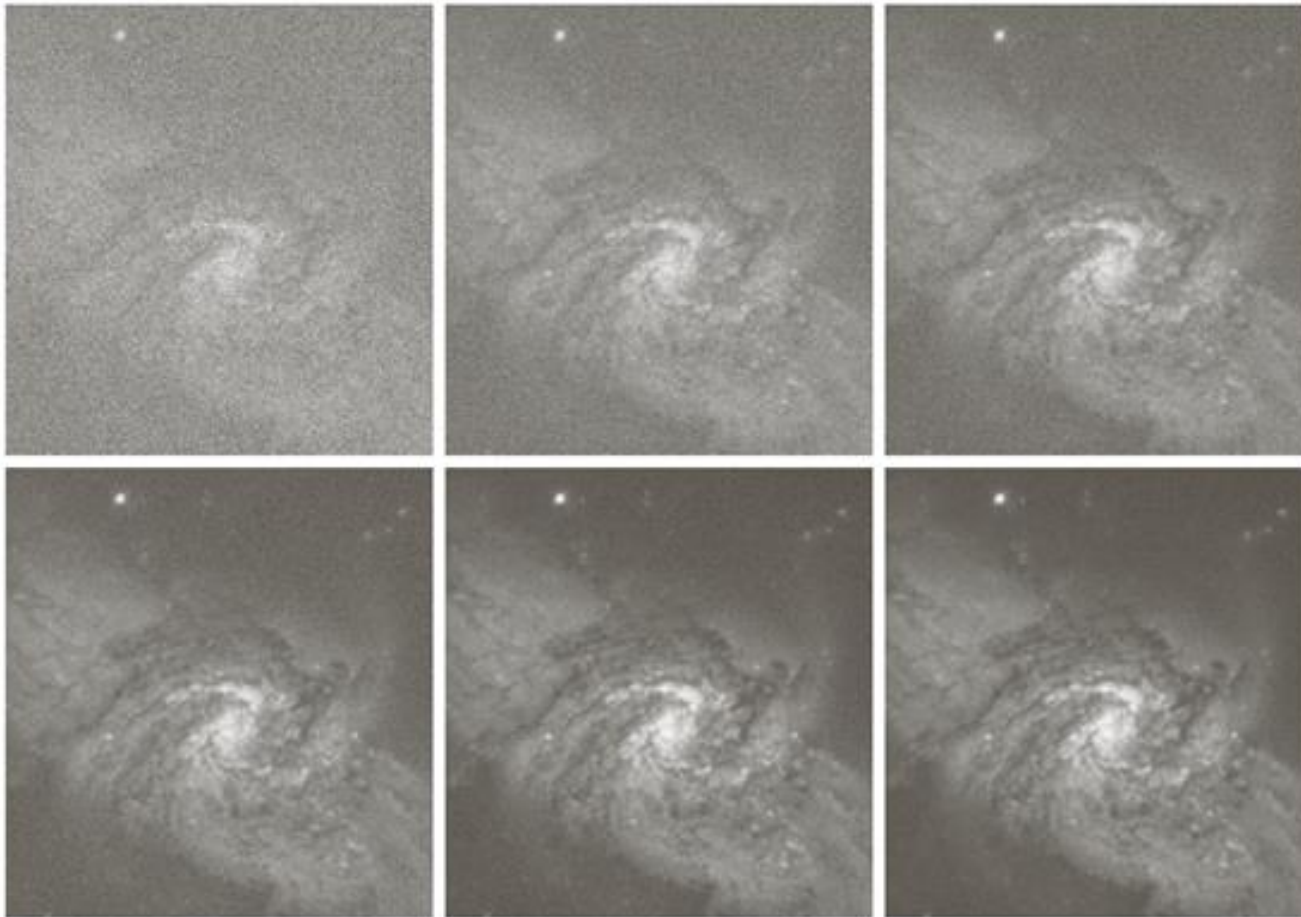
The standard deviation at any point in the average image is

$$\sigma_{\bar{g}(x,y)} = \frac{1}{\sqrt{K}} \sigma_{\eta(x,y)}$$

As  $K$  increases the variability of the pixel values at each location  $(x,y)$  decreases.

$\bar{g}(x,y)$  approaches  $f(x,y)$  as the number of noisy images used in the averaging process increases.

# Averaging Noisy Images

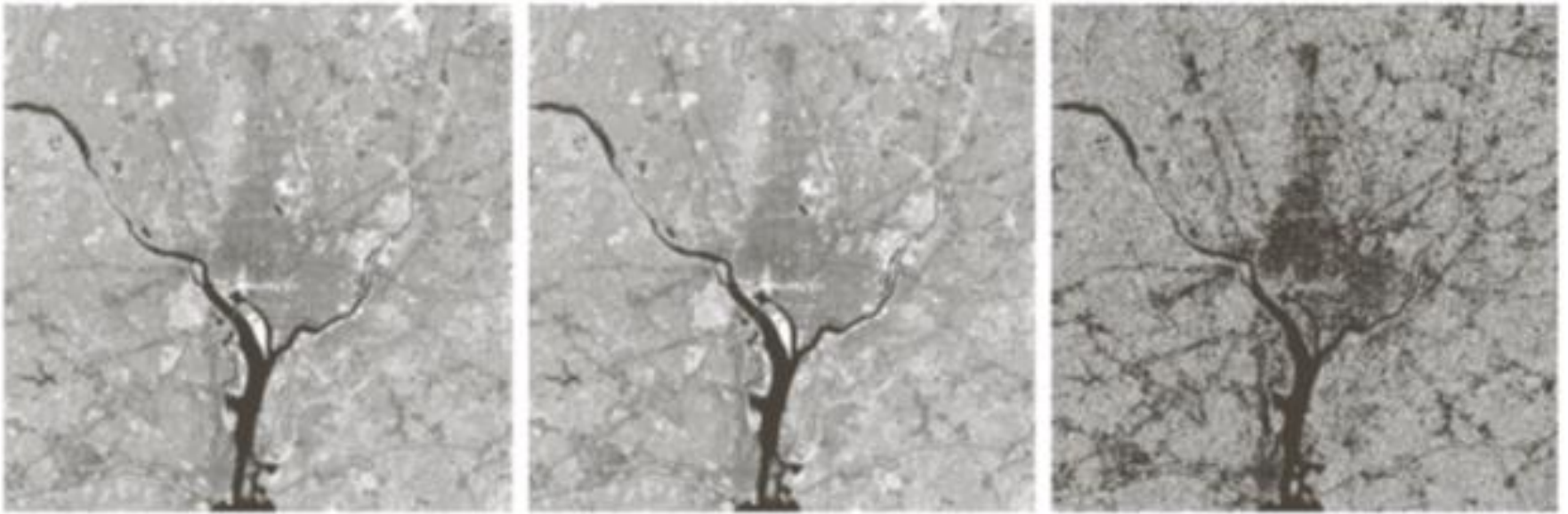


a b c  
d e f

**FIGURE 2.26** (a) Image of Galaxy Pair NGC 3314 corrupted by additive Gaussian noise. (b)–(f) Results of averaging 5, 10, 20, 50, and 100 noisy images, respectively. (Original image courtesy of NASA.)

# Image Subtraction

$$g(x, y) = f(x, y) - h(x, y)$$



a b c

**FIGURE 2.27** (a) Infrared image of the Washington, D.C. area. (b) Image obtained by setting to zero the least significant bit of every pixel in (a). (c) Difference of the two images, scaled to the range [0, 255] for clarity.

# Image Multiplication

$$g(x, y) = f(x, y)h(x, y)$$

$h(x, y)$  is the shading function, find  $f(x, y)$



a b c

**FIGURE 2.29** Shading correction. (a) Shaded SEM image of a tungsten filament and support, magnified approximately 130 times. (b) The shading pattern. (c) Product of (a) by the reciprocal of (b). (Original image courtesy of Mr. Michael Shaffer, Department of Geological Sciences, University of Oregon, Eugene.)

# Image Multiplication

ROI: Region Of Interest

Multiply the image by a mask containing 1s in the specified location.



a b c

**FIGURE 2.30** (a) Digital dental X-ray image. (b) ROI mask for isolating teeth with fillings (white corresponds to 1 and black corresponds to 0). (c) Product of (a) and (b).



# Image Scaling

$$255 + 255 = 510$$

$$0 - 255 = -255$$

$$f_m = f - \min(f)$$

$$f_s = K[f_m / \max(f_m)]$$

$K = 255$  for 8 – *bit* images. What is the solution to division by 0?

# Logical Operations: Binary Images



**FIGURE 2.33**  
Illustration of logical operations involving foreground (white) pixels. Black represents binary 0s and white binary 1s. The dashed lines are shown for reference only. They are not part of the result.

# Geometric Transformations

- a spatial transformation of coordinates
- intensity interpolation

$$(x, y) = T\{(v, w)\}$$

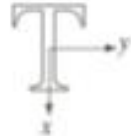





Affine transform can scale, rotate, translate or sheer a set of coordinate points depending on the value for the elements of  $T$ .

$$\begin{bmatrix} x & y & 1 \end{bmatrix} = \begin{bmatrix} v & w & 1 \end{bmatrix} T = \begin{bmatrix} v & w & 1 \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} & 0 \\ t_{21} & t_{22} & 0 \\ t_{31} & t_{32} & 1 \end{bmatrix}$$

# Affine Transformations

**TABLE 2.2**

Affine transformations based on Eq. (2.6-23).

Transformation Name	Affine Matrix, T	Coordinate Equations	Example
Identity	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x = v$ $y = w$	
Scaling	$\begin{bmatrix} c_x & 0 & 0 \\ 0 & c_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x = c_x v$ $y = c_y w$	
Rotation	$\begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x = v \cos \theta - w \sin \theta$ $y = v \sin \theta + w \cos \theta$	
Translation	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix}$	$x = v + t_x$ $y = w + t_y$	
Shear (vertical)	$\begin{bmatrix} 1 & 0 & 0 \\ s_x & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x = v + s_x w$ $y = w$	
Shear (horizontal)	$\begin{bmatrix} 1 & s_h & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$x = v$ $y = s_h v + w$	

# Geometric Transformations and Interpolation



a b c d

**FIGURE 2.36** (a) A 300 dpi image of the letter T. (b) Image rotated  $21^\circ$  clockwise using nearest neighbor interpolation to assign intensity values to the spatially transformed pixels. (c) Image rotated  $21^\circ$  using bilinear interpolation. (d) Image rotated  $21^\circ$  using bicubic interpolation. The enlarged sections show edge detail for the three interpolation approaches.