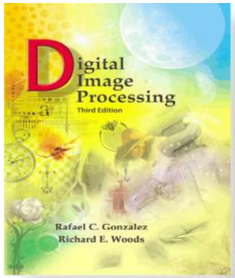


Filtering in Frequency Domain

Any function can be represented as a weighted sum of sines and cosines.

Component sinusoidal

Original Function

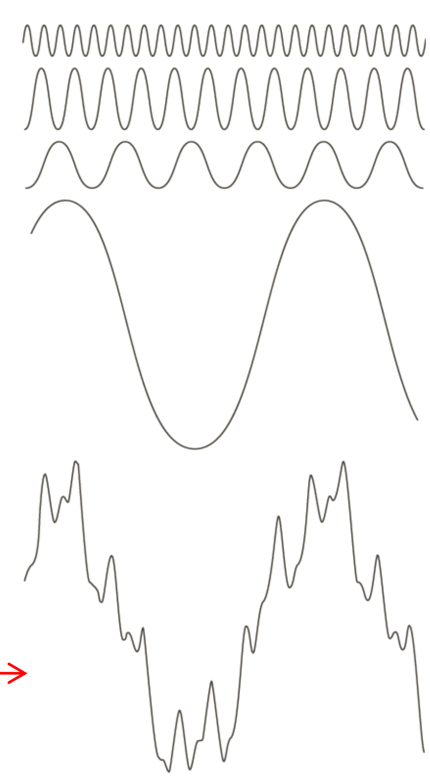
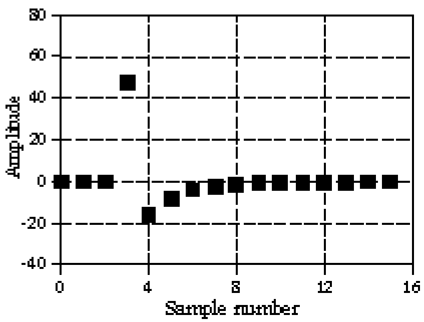
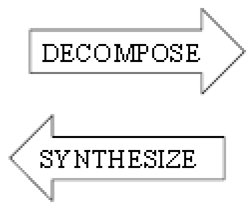


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

Discrete Fourier Transform

Any continuous periodic signal can be represented as the sum of properly chosen sinusoidal waves.





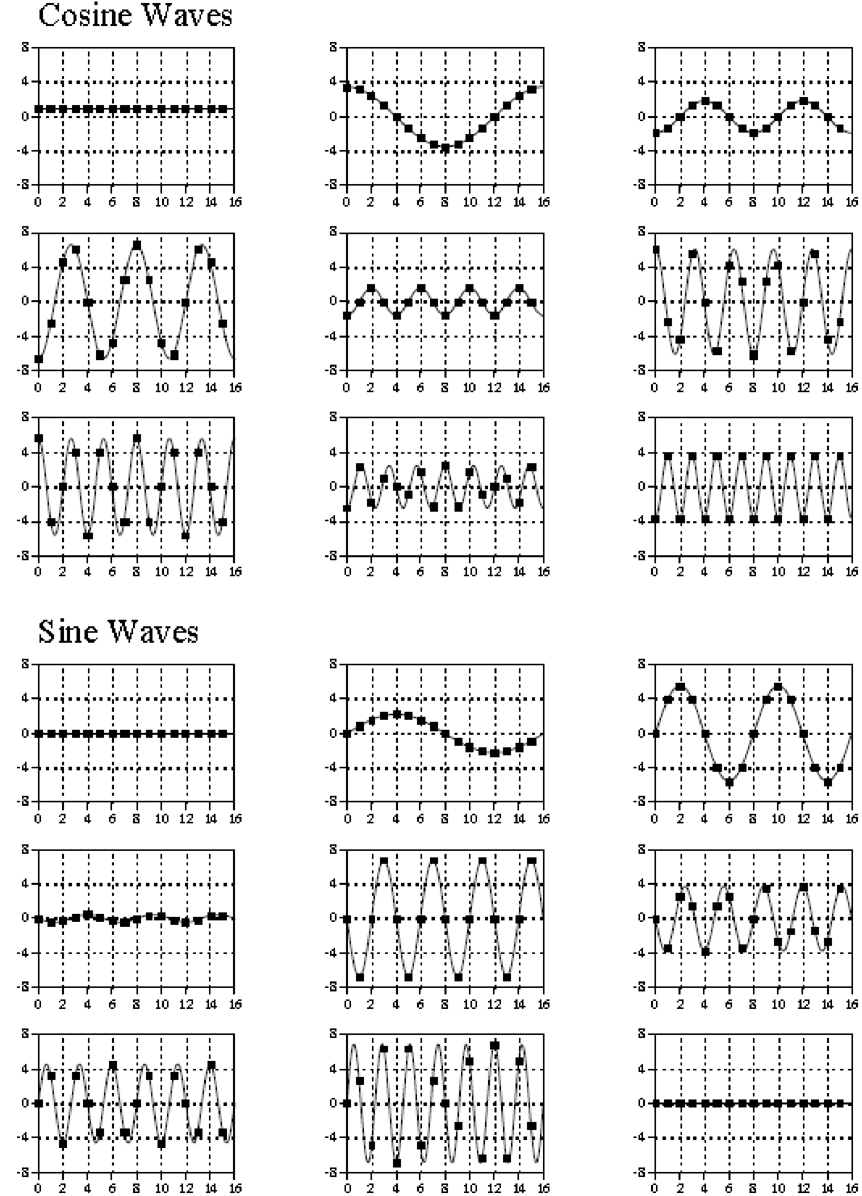


FIGURE 8-1b
Example of Fourier decomposition. A 16 point signal (opposite page) is decomposed into 9 cosine waves and 9 sine waves. The frequency of each sinusoid is fixed; only the amplitude is changed depending on the shape of the waveform being decomposed.

Fourier Transfer of a Simple Function

$$F(\mu)=\int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}dt=\int_{-W/2}^{W/2} Ae^{-j2\pi\mu t}dt$$

$$=\frac{-A}{j2\pi\mu}[e^{-j2\pi\mu t}]_{-W/2}^{W/2}=\frac{-A}{j2\pi\mu}[e^{-j\pi\mu W}-e^{j\pi\mu W}]$$

$$=\frac{A}{j2\pi\mu}[e^{j\pi\mu W}-e^{-j\pi\mu W}]=AW\frac{\sin(\pi\mu W)}{(\pi\mu W)}$$

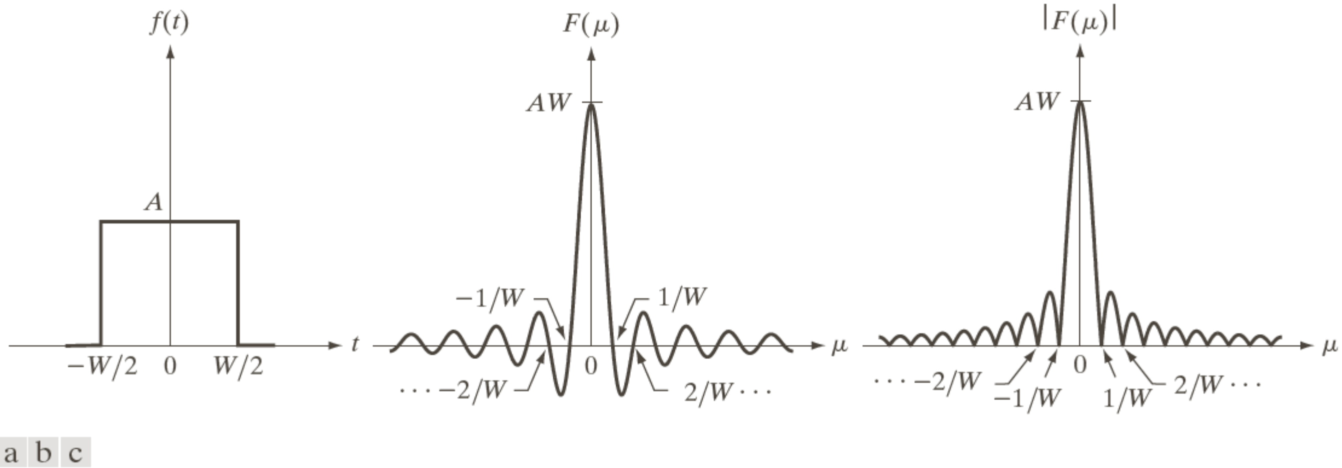
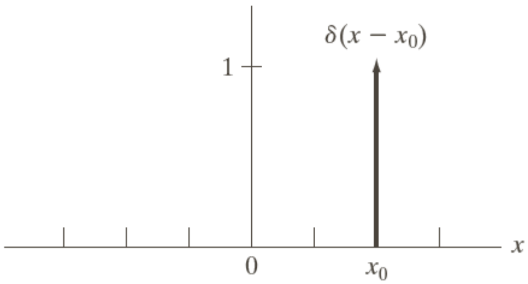


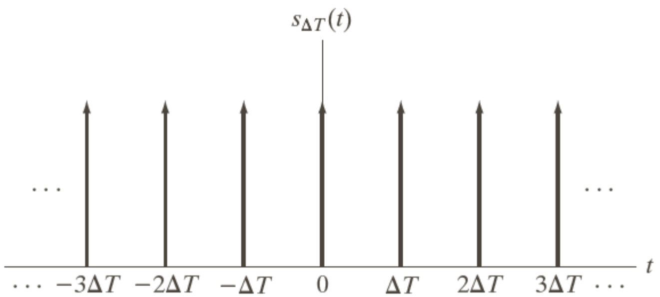
FIGURE 4.4 (a) A simple function; (b) its Fourier transform; and (c) the spectrum. All functions extend to infinity in both directions.

Discrete Impulse



Unit discrete impulse at x_0.

Fourier transform (FT) of an impulse is constant.



Impulse train

$$s_{\Delta T}(t) = \sum_{-\infty}^{\infty} \delta(t - n\Delta T)$$

Fourier transform (FT) of an impulse train is also an impulse train.

Delta Function Pairs in Polar Form

Delta Function

Shifted Delta Function

Same Magnitude, Different Phase

Shifted Delta Function

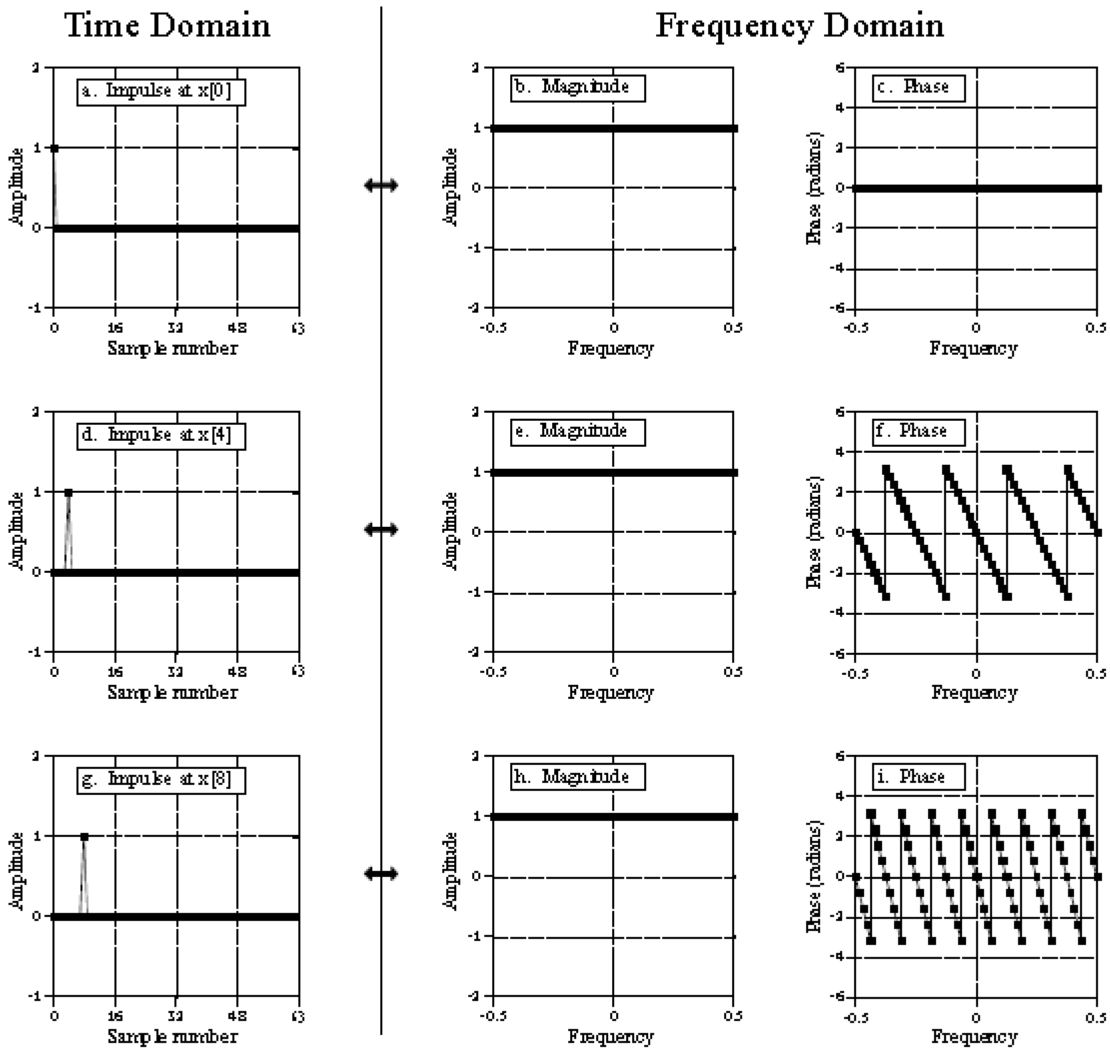


FIGURE 11-1
Delta function pairs in *polar form*. An impulse in the time domain corresponds to a constant magnitude and a linear phase in the frequency domain.

Delta Function Pairs in Rectangular Form

Delta Function

Shifted Delta Function

Shifted Delta Function

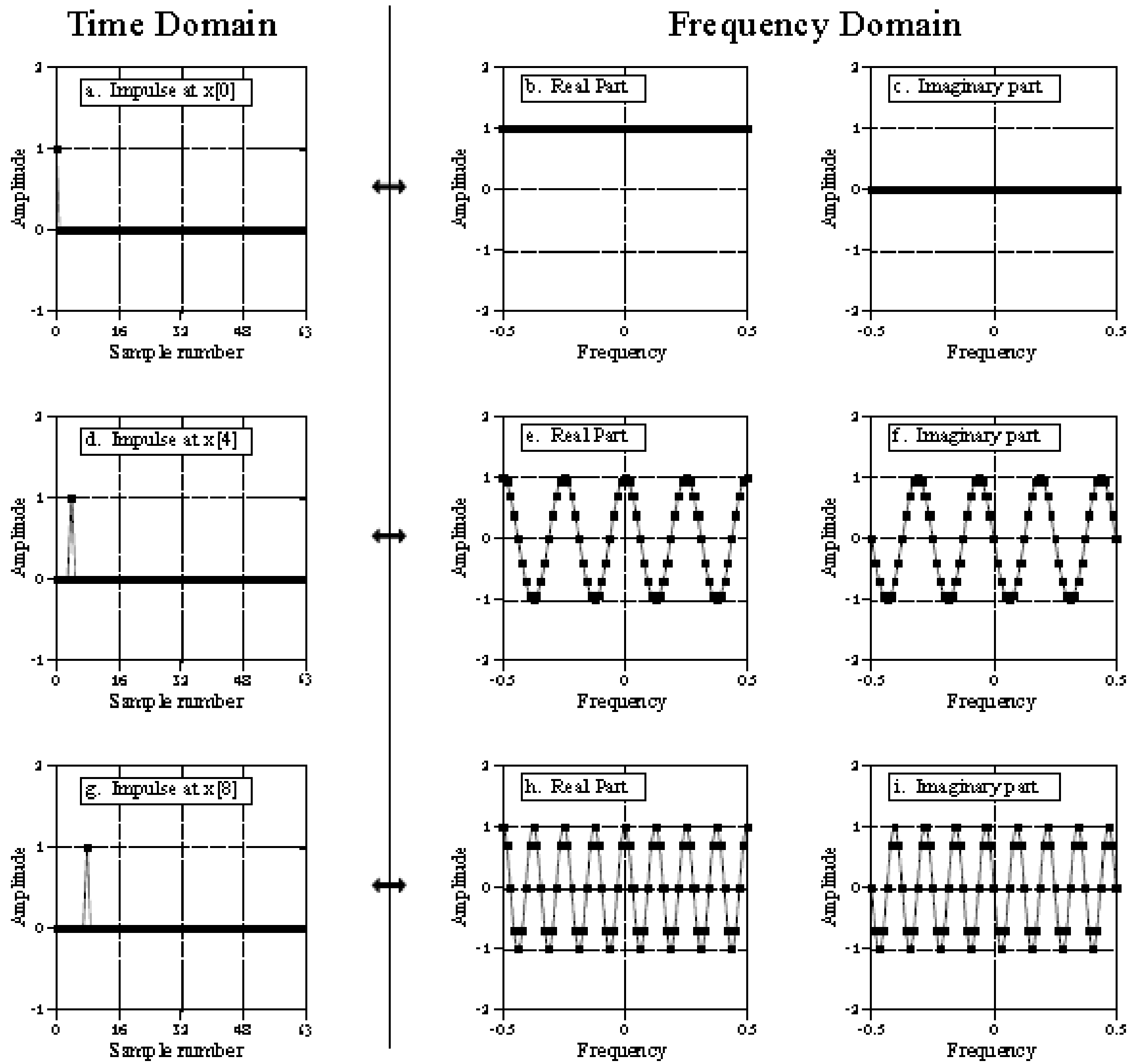


FIGURE 11-2
Delta function pairs in *rectangular form*. Each sample in the time domain results in a cosine wave in the real part, and a negative sine wave in the imaginary part of the frequency domain.

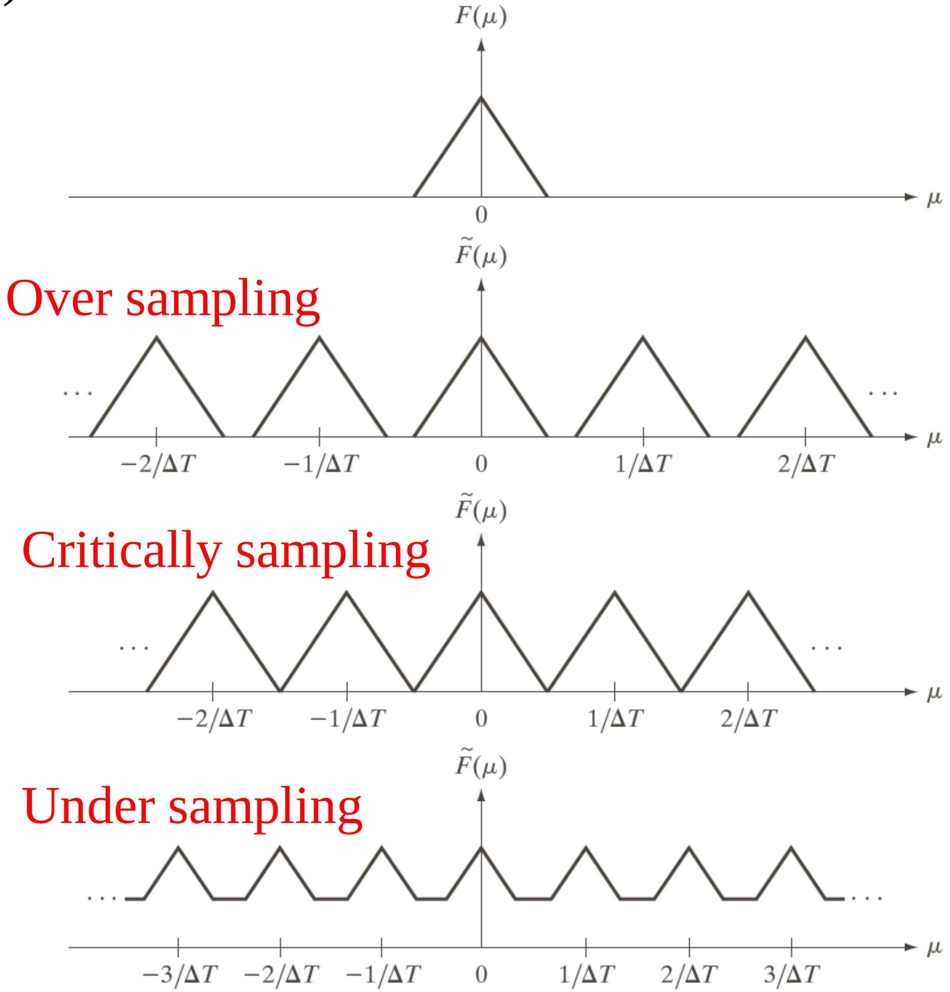
FT of Sampled Functions

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

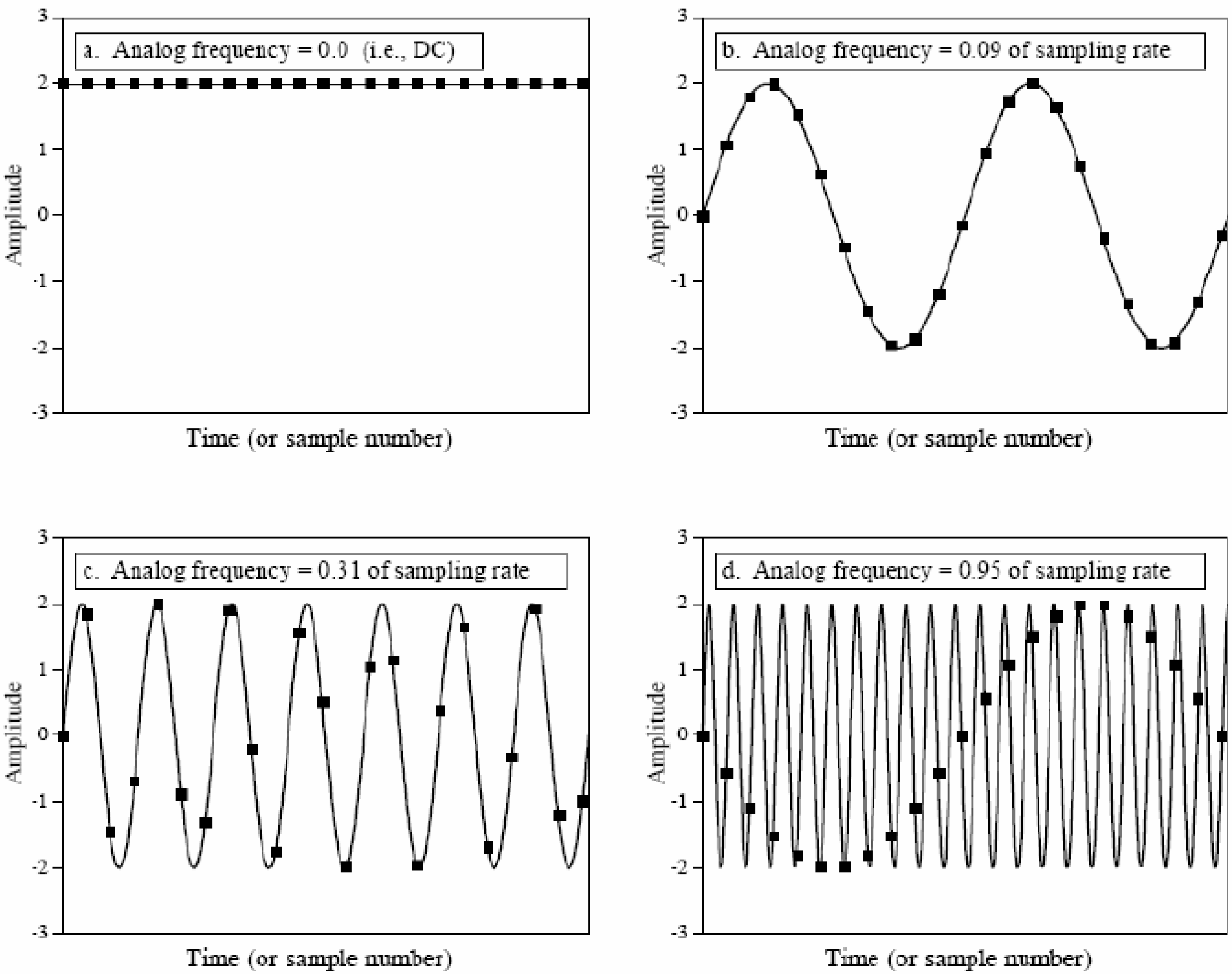
$$\tilde{F}(\mu) = F(\mu) \otimes S(\mu)$$

$$= \frac{1}{\Delta T}\sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$

$\tilde{F}(\mu)$ is infinite, periodic sequence of copies of $F(\mu)$



Sampling Theory

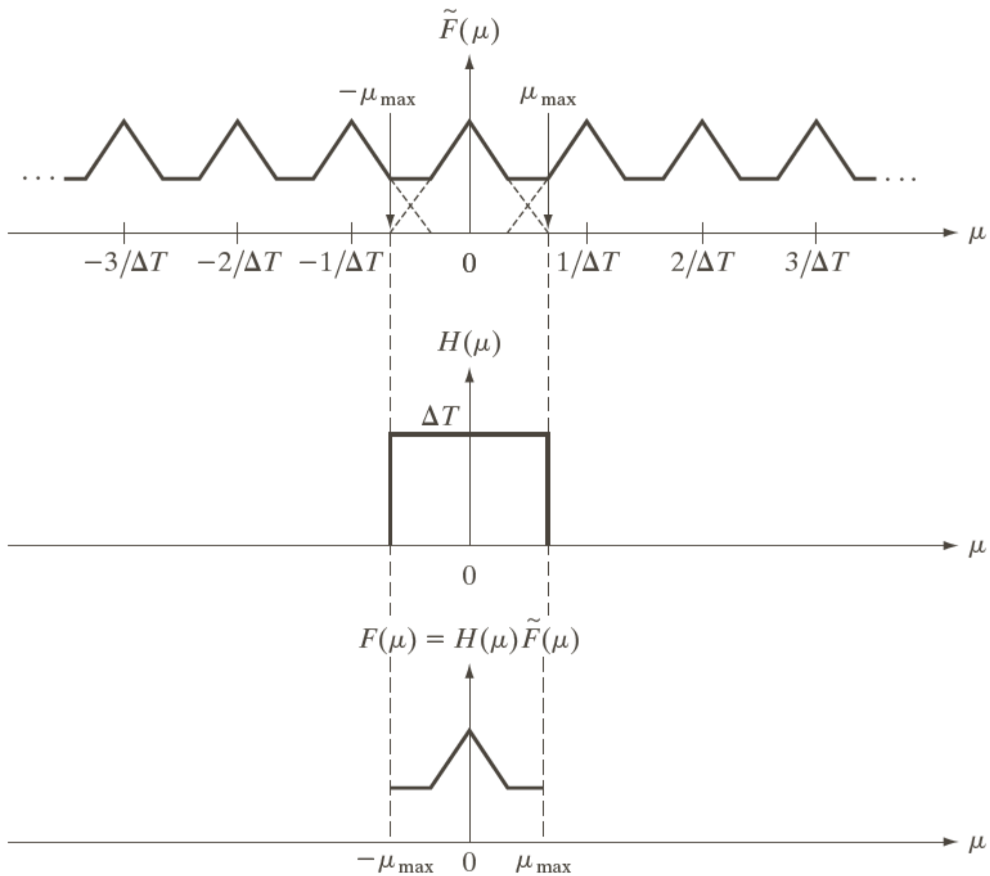


Proper sampling:

Frequency components are not above half the sampling rate.

Nyquist Theorem

Aliasing



a
b
c

FIGURE 4.9 (a) Fourier transform of an under-sampled, band-limited function. (Interference from adjacent periods is shown dashed in this figure). (b) The same ideal lowpass filter used in Fig. 4.8(b). (c) The product of (a) and (b). The interference from adjacent periods results in aliasing that prevents perfect recovery of $F(\mu)$ and, therefore, of the original, band-limited continuous function. Compare with Fig. 4.8.

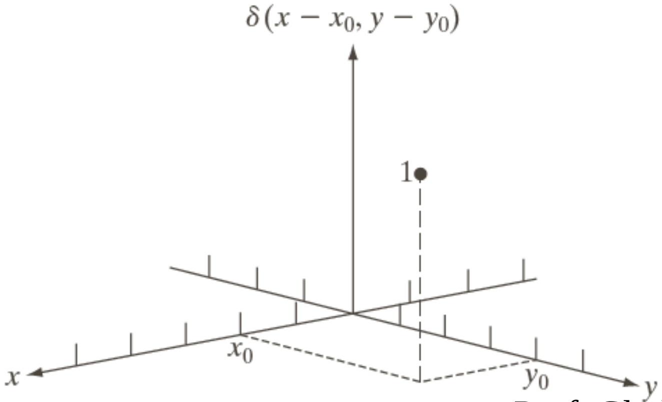
FT in One Dimension

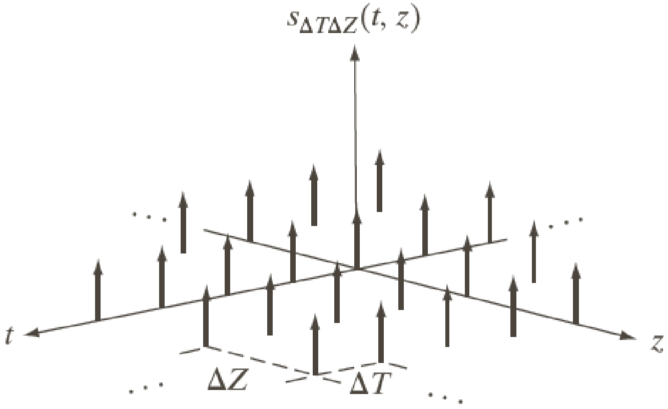
Fourier Transform, $F(u) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M}$ $\quad u = 0, 1, 2, \; M\text{-}1$

Inverse Fourier Transform, $f(x) = \frac{1}{M} \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M}$

Drill Example: 4.4

2-D Impulse





Impulse train

Aliasing in Image - I

Aliasing in resampled (resized) image.

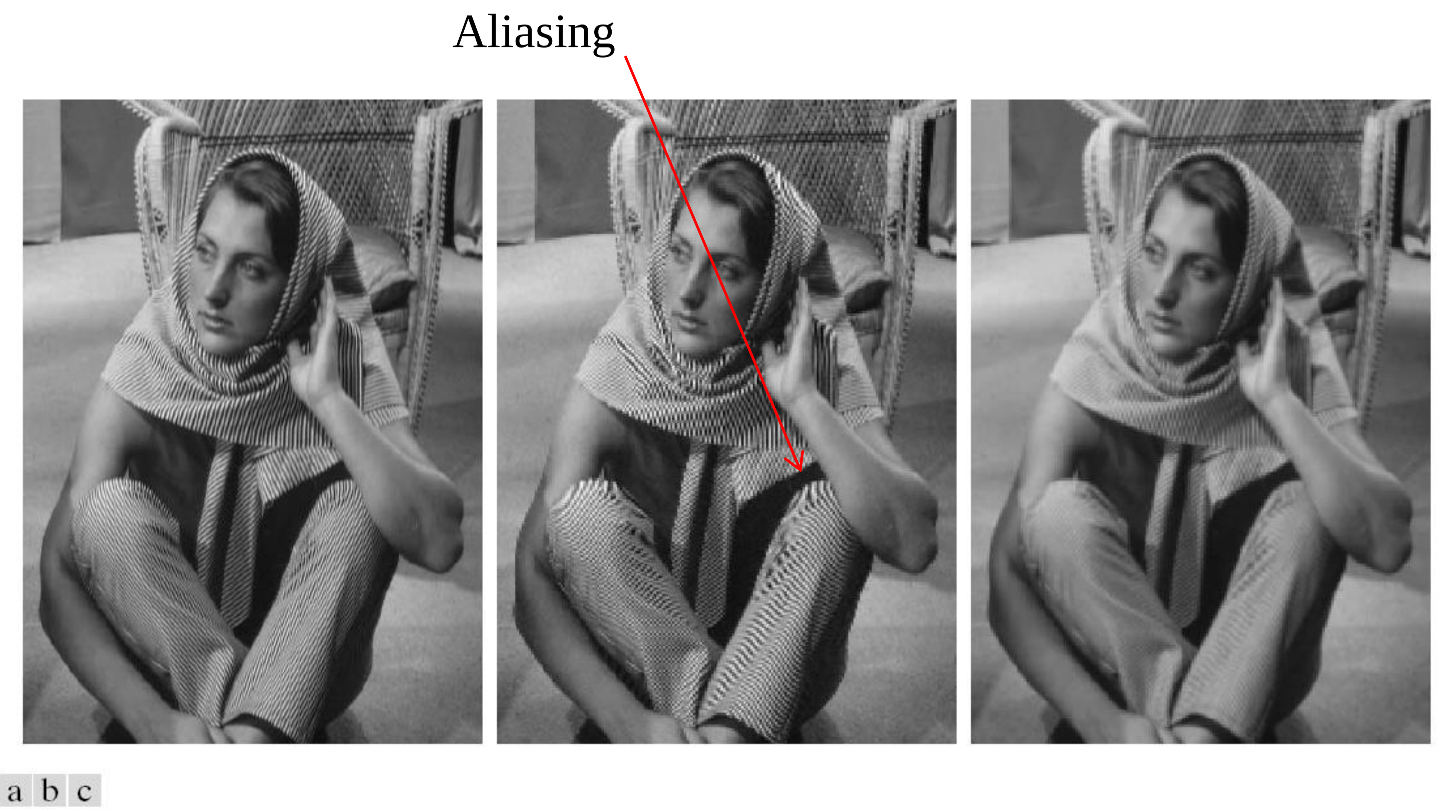
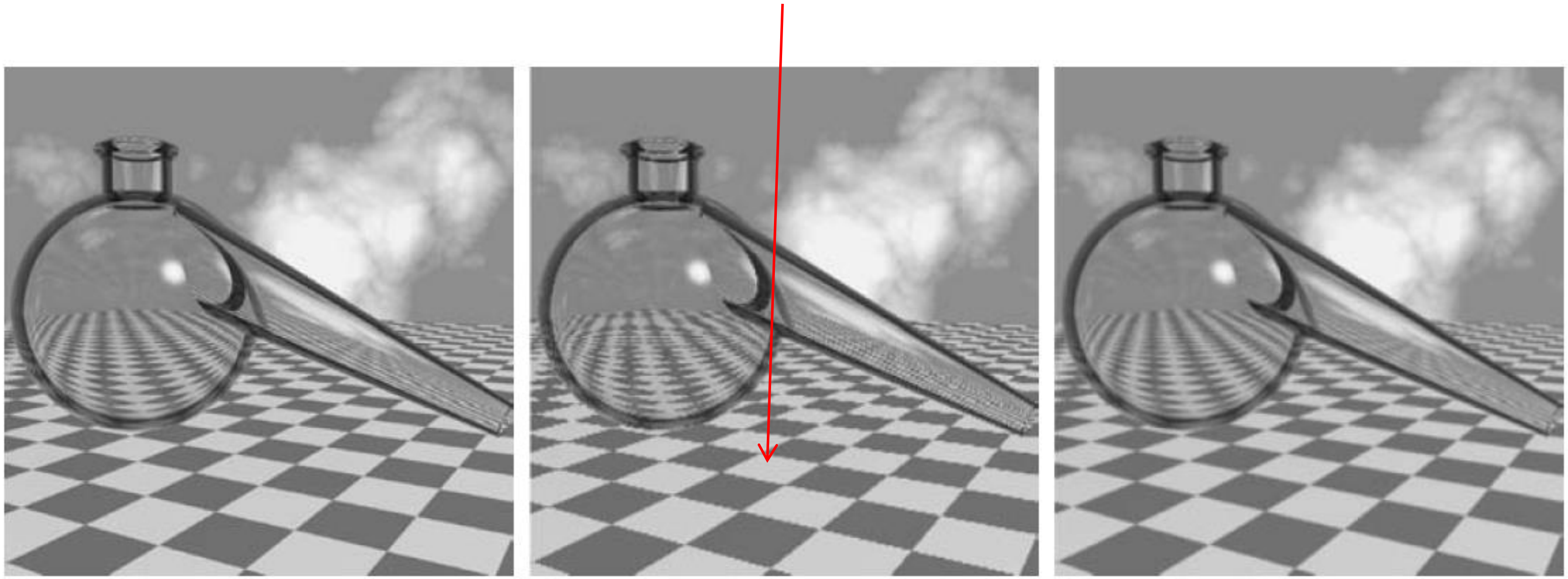


FIGURE 4.17 Illustration of aliasing on resampled images. (a) A digital image with negligible visual aliasing. (b) Result of resizing the image to 50% of its original size by pixel deletion. Aliasing is clearly visible. (c) Result of blurring the image in (a) with a 3×3 averaging filter prior to resizing. The image is slightly more blurred than (b), but aliasing is not longer objectionable. (Original image courtesy of the Signal Compression Laboratory, University of California, Santa Barbara.)

Aliasing in Image - II

Jaggies: The effect of aliasing in edges.



a b c

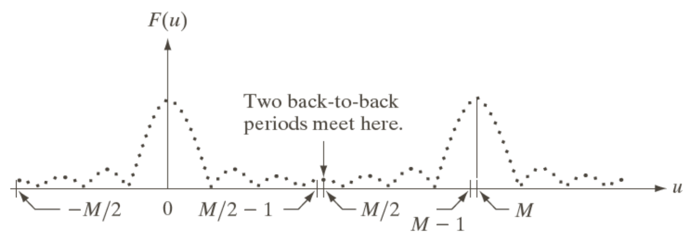
FIGURE 4.18 Illustration of jaggies. (a) A 1024 × 1024 digital image of a computer-generated scene with negligible visible aliasing. (b) Result of reducing (a) to 25% of its original size using bilinear interpolation. (c) Result of blurring the image in (a) with a 5 × 5 averaging filter prior to resizing it to 25% using bilinear interpolation. (Original image courtesy of D. P. Mitchell, Mental Landscape, LLC.)

FT in 2-D

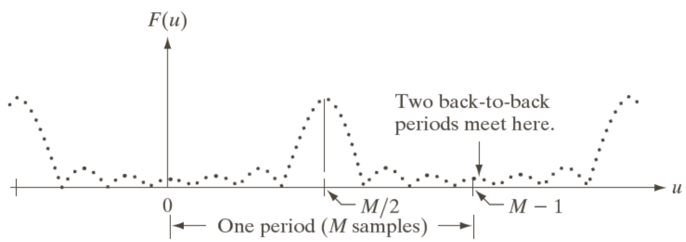
Fourier Transform, $$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

Inverse Fourier Transform, $$f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$$

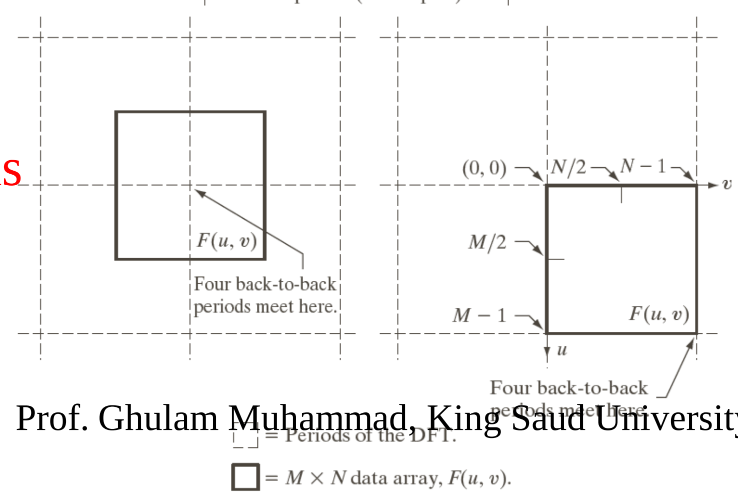
DFT: infinite # periods



Shifted DFT:
(-1)^{x}*f(x)



2-D DFT: infinite # periods



Shifted 2-D DFT:
(-1)$^{x+y}$*f(x,y)

Translation and rotation of the 2-D discrete Fourier transform and its inverse

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

$$f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$$

The 2-D discrete Fourier transform pair satisfies the following translation propertie

$$f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$$

$$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M+vy_0/N)}$$

Multiplying $f(x,y)$ by the exponential $e^{j2\pi(u_0x/M+v_0y/N)}$, shifts the origin of the 2D-DFT to (u_0, v_0)

Multiplying $F(u,v)$ by the exponential $e^{-j2\pi(ux_0/M+vy_0/N)}$, shifts the origin of $f(x,y)$ to (x_0, y_0)

This means that translation has no effect on the magnitude (spectrum) of $F(u,v)$

Translation and rotation of the 2-D discrete Fourier transform and its inverse

Using polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, $u = w\cos\phi$, $v = w\sin\phi$

$$f(r, \theta + \theta_0) \Leftrightarrow F(w, \phi + \theta_0)$$

Rotating $f(x, y)$ by an angle θ_0 rotates $F(u, v)$ by the same angle.

And rotating $F(u, v)$ by an angle θ_0 rotates $f(x, y)$ by the same angle.

Even and Odd Functions

Consider the example:

$$f = \{f(0), f(1), f(2), f(3)\} = \{2,1,1,1\}$$ M = 4 (even)

For evenness: $f(x) = f(4-x)$ The form is: $\{a \ \ b \ \ c \ \ b\}$

$$f(0) = 2+1+1+1 = 5, \quad f(4) = \text{outside the range}$$

Consider another example: $g = \{g(0), g(1), g(2), g(3)\} = \{0,-1,0,1\}$

For oddness: $g(x) = -g(4-x)$ The form is: $\{0 \ \ -b \ \ 0 \ \ b\}$

For M = even, $g(0)$ and $g(M/2)$ are zero.

For M = odd, only $g(0)$ is zero.

{0 -1 0 1} is odd, but {0 -1 0 1 0} is neither odd nor even.

Fourier Spectrum & Phase Angle

2-D complex DFT: $F(u,v) = |F(u,v)| e^{j\phi(u,v)}$

(1) Frequency spectrum: $|F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$

(2) Phase angle: $\phi(u,v) = \arctan\left[\frac{I(u,v)}{R(u,v)}\right]$

(3) Power spectrum: $P(u,v) = |F(u,v)|^2$

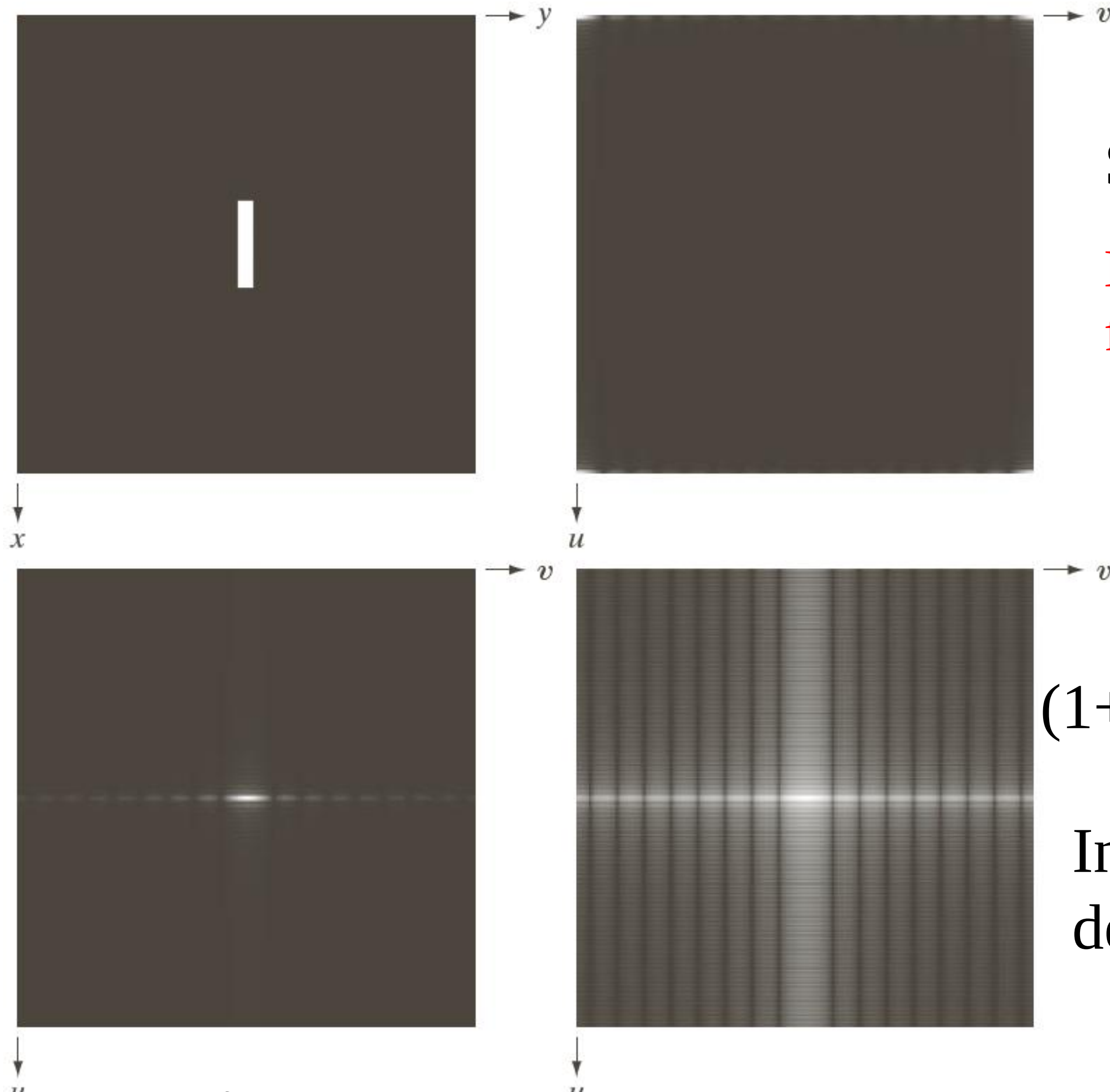
(1), (2), (3) are of M × N size.

Fourier Spectrum of An Image

Image

Spectrum

Bright spots at four corners

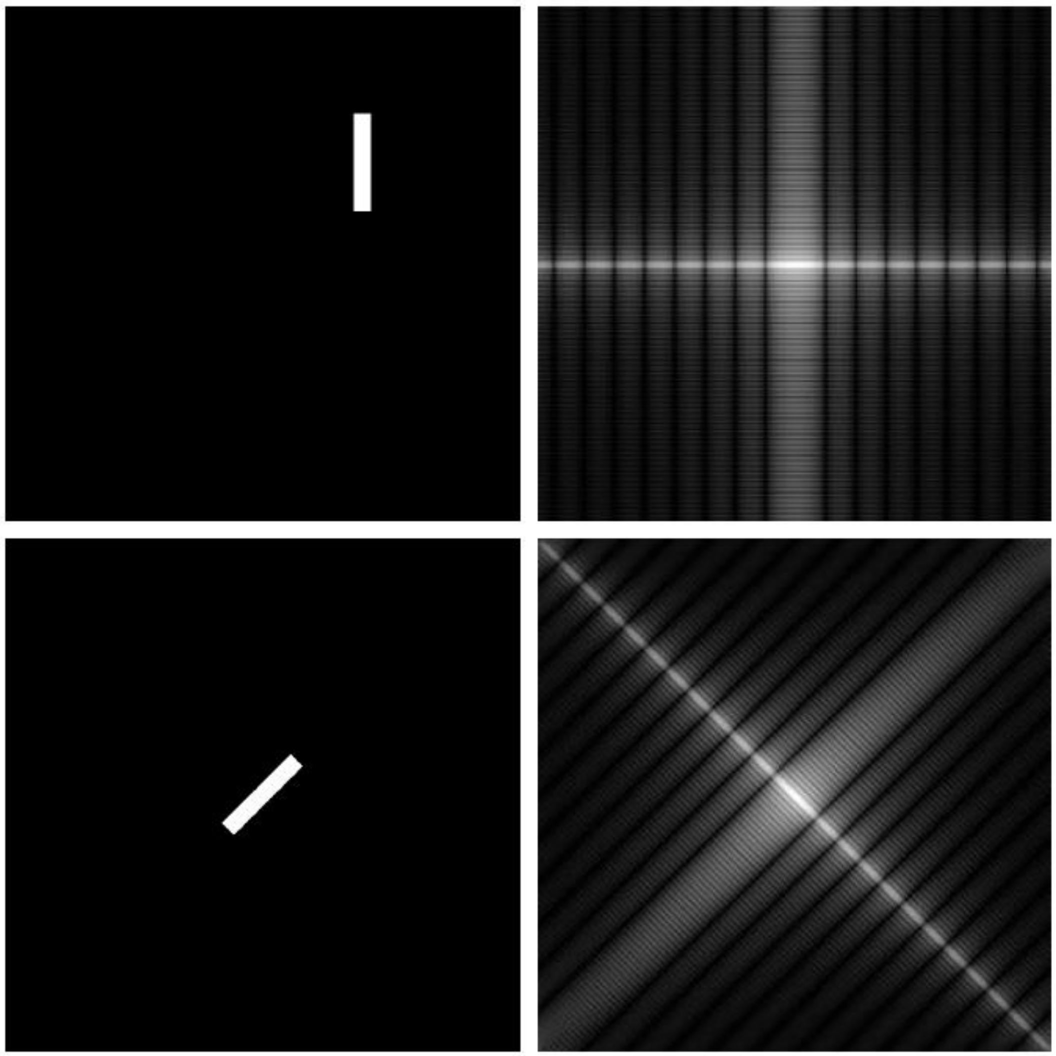


Centered spectrum (obtained multiplying the image by $(-1)^{x+y}$ before DFT)

$(1+\log|F(u,v)|)$

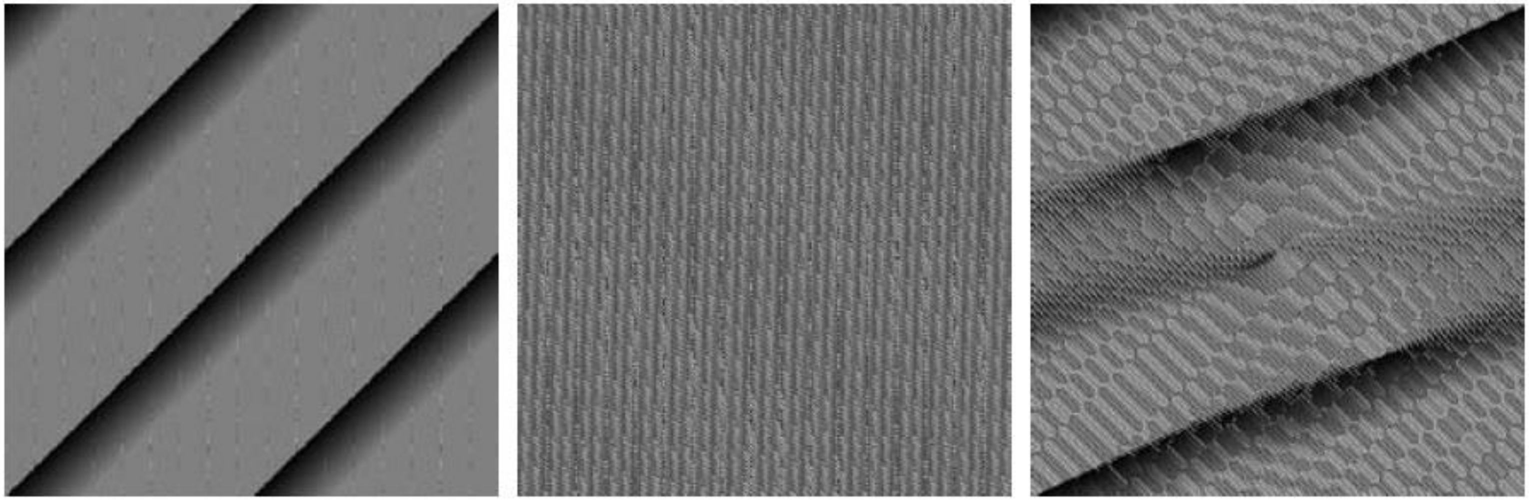
Increased detail

Translation effect on Spectrum



No effect on spectrum for image translation.

Translation effect on Phase Angle

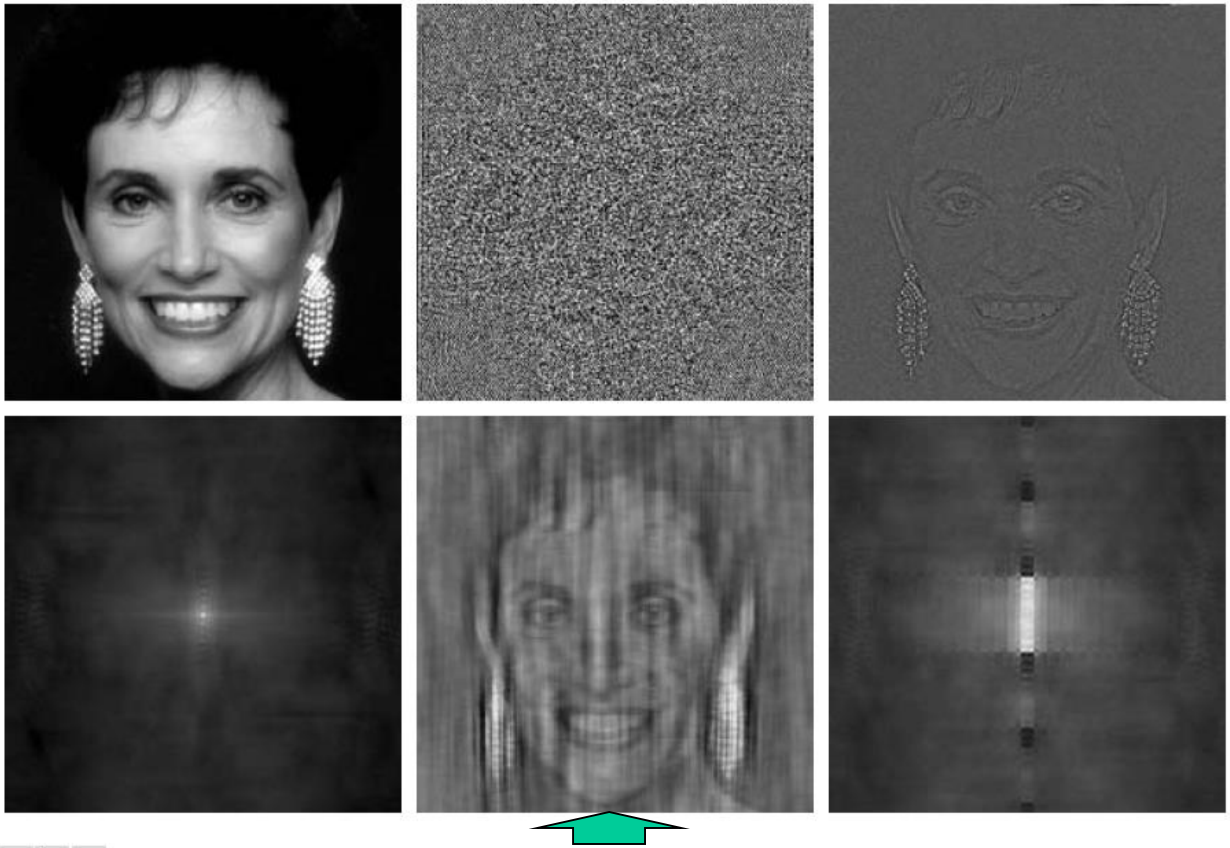


a b c

FIGURE 4.26 Phase angle array corresponding (a) to the image of the centered rectangle in Fig. 4.24(a), (b) to the translated image in Fig. 4.25(a), and (c) to the rotated image in Fig. 4.25(c).

Translation has effect on phase angle.

Importance of Phase Angle



Reconstructed using only phase angle

Reconstructed using only spectrum

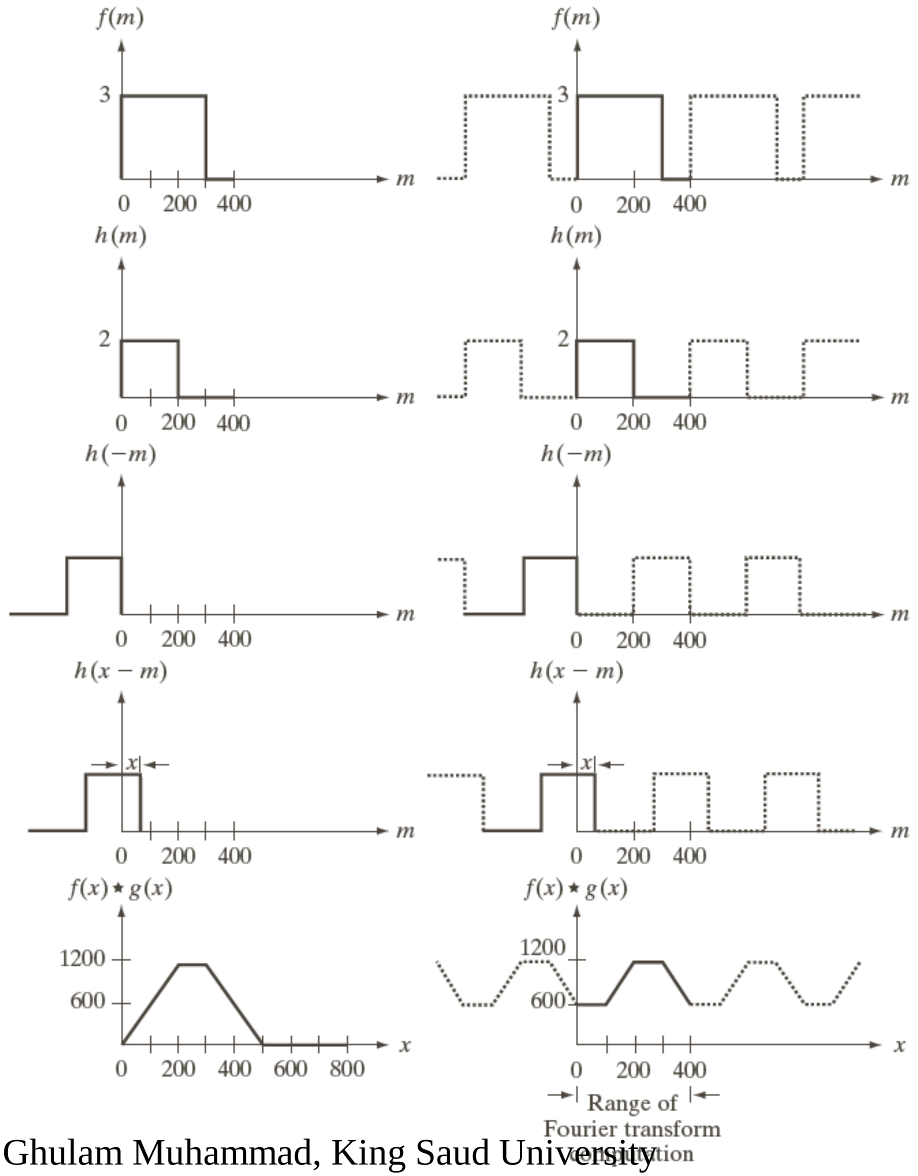
a b c
d e f

Reconstruction: Woman phase + rectangle spectrum

FIGURE 4.27 (a) Woman. (b) Phase angle. (c) Woman reconstructed using only the phase angle. (d) Woman reconstructed using only the spectrum. (e) Reconstruction using the phase angle corresponding to the woman and the spectrum corresponding to the rectangle in Fig. 4.24(a). (f) Reconstruction using the phase of the rectangle and the spectrum of the woman.

1-D Convolution

$$f(x) \otimes h(x) = \sum_{m=0}^{399} f(x)h(x-m)$$



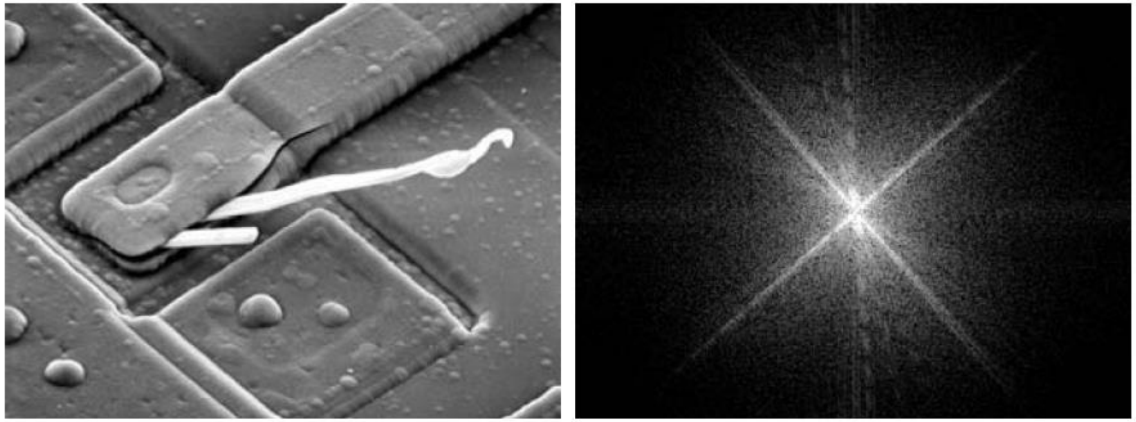
For DFT, we need to append zero: zero padding.

f(x), h(x) and output must be of length P.

$P \geq A + B - 1$

Filtering in Frequency Domain - I

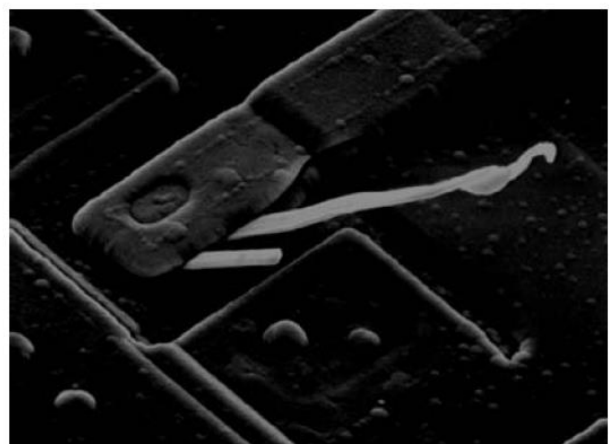
Edges are 45°



a b

FIGURE 4.29 (a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

$$g(x,y) = \Im^{-1}\left[H(u,v)F(u,v)\right]$$

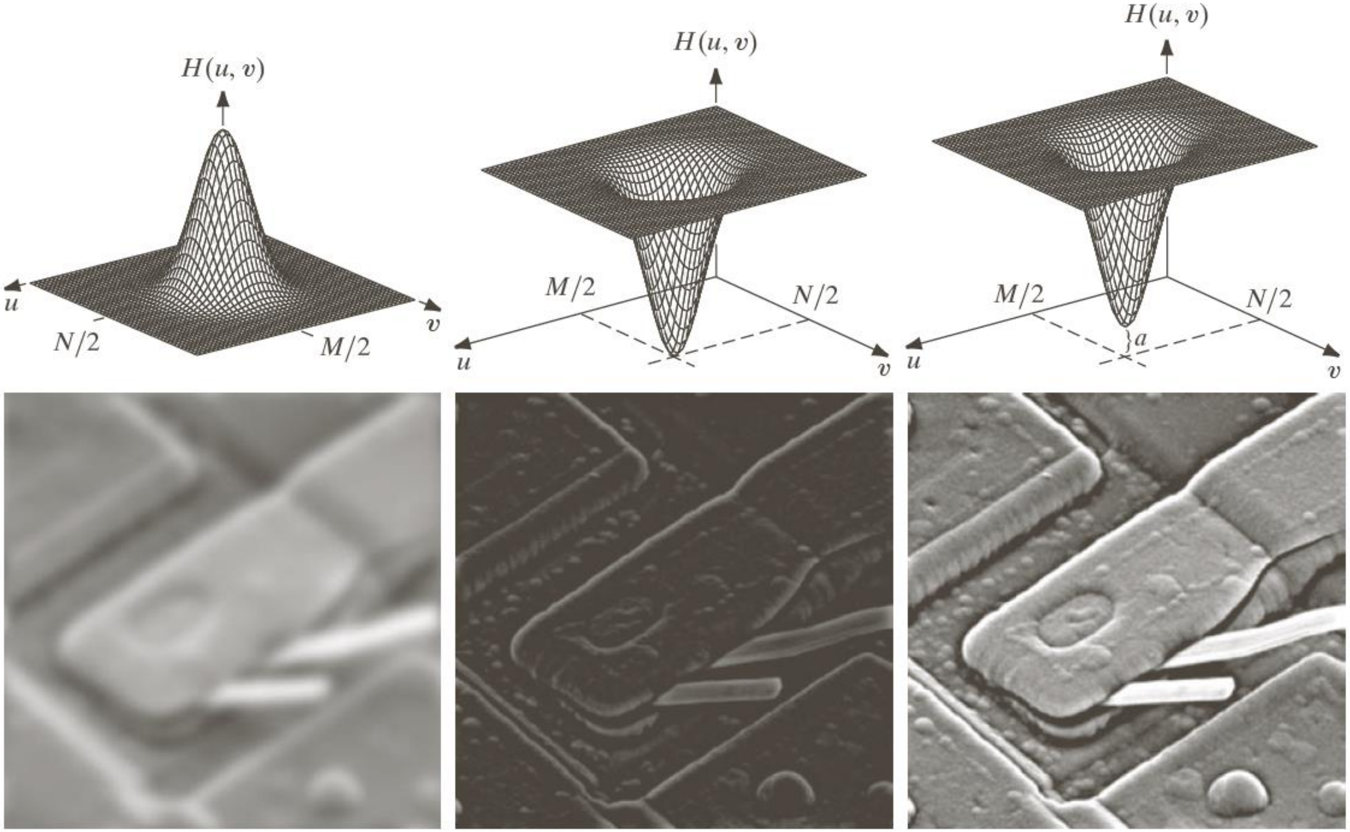


Simplest filter: removing dc component (zero freq). Put H(M/2, N/2) = 0.

Zero averaging.

Filtering in Frequency Domain - II

Low pass | High pass | High pass: without eliminating dc.



a b c
d e f

FIGURE 4.31 Top row: frequency domain filters. Bottom row: corresponding filtered images obtained using Eq. (4.7-1). We used $a = 0.85$ in (c) to obtain (f) (the height of the filter itself is 1). Compare (f) with Fig. 4.29(a).

Zero Padding Effect - I

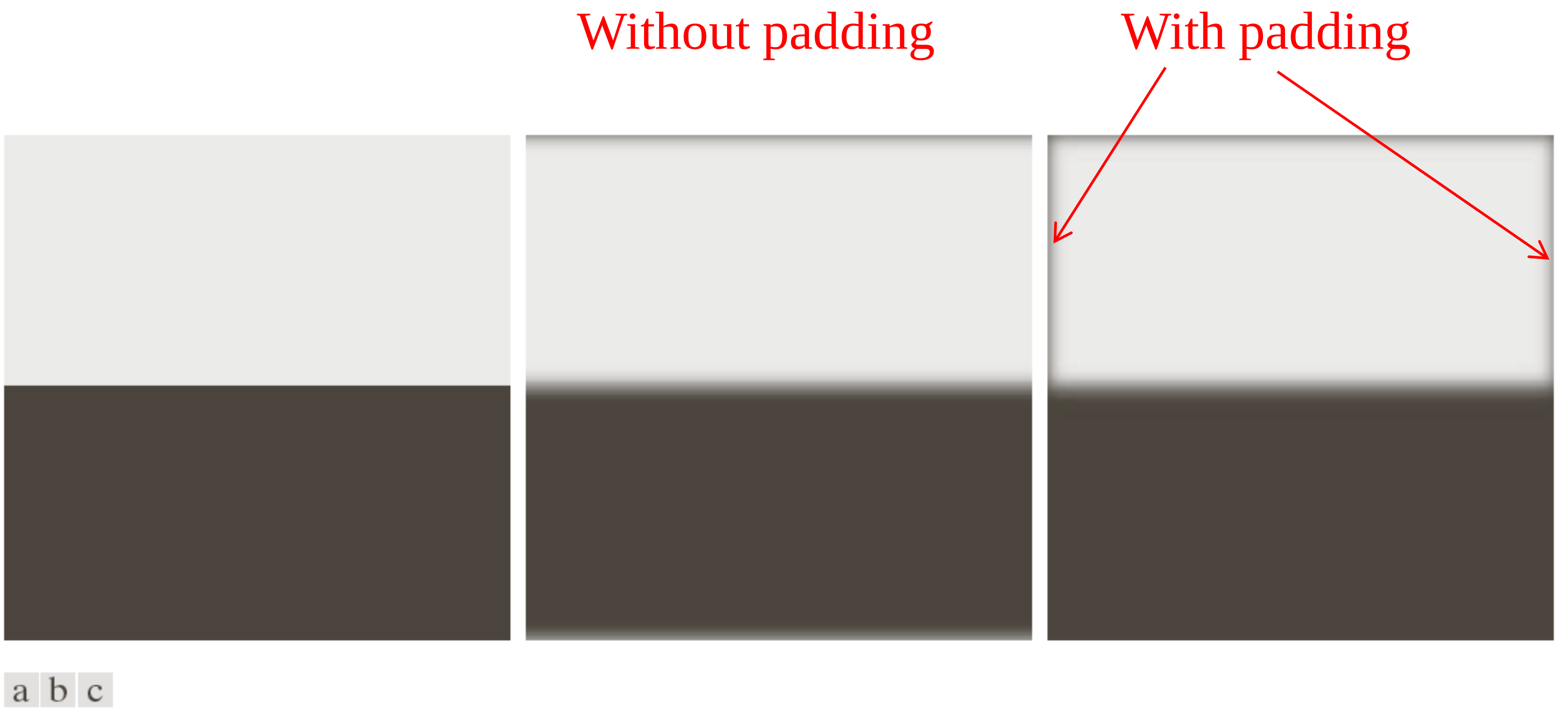
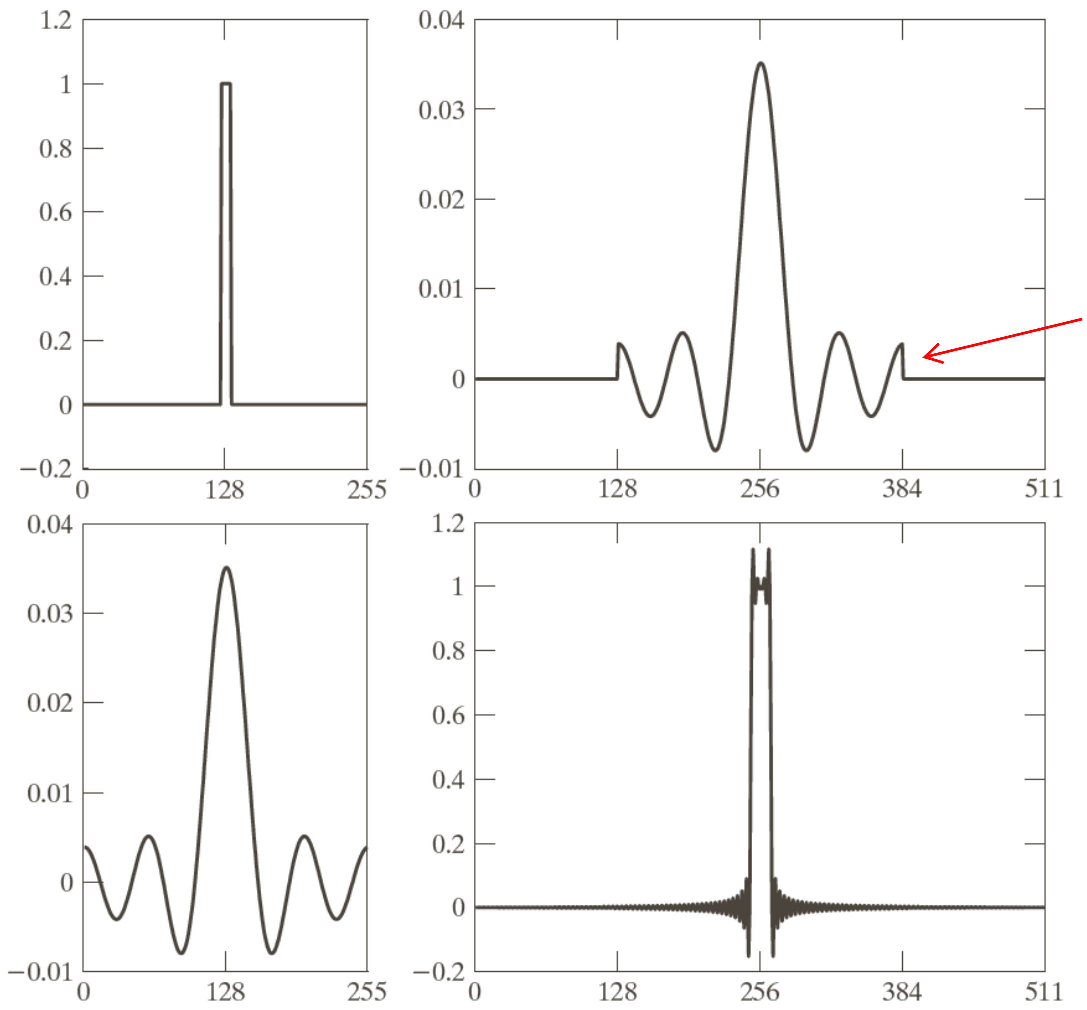


FIGURE 4.32 (a) A simple image. (b) Result of blurring with a Gaussian lowpass filter without padding. (c) Result of lowpass filtering with padding. Compare the light area of the vertical edges in (b) and (c).

Zero Padding Effect - II

Filter in f-domain

Filter in spatial domain



Zero padding

Discontinuous

IDFT

Steps for Frequency Domain Filtering

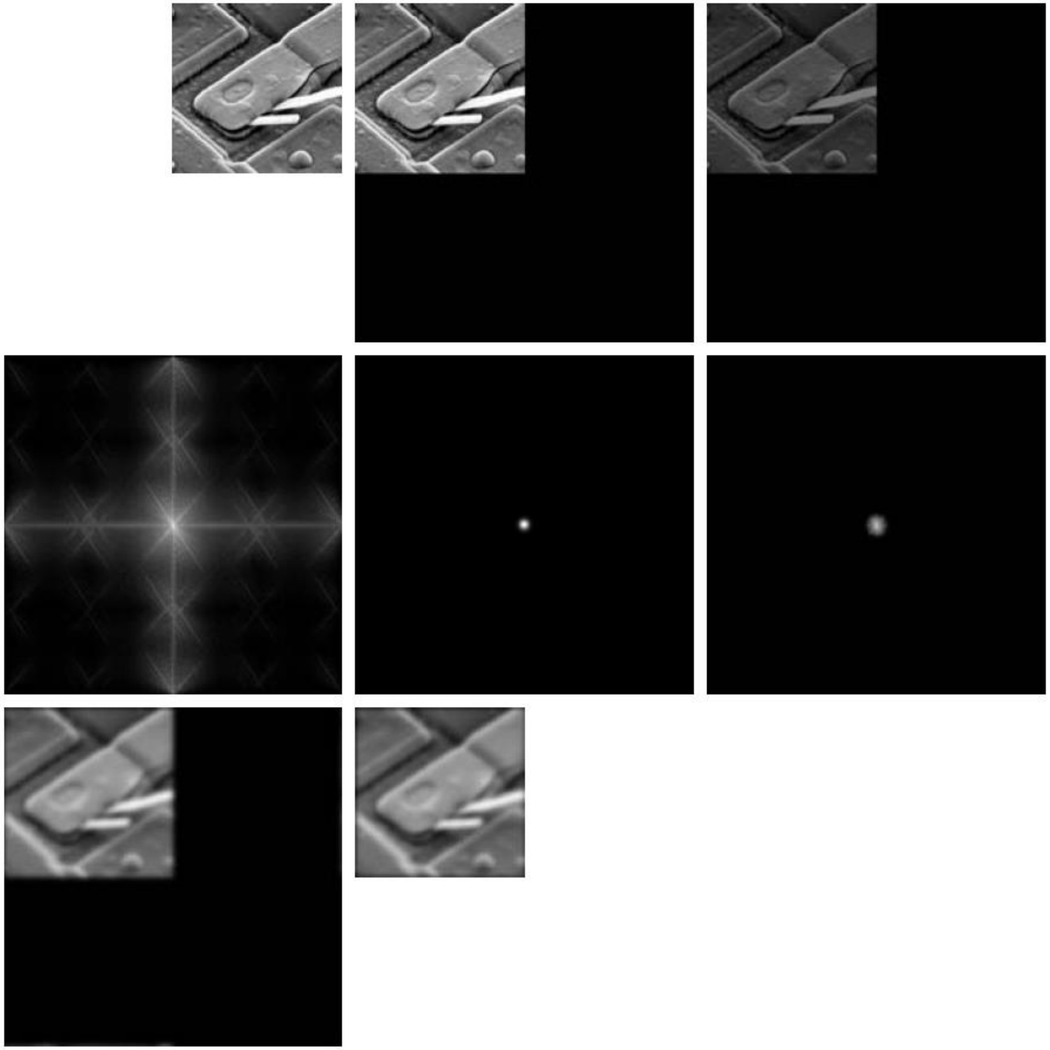


FIGURE 4.36 (a) An $M \times N$ image, f. (b) Padded image, f_p of size $P \times Q$. (c) Result of multiplying f_p by $(-1)^{x+y}$. (d) Spectrum of F_p. (e) Centered Gaussian lowpass filter, H, of size $P \times Q$. (f) Spectrum of the product HF_p. (g) g_p, the product of $(-1)^{x+y}$ and the real part of the IDFT of HF_p. (h) Final result, g, obtained by cropping the first M rows and N columns of g_p.