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BLOW UP RESULTS FOR EVOLUTION INEQUALITIES IN SOME SPECIFIC DOMAINS

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Notations

We introduce a set of notations that will be consistently applied throughout the thesis.

- $\mathbb{R}^N := \{(x_1, x_2, ..., x_N) : x_i \in \mathbb{R}, \quad \text{for all } i = 1, 2, ..., N\}$
- $L^p_{loc}(\Omega) := \left\{ f : \Omega \longrightarrow \mathbb{R} \text{ measurable} : \int_K |f|^p d\sigma < \infty \text{ for all compact sets } K \subset\subset \Omega \right\}, 1 < p \leq \infty$
- $\overline{A} := A \cup \{x \in X : x \text{ is a limit point of } A, \ A \subseteq X\}$
- $\partial A := \overline{A} \cap \overline{X \setminus A}$
- $|x| := \sqrt{x_1^2 + ... + x_N^2} \quad \text{for } x \in \mathbb{R}^N$
- $\Delta u(x) := \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$ (the Laplacian of u)
- $\Delta^m u :=$ the m-th iteration of (Δu)
- $\text{supp}(f) := \overline{\left\{ x \in X : f(x) \neq 0 \right\}}$
- $\text{sgn}(f) := \begin{cases} 1 & \text{if} \quad f > 0 \\ -1 & \text{if} \quad f < 0 \\ 0 & \text{if} \quad f = 0 \end{cases}$
- $C^{k,m}_{t,x}(D) :=$ the space of functions $u(t, x) \in C^k$in t and C^m in x on domain D

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Contents

Chapter

1

Introduction

Introduction

The study of the nonexistence of a solution, or, in other words, the necessary conditions for the existence of a solution, constitutes an important topic of the theory of partial differential equations, which was initiated by the well-known Liouville theorem for harmonic functions. The literature includes numerous results devoted to this subject. For instance, we refer to the monographs of Samarskii, Galaktionov, Kurdymov, and Mikhailov [18], Straugham [19], and the survey papers by Galaktionov and Vázquez [20], and Levine [11]. The mentioned references contain the main bibliography related to the blow-up theory for nonlinear parabolic equations. Interesting studies related to the necessary conditions for the existence of global solutions to nonlinear hyperbolic problems can be found in the monographs of John [9] and Strauss [16]. A consistent list of references related to the nonexistence of solutions to nonlinear elliptic problems can be found in the papers by Serrin [14], Mitidieri and Pohozaev [12], and Serrin and Zhou [15]. In the mentioned references, the blow-up of a solution in a finite time for evolution equations was proved by the construction of lower blow-up solutions and the use of the comparison principle. The same approach was used for the corresponding nonlinear elliptic problems.

The pioneering ideas related to capacity methods were first introduced by Baras and Pierre [54]. Subsequently, Mitidieri and Pohozaev [13] developed these ideas into a systematic approach, known as the nonlinear capacity (test function) method. This approach does not employ the comparison principle. It is essentially based on a priori estimates. Namely, the method consists of deriving an a priori estimate for a solution to the considered nonlinear problem and determining an asymptotics for the obtained a priori estimate with respect to a certain parameter that tends to $+\infty$. Next, arguing by contradiction, the nonexistence follows by proving that the corresponding a priori estimate admits the zero limit. The derivation of an a priori estimate is essentially based on an optimal choice of a test function. This choice depends on the problem under study and the considered domain.

In this thesis, we are concerned with the study of the nonexistence of a weak solution to some nonlinear evolution problems posed outside an obstacle, under inhomogeneous boundary conditions. Our approach is based on the nonlinear capacity method described above.

The first work dealing with the study of the nonexistence of a solution to exterior problems with inhomogeneous boundary conditions is the paper of Zhang [17], where several exterior boundary-value problems covering semilinear parabolic and hyperbolic equations were studied. Namely, Zhang considered the evolution problems:

$$\begin{cases} \partial_t u - \Delta u = V u^p \;\; (u \geq 0) & \text{in} \quad (0,\infty) \times D^c, \\ u(t,x) = f(x) & \text{in} \quad (0,\infty) \times \partial D, \\ u(0,x) = u_0(x) \geq 0 & \text{in} \quad D^c, \end{cases} \tag{1.0.1}$$

$$\begin{cases} \partial_t u - \Delta u = V u^p \;\; (u \geq 0) & \text{in} \quad (0,\infty) \times D^c, \\ \dfrac{\partial u}{\partial \nu}(t,x) = f(x) & \text{in} \quad (0,\infty) \times \partial D, \\ u(0,x) = u_0(x) \geq 0 & \text{in} \quad D^c, \end{cases} \tag{1.0.2}$$

and

$$\begin{cases} \partial_{tt} u - \Delta u = V |u|^p & \text{in} \quad (0,\infty) \times D^c, \\ \dfrac{\partial u}{\partial \nu}(t,x) = f(x) & \text{in} \quad (0,\infty) \times \partial D, \\ (u(0,x), \partial_t u(0,x)) = (u_0(x), u_1(x)) & \text{in} \quad D^c, \end{cases} \tag{1.0.3}$$

where $D^c = \mathbb{R}^N \backslash \overline{D}$, $N \geq 3$, D is a bounded Lipschitz domain of $\mathbb{R}^N$, and $f = f(x) \geq 0$ is a nontrivial $L^1(\partial D)$ function, and $V(x) \sim |x|^m$ as $|x| \to \infty$ with $m > -2$. Here, ν denotes the outward (relative to D^c) unit normal of ∂D. It was shown that problems (1.0.1), (1.0.2), and (1.0.3) share the same critical exponent given by

$$p^*(N,m) = \frac{N+m}{N-2}.$$

More precisely, the following results were obtained:

(i) If $1 < p < p^*(N,m)$ ($1 < p \leq p^*(N,m)$ for (1.0.2)), then (1.0.1), (1.0.2), and (1.0.3) admit no global solutions.

(ii) If $p > p^*(N,m)$, then (1.0.1), (1.0.2), and (1.0.3) admit global solutions for some initial values and $f > 0$.

It is interesting to mention that in the homogeneous case $f \equiv 0$ (see [11]), the critical exponent for (1.0.1) and (1.0.2) with $m = 0$, coincides with the Fujita critical exponent [2]

$$p_F^* = 1 + \frac{2}{N}\left(< p^*(N,0) = \frac{N}{N-2}\right)$$

for the semilinear heat equation posed in the whole space $\mathbb{R}^N$. So, from the above results of Zhang, an interesting phenomenon appears: The additional inhomogeneous term f, no matter how small it is, has the effect of increasing the critical exponent. We also mention that the critical case $p = p^*(N,m)$ for problems (1.0.1) and (1.0.3) was left as an open question in the paper [17] of Zhang. Also, the two-dimensional case was not studied in that paper. All these questions were completely solved in the papers by Jleli and Samet [5], Jleli, Samet, and Ye [8], and Jleli, Kirane, and Samet [4]. Namely, for $N \geq 3$, it was shown that $p = p^*(N,m)$ belongs to the nonexistence case. Furthermore, if $N = 2$, then for all $p > 1$, problems (1.0.1), (1.0.2), and (1.0.3) admit no global solutions ($p^*(2,m) = \infty$).

The first chapter of this thesis is concerned with the study of the existence and nonexistence of solutions to higher order evolution inequalities posed outside the half-ball. Namely, we consider the problem

$$\partial_t^k u + L_\lambda u \geq |x|^\tau |u|^p \quad \text{in } (0,\infty)\times\Omega, \tag{1.0.4}$$

where

$$\Omega = \{x \in \mathbb{R}_+^N : |x| > 1\}, \quad \mathbb{R}_+^N = \{x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N : x_N > 0\}, \quad N \geq 2,$$

$u = u(t,x)$, $k \geq 1$ (k is a natural number), $\partial_t^k = \frac{\partial^k}{\partial t^k}$, $\tau \in \mathbb{R}$, $p > 1$, and

$$L_\lambda = -\Delta + \frac{\lambda}{|x|^2}, \quad \lambda \geq -\frac{N^2}{4}.$$

Problem (1.0.4) is studied under the boundary conditions

$$\begin{cases} u \geq 0 & \text{on } (0,\infty)\times\Sigma_0, \\ u \geq w & \text{on } (0,\infty)\times\Sigma_1, \end{cases} \tag{1.0.5}$$

where $w = w(x) \in L^1(\Sigma_1)$ and

$$\Sigma_0 = \{x \in \mathbb{R}^N : x_N = 0, |x| > 1\}, \quad \Sigma_1 = \{x \in \mathbb{R}^N_+ : |x| = 1\}.$$

We show that the dividing line with respect to existence or nonexistence for problem (1.0.4) under the boundary conditions (1.0.5), is given by a Fujita-type critical exponent that depends on λ, N, and τ, but does not depend on the order of the time-derivative. Hamidi and Laptev [3] (see also Jleli and Samet [33]) investigated problem (1.0.4) in the case $\Omega = \mathbb{R}^N$, $N \geq 3$, $\lambda \geq -\left(\frac{N-2}{2}\right)^2$, and $\tau = 0$. Namely, they studied the problem

$$\partial_t^k u + L_\lambda u \geq |u|^p \quad \text{in } (0, \infty) \times \mathbb{R}^N \tag{1.0.6}$$

subject to the initial condition

$$\partial_t^{k-1} u(0, x) \geq 0 \quad \text{in } \mathbb{R}^N. \tag{1.0.7}$$

It was shown that, if one of the following assumptions is satisfied:

$$\lambda \geq 0,\ 1 < p \leq 1 + \frac{2}{\frac{2}{k} + s^*};$$

or

$$-\left(\frac{N-2}{2}\right)^2 \leq \lambda < 0,\ 1 < p \leq 1 + \frac{2}{\frac{2}{k} - s_*},$$

where

$$s^* = \frac{N-2}{2} + \sqrt{\lambda + \left(\frac{N-2}{2}\right)^2},\ s_* = s^* + 2 - N,$$

then problem (1.0.6) subject to the initial condition (1.0.7) admits no nontrivial weak solution. For other results related the study of evolution problems (posed in the whole space $\mathbb{R}^N$) involving the differential operator L_λ, we refer to the papers by Abdellaoui, Attar, Bentifour, and Peral [35], Abdellaoui, Medina, Peral, and Primo [36], Abdellaoui, Miri, Peral, and Touaoula [31], and Abdellaoui, Peral, and Primo [34, 37]. We also mention that problem (1.0.4) under the boundary conditions (1.0.5) was previously studied by Jleli, Samet, and Vetro [7] in the special case $k = 2$ and $\lambda = \tau = 0$. Namely, they considered the hyperbolic differential inequality

$$\partial_t^2 u - \Delta u \geq |u|^p \quad \text{in } (0,\infty)\times\Omega \tag{1.0.8}$$

under the boundary conditions (1.0.5). It was shown that,

(i) if $1 < p \leq \frac{N+1}{N-1}$, then problem (1.0.8) under the boundary conditions (1.0.5) admits no weak solution satisfying

$$\int_{\Sigma_1} x_N w(x)\, dS_x > 0; \tag{1.0.9}$$

(ii) if $p > \frac{N+1}{N-1}$, then problem (1.0.8) under the boundary conditions (1.0.5) admits solutions for some w satisfying (1.0.9).

It is interesting to mention that $\frac{N+1}{N-1}$ is the Kato critical exponent [10] for the wave inequality

$$\partial_t^2 u - \Delta u \geq |u|^p \quad \text{in } (0,\infty)\times\mathbb{R}^N.$$

In the first chapter, our main motivation is to study the influence of the inverse-square potential $\frac{\lambda}{|x|^2}$ as well as the order of the time-derivative k on the critical behavior of Problem (1.0.8) under the boundary conditions (1.0.5).

The aim of the second chapter is to study the questions of existence and nonexistence of weak solutions to the system of polyharmonic wave inequalities

$$\begin{cases} u_{tt} + (-\Delta)^m u \geq |x|^a |v|^p, & (t,x)\in(0,\infty)\times\mathbb{R}^N\backslash\overline{B_1}, \\ v_{tt} + (-\Delta)^m v \geq |x|^b |u|^q, & (t,x)\in(0,\infty)\times\mathbb{R}^N\backslash\overline{B_1}. \end{cases} \tag{1.0.10}$$

Here, $(u,v) = (u(t,x), v(t,x))$, $N \geq 2$, B_1 is the open unit ball of $\mathbb{R}^N$, $m \geq 1$ is an integer, $a, b \geq -2m$, $(a,b) \neq (-2m,-2m)$, and $p,q > 1$. We investigate (1.0.10) under the Navier-type boundary conditions

$$\begin{cases} (-\Delta)^i u \geq f_i(x),\ i = 0,\cdots,m-1,\ (t,x)\in(0,\infty)\times\partial B_1, \\ (-\Delta)^i v \geq g_i(x),\ i = 0,\cdots,m-1,\ (t,x)\in(0,\infty)\times\partial B_1, \end{cases} \tag{1.0.11}$$

where $f_i, g_i \in L^1(\partial B_1)$ and $(-\Delta)^0$ is the identity operator. We establish a sharp criterium for the nonexistence of weak solutions to (1.0.10)–(1.0.11). Next, we deduce an optimal nonexistence result for the corresponding stationary problem. This work is motivated by the papers by Jleli, Samet, and Ye [8], and Jleli and Samet [6]. In [8], the authors studied (1.0.10)–(1.0.11) in the special case $m = 1$. Namely, they considered the system of wave inequalities

$$\begin{cases} u_{tt} - \Delta u \geq |x|^a |v|^p, & (t,x) \in (0,\infty) \times \mathbb{R}^N \backslash \overline{B_1}, \\ v_{tt} - \Delta v \geq |x|^b |u|^q, & (t,x) \in (0,\infty) \times \mathbb{R}^N \backslash \overline{B_1} \end{cases} \tag{1.0.12}$$

under the Dirichlet-type boundary conditions

$$\begin{cases} u \geq f_0(x), \ (t,x) \in (0,\infty) \times \partial B_1, \\ v \geq g_0(x), \ (t,x) \in (0,\infty) \times \partial B_1, \end{cases} \tag{1.0.13}$$

where $a, b \geq -2$, $(a,b) \neq (-2,-2)$, and $p, q > 1$. The following result was proved: Assume that

$$I_{f_0} := \int_{\partial B_1} f_0 \, dS_x \geq 0, \ I_{g_0} := \int_{\partial B_1} g_0 \, dS_x \geq 0, \quad (I_{f_0}, I_{g_0}) \neq (0,0).$$

If $N = 2$; or $N \geq 3$ and

$$N < \max\left\{ \operatorname{sgn}(I_{f_0}) \frac{2p(q+1) + pb + a}{pq - 1}, \operatorname{sgn}(I_{g_0}) \frac{2q(p+1) + qa + b}{pq - 1} \right\},$$

then (1.0.12)-(1.0.13) admits no weak solution. Moreover, Jleli, Samet, and Ye pointed out the sharpness of the above condition. In [6], Jleli and Samet studied (1.0.10) in the special case $m = 2$, under different types of boundary conditions. In particular, they considered the system of biharmonic wave inequalities

$$\begin{cases} u_{tt} + \Delta^2 u \geq |x|^a |v|^p, & (t,x) \in (0,\infty) \times \mathbb{R}^N \backslash \overline{B_1}, \\ v_{tt} + \Delta^2 v \geq |x|^b |u|^q, & (t,x) \in (0,\infty) \times \mathbb{R}^N \backslash \overline{B_1} \end{cases} \tag{1.0.14}$$

under the boundary conditions

$$\begin{cases} u \geq 0, \ -\Delta u \geq f_1(x), & (t,x) \in (0,\infty) \times \partial B_1, \\ v \geq 0, \ -\Delta v \geq g_1(x), & (t,x) \in (0,\infty) \times \partial B_1, \end{cases} \tag{1.0.15}$$

where $a, b \geq -4$, $(a,b) \neq (-4,-4)$, $\int_{\partial B_1} f_1 \, dS_x > 0$, $\int_{\partial B_1} g_1 \, dS_x > 0$, and $p, q > 1$. Namely, the following result was obtained: If $N \in \{2,3,4\}$; or

$$N \geq 5, \ N < \max\left\{ \frac{4p(q+1)+pb+a}{pq-1}, \frac{4q(p+1)+qa+b}{pq-1} \right\},$$

then the system (1.0.14) under the boundary conditions (1.0.15) admits no weak solution. Moreover, it was shown that the above condition is sharp. In the second chapter, we extend the obtained results in [6, 8] from $m \in \{1,2\}$ to $m \geq 1$.

Chapter

2

Existence and nonexistence results for higher order evolution inequalities posed outside the half-ball

The result of this chapter is published in: Manal Alfulaij and Mohamed Jleli. Existence and nonexistence results for higher order evolution inequalities posed outside the half-ball.Discrete and Continuous Dynamical Systems-S 17.10 (2024): 3046-3063.

This chapter focuses on existence and nonexistence of weak solutions to semilinear higher order (in time) evolution inequalities with Hardy potential and posed outside the half-ball of $\mathbb{R}^N$ ($N \geq 2$). We impose inhomogeneous Dirichlet-type boundary conditions, and show the dividing line with respect to existence or nonexistence is given by a Fujita-type critical exponent which depends on a suitable parameter λ, but does not depend on the order of the time derivative. We conclude naturally optimal nonexistence results for the corresponding elliptic inequality.

2.1 Introduction and main results

For $N \geq 2$, let $\mathbb{R}^N_+$ be the half-space, i.e. $\mathbb{R}^N_+ = \{x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N : x_N > 0\}$. We consider the exterior domain $\Omega = \{x \in \mathbb{R}^N_+ : |x| > 1\}$. Let $\Sigma_0 = \{x \in \mathbb{R}^N : x_N = 0, |x| > 1\}$ and $\Sigma_1 = \{x \in \mathbb{R}^N_+ : |x| = 1\}$. For $\lambda \geq -\frac{N^2}{4}$, we focus on the differential operator

$$L_\lambda = -\Delta + \frac{\lambda}{|x|^2},$$

and we point out that the term $-\frac{N^2}{4}$ appears in the Hardy-type inequality

$$\int_{\mathbb{R}^N_+} |\nabla \varphi|^2 \, dx - \frac{N^2}{4} \int_{\mathbb{R}^N_+} \frac{\varphi^2}{|x|^2} \, dx \geq 0$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N_+)$, see e.g. [30].

In this chapter, we are concerned with the higher order (in time) evolution inequality

$$\partial_t^k u + L_\lambda u \geq |x|^\tau |u|^p \quad \text{in } (0, \infty) \times \Omega, \tag{2.1.1}$$

where $u = u(t, x)$, $k \geq 1$ is an integer, $\tau \in \mathbb{R}$ and $p > 1$. Here, for any nonnegative integer i,

$\partial_t^i = \frac{\partial^i}{\partial t^i}$. Problem (2.1.1) is considered under the Dirichlet-type boundary conditions

$$\begin{cases} u \geq 0 & \text{on } (0,\infty) \times \Sigma_0, \\ u \geq w & \text{on } (0,\infty) \times \Sigma_1, \end{cases} \tag{2.1.2}$$

where $w = w(x) \in L^1(\Sigma_1)$. Namely, our goal is to study the existence and nonexistence of weak solutions to (2.1.1)–(2.1.2). We recall below some related works from the literature.

Among other problems, Abdellaoui et al. [31] (see also [34]) considered problems of the form

$$\partial_t(u^{p-1}) - \Delta_p u = \lambda \frac{u^{p-1}}{|x|^p} + u^q \ (u > 0) \quad \text{in } (0,\infty) \times \mathbb{R}^N, \tag{2.1.3}$$

where $1 < p < N$, $q > 0$ and $0 \leq \lambda < \left(\frac{N-p}{p}\right)^p$. Namely, it was proven that there exist two exponents $q^+(p,\lambda)$ and $F(p,\lambda)$ such that

(i) if $p - 1 < q < F(p,\lambda) < q^+(p,\lambda)$ and u is a solution to (2.1.3) satisfying a certain behavior, then u blows-up in a finite time;

(ii) if $F(p,\lambda) < q < q^+(p,\lambda)$, then under suitable condition on $u(0,\cdot)$, (2.1.3) admits a global in time positive solution.

Hamidi and Laptev [32] (see also [33]) investigated problem (2.1.1) in the case $\Omega = \mathbb{R}^N$, $N \geq 3$, $\lambda \geq -\left(\frac{N-2}{2}\right)^2$ and $\tau = 0$, namely they studied the following problem

$$\partial_t^k u + L_\lambda u \geq |u|^p \quad \text{in } (0,\infty) \times \mathbb{R}^N \tag{2.1.4}$$

subject to the initial condition

$$\partial_t^{k-1} u(0,x) \geq 0 \quad \text{in } \mathbb{R}^N. \tag{2.1.5}$$

It was shown that, if one of the following assumptions is satisfied:

$$\lambda \geq 0,\ 1 < p \leq 1 + \frac{2}{\frac{2}{k} + s^*};$$

or

$$-\left(\frac{N-2}{2}\right)^2 \leq \lambda < 0,\ 1 < p \leq 1 + \frac{2}{\frac{2}{k} - s_*},$$

where

$$s^* = \frac{N-2}{2} + \sqrt{\lambda + \left(\frac{N-2}{2}\right)^2},\ s_* = s^* + 2 - N,$$

then (2.1.4)–(2.1.5) admits no nontrivial weak solution. Other results related to evolution equations with a Hardy potential in the whole space $\mathbb{R}^N$ can be found in [36, 35, 37, 38] (see also the references therein).

Evolution equations and inequalities have been also studied in exterior domains of $\mathbb{R}^N$ under different types of boundary conditions: semilinear heat equation under homogeneous Dirichlet and Neumann boundary conditions [39, 24], semilinear wave equations with homogeneous Dirichlet boundary condition [40, 51, 42, 43, 44], semilinear evolution equations with inhomogeneous boundary conditions [17], systems of semilinear parabolic inequalities with inhomogeneous Dirichlet-type boundary conditions [25] and systems of wave inequalities with various types of inhomogeneous boundary conditions [8, 45]. Other related results can be found in [46, 47].

In [22], among other problems, the authors studied the special case of (2.1.1)–(2.1.2) when $k = 2$, $\lambda = \tau = 0$, namely the problem

$$\partial_t^2 u - \Delta u \geq |u|^p \quad \text{in } (0, \infty) \times \Omega \tag{2.1.6}$$

under the boundary conditions (2.1.2). It was proven that,

(i) if $1 < p \leq \frac{N+1}{N-1}$, then (2.1.6), (2.1.2) admits no weak solution satisfying

$$\int_{\Sigma_1} x_N w(x)\, dS_x > 0; \tag{2.1.7}$$

(ii) if $p > \frac{N+1}{N-1}$, then (2.1.6), (2.1.2) admits solutions for some w satisfying (2.1.7).

Notice that $\frac{N+1}{N-1}$ is the Kato critical exponent for the wave inequality

$$\partial_t^2 u - \Delta u \geq |u|^p \quad \text{in } (0, \infty) \times \mathbb{R}^N.$$

We refer to [10, 48, 49] for more details about the above problem.

The main motivation of this work is to study the influence of the inverse-square potential $\frac{\lambda}{|x|^2}$ as well as the order of the time-derivative k on the critical behavior of Problem (2.1.6) under the boundary conditions (2.1.2).

Before stating our obtained results, let us define weak solutions to (2.1.1)–(2.1.2). Hence, we introduce the sets

$$Q = (0, \infty) \times \overline{\Omega} \quad \text{and} \quad \Sigma_Q^i = (0, \infty) \times \Sigma_i,\ i = 0, 1.$$

Notice that $\Sigma_Q^i \subset Q$ for all $i = 0, 1$. Further, by Φ we mean the set of functions $\varphi = \varphi(t, x)$ satisfying the following properties:

(A_1) $\varphi \geq 0, \varphi \in C_{t,x}^{k,2}(Q)$;

(A_2) $\text{supp}(\varphi) \subset\subset Q$;

(A_3) $\varphi_{|\Sigma_Q^i} = 0, i = 0, 1$;

(A$_4$) $\frac{\partial \varphi}{\partial \nu_i}|_{\Sigma_Q^i} \le 0$, $i = 0, 1$, where ν_i denotes the outward unit normal vector to Σ_i, relative to Ω.

We can now define weak solutions to (2.1.1)–(2.1.2) as follows.

Definition 2.1.1. *Let $k \ge 1$, $\tau \in \mathbb{R}$, $N \ge 2$, $\lambda \ge -\frac{N^2}{4}$, $p > 1$ and $w = w(x) \in L^1(\Sigma_1)$. We say that $u \in L^p_{loc}(Q)$ is a weak solution to* (2.1.1)–(2.1.2)*, if*

$$\int_Q |x|^\tau |u|^p \varphi \, dx \, dt - \int_{\Sigma_Q^1} \frac{\partial \varphi}{\partial \nu_1} w(x) \, dS_x \, dt \le (-1)^k \int_Q u \partial_t^k \varphi \, dx \, dt + \int_Q u L_\lambda \varphi \, dx \, dt \tag{2.1.8}$$

for all $\varphi = \varphi(t, x) \in \Phi$.

Using standard integrations by parts, it can be easily seen that any smooth solution to (2.1.1)–(2.1.2) is a weak solution, in the sense of Definition 2.1.1.

We now introduce some further notations. For $\lambda \ge -\frac{N^2}{4}$, let us define the parameter

$$\mu = -\frac{N}{2} + \sqrt{\lambda + \frac{N^2}{4}}. \tag{2.1.9}$$

For $w \in L^1(\Sigma_1)$, let

$$I_w = \int_{\Sigma_1} w(x) x_N \, dS_x.$$

Finally, we consider the set

$$L^{1,+}(\Sigma_1) = \left\{ w \in L^1(\Sigma_1) : I_w > 0 \right\}.$$

Our main results are stated in the two theorems below.

Theorem 2.1.1. *Let $k \ge 1$, $\tau \in \mathbb{R}$, $N \ge 2$, $\lambda \ge -\frac{N^2}{4}$ and $p > 1$.*

(I) *Let $w \in L^{1,+}(\Sigma_1)$. If*

$$(N + \mu - 1)p < N + \mu + 1 + \tau, \tag{2.1.10}$$

then (2.1.1)–(2.1.2) *admits no weak solution.*

(II) *If*

$$(N+\mu-1)p > N+\mu+1+\tau, \tag{2.1.11}$$

then (2.1.1)–(2.1.2) *admits nonnegative (stationary) solutions for some* $w \in L^{1,+}(\Sigma_1)$.

When $\lambda > -\frac{N^2}{4}$ and $(N+\mu-1)p = N+\mu+1+\tau$, we have the following nonexistence result.

Theorem 2.1.2. *Let* $k \geq 1$, $\tau \in \mathbb{R}$, $N \geq 2$, $\lambda > -\frac{N^2}{4}$ *and* $p > 1$. *If* $w \in L^{1,+}(\Sigma_1)$ *and*

$$(N+\mu-1)p = N+\mu+1+\tau, \tag{2.1.12}$$

then (2.1.1)–(2.1.2) *admits no weak solution.*

The proof of the nonexistence results given by Theorem 2.1.1 (I) and Theorem 2.1.2, relies on nonlinear capacity estimates specifically adapted to the operator L_λ, the domain Ω and the considered boundary conditions. The existence result provided by Theorem 2.1.1 (II) is established by the construction of explicit solutions.

Remark 2.1.1. Theorems 2.1.1 and 2.1.2 leave open the issue of existence and nonexistence in the critical case:

$$\lambda = -\frac{N^2}{4},\ (N+\mu-1)p = N+\mu+1+\tau.$$

Remark 2.1.2. We note the following facts:

(i) If $\lambda > -\frac{N^2}{4}$, then

$$N+\mu-1 = \frac{N-2}{2} + \sqrt{\lambda + \frac{N^2}{4}} > 0.$$

Hence, (2.1.10) and (2.1.12) reduce to

$$\tau > -2,\ 1 < p \leq 1 + \frac{\tau + 2}{N + \mu - 1},$$

and (2.1.11) reduces to

$$\tau \leq -2,\ p > 1; \quad \text{or} \quad \tau > -2,\ p > 1 + \frac{\tau + 2}{N + \mu - 1}.$$

Consequently, when $\tau > -2$, (2.1.1)–(2.1.2) admits a Fujita-type critical exponent given by

$$p^*(\lambda, N, \tau) = 1 + \frac{\tau + 2}{N + \mu - 1}.$$

It is interesting to observe that $p^*(\lambda, N, \tau)$ is independent of the value of k. Furthermore, if $\lambda = 0$ and $\tau = 0$, then $p^*(\lambda, N, \tau) = \frac{N+1}{N-1}$, which is the critical exponent obtained in [7] for problem (2.1.6) under the boundary conditions (2.1.2).

(ii) When $\lambda = -\frac{N^2}{4}$, we have

$$N + \mu - 1 = \frac{N - 2}{2} \geq 0.$$

(a) If $N = 2$, we deduce from Theorem 2.1.1 that (2.1.1)–(2.1.2) admits a critical value $\tau^* = -2$ in the following sense:

- if $w \in L^{1,+}(\Sigma_1)$ and $\tau > \tau^*$, then (2.1.1)–(2.1.2) admits no weak solution;
- if $\tau < \tau^*$, then (2.1.1)–(2.1.2) admits solutions for some $w \in L^{1,+}(\Sigma_1)$.

(b) If $N \geq 3$, then $N + \mu - 1 > 0$. Hence, when $\tau > -2$, (2.1.1)–(2.1.2) admits as Fujita-type critical exponent the real number

$$p^*(\lambda, N, \tau) = p^*\left(-\frac{N^2}{4}, N, \tau\right) = 1 + \frac{2(\tau+2)}{N-2}.$$

Clearly, Theorems 2.1.1 and 2.1.2 yield existence and nonexistence results for the corresponding elliptic inequality

$$L_\lambda u \geq |x|^\tau |u|^p \quad \text{in } \Omega \tag{2.1.13}$$

under the boundary conditions

$$\begin{cases} u \geq 0 & \text{on } (0,\infty) \times \Sigma_0, \\ u \geq w & \text{on } (0,\infty) \times \Sigma_1. \end{cases} \tag{2.1.14}$$

For readers' convenience, we now give the analogous of main theorems in the case of elliptic problem (2.1.13).

Corollary 2.1.1. *Let $N \geq 2$, $\tau \in \mathbb{R}$, $\lambda \geq -\frac{N^2}{4}$ and $p > 1$.*

(I) *Let $w \in L^{1,+}(\Sigma_1)$. If* (2.1.10) *holds, then* (2.1.13)–(2.1.14) *admits no weak solution.*

(II) *If* (2.1.11) *holds, then* (2.1.13)–(2.1.14) *admits nonnegative solutions for some $w \in L^{1,+}(\Sigma_1)$.*

Corollary 2.1.2. *Let $N \geq 2$, $\tau \in \mathbb{R}$, $\lambda > -\frac{N^2}{4}$ and $p > 1$. If $w \in L^{1,+}(\Sigma_1)$ and* (2.1.12) *holds, then* (2.1.13)–(2.1.14) *admits no weak solution.*

The rest of the paper is organized as follows. In Section 2.2, some auxiliary results useful for the proofs of Theorem 2.1.1 (I) and Theorem 2.1.2 are established. Namely, we first establish an a priori estimate for problem (2.1.1)–(2.1.2) (see Lemma 2.2.1). Next, we introduce two families of test functions belonging to Φ and depending on three sufficiently large parameters

T, R, ℓ, and a nonnegative function H, solution to the problem

$$L_\lambda H = 0 \text{ in } \Omega, \ \ H = 0 \text{ on } \Sigma_0 \cup \Sigma_1.$$

The first family of test functions will be used for proving Theorem 2.1.1 (I). The second one will be used for proving Theorem 2.1.2. For both families of the introduced test functions, some useful estimates are provided. Finally, Section 2.3 is devoted to the proofs of Theorems 2.1.1 and 2.1.2.

2.2 Auxiliary results

2.2.1 A priori estimates

Throughout this chapter, the symbols C, C_i denote always generic positive constants, which are independent on the scaling parameters T, R and the solution u. Their values could be changed from one line to another. The notation $R \gg 1$ means that R is sufficiently large.

The first step of our approach here is the establishment of useful estimates and bounds. In this setting, we point out the following basic estimate directly linked to the notion of weak solution (see Definition 2.1.1). Further estimates, needed to conclude the proofs of main results, will be proved in a forthcoming subsection.

Let $k \geq 1$, $N \geq 2$, $\tau \in \mathbb{R}$, $p > 1$ and $\lambda \geq -\frac{N^2}{4}$. For $\varphi \in \Phi$, we introduce the following functionals:

$$J_1(\varphi) = \int_{\mathrm{supp}(\partial_t^k \varphi)} |x|^{\frac{-\tau}{p-1}} \varphi^{\frac{-1}{p-1}} |\partial_t^k \varphi|^{\frac{p}{p-1}} \, dx \, dt, \tag{2.2.1}$$

$$J_2(\varphi) = \int_{\mathrm{supp}(L_\lambda \varphi)} |x|^{\frac{-\tau}{p-1}} \varphi^{\frac{-1}{p-1}} \left| L_\lambda \varphi \right|^{\frac{p}{p-1}} \, dx \, dt. \tag{2.2.2}$$

Then, we establish a priori estimate for $J_i(\varphi)$, $i = 1, 2$.

Lemma 2.2.1. *Let $u \in L^p_{loc}(Q)$ be a weak solution to* (2.1.1)–(2.1.2)*. Then, we have*

$$-\int_{\Sigma_Q^1} \frac{\partial \varphi}{\partial \nu_1} w(x)\, dS_x\, dt \le C \sum_{i=1}^{2} J_i(\varphi) \tag{2.2.3}$$

for all $\varphi \in \Phi$, provided $J_i(\varphi) < \infty$, $i = 1, 2$.

Proof. Let $u \in L^p_{\mathrm{loc}}(Q)$ be a weak solution to (2.1.1)–(2.1.2), and let $\varphi \in \Phi$ be such that $J_i(\varphi) < \infty$, $i = 1, 2$. By (2.1.8) (i.e., the definition of weak solution), we have

$$\int_Q |x|^\tau |u|^p \varphi\, dx\, dt - \int_{\Sigma_Q^1} \frac{\partial \varphi}{\partial \nu_1} w(x)\, dS_x\, dt \le \int_Q |u||\partial_t^k \varphi|\, dx\, dt + \int_Q |u| \left| L_\lambda \varphi \right|\, dx\, dt. \tag{2.2.4}$$

By means of Young's inequality, we deduce easily that

$$\begin{aligned}
\int_Q |u||\partial_t^k \varphi|\, dx\, dt &= \int_Q \left(|x|^{\frac{\tau}{p}} |u| \varphi^{\frac{1}{p}}\right)\left(|x|^{\frac{-\tau}{p}} \varphi^{\frac{-1}{p}} |\partial_t^k \varphi|\right) dx\, dt \\
&\le \frac{1}{2} \int_Q |x|^\tau |u|^p \varphi\, dx\, dt + C J_1(\varphi).
\end{aligned} \tag{2.2.5}$$

Similarly, we obtain the inequality

$$\int_Q |u| \left| L_\lambda \varphi \right|\, dx\, dt \le \frac{1}{2} \int_Q |x|^\tau |u|^p \varphi\, dx\, dt + C J_2(\varphi). \tag{2.2.6}$$

Using (2.2.5) and (2.2.6) in the right-hand side of the principal inequality (2.2.4), we conclude that the estimate (2.2.3) holds true. □

2.2.2 Test functions

As mentioned in Section 2.1 a crucial ingredient of our finding is the appropriate definition of test functions to manipulate our integral functionals. So, we first introduce the function H

defined in $\overline{\Omega}$ by

$$H(x) = x_N h(|x|) = x_N \begin{cases} |x|^{\mu}\left(1 - |x|^{-N-2\mu}\right) & \text{if } \lambda > -\frac{N^2}{4}, \\ |x|^{\frac{-N}{2}} \ln|x| & \text{if } \lambda = -\frac{N^2}{4}, \end{cases} \tag{2.2.7}$$

where the parameter μ is defined by (2.1.9). We collect below some useful properties satisfied by H.

Lemma 2.2.2. *The function H defined by* (2.2.7) *fulfils the following properties:*

(i) $H \geq 0$, $L_\lambda H = 0$ *in* Ω, $H_{|\Sigma_i} = 0$, $i = 0, 1$;

(ii) $R \gg 1$, $x \in \Omega$, $|x| < R \implies H(x) \leq x_N |x|^{\mu} \ln R$;

(iii) $R \gg 1$, $x \in \Omega$, $|x| > R \implies H(x) \geq C x_N |x|^{\mu}$;

(iv) $\dfrac{\partial H}{\partial \nu_0}(x) = -h(|x|)$, $x \in \Sigma_0$;

(v) $\dfrac{\partial H}{\partial \nu_1}(x) = -C x_N$, $x \in \Sigma_1$.

Proof. Property (i) follows from elementary calculations, hence we omit the details. Properties (ii) and (iii) follow directly from the definition of H. On the other hand, for all $x \in \Omega$, we have

$$\nabla H(x) = h(|x|) e_N + x_N \begin{cases} |x|^{\mu-2}\left(\mu + (N+\mu)|x|^{-N-2\mu}\right) x & \text{if } \quad \lambda > -\frac{N^2}{4}, \\ \left(\mu |x|^{\mu-2} \ln|x| + |x|^{\mu-2}\right) x & \text{if } \quad \lambda = -\frac{N^2}{4}, \end{cases} \tag{2.2.8}$$

where $e_N = (0, \cdots, 0, 1) \in \mathbb{R}^N$. Using (2.2.8), we obtain

$$\frac{\partial H}{\partial \nu_0}(x) = -\nabla H(x) \cdot e_N|_{x_N=0} = -h(|x|),$$

where $\cdot$ denotes the inner product in $\mathbb{R}^N$, which proves property (iv). Again, using (2.2.8), we get

$$\begin{aligned} \frac{\partial H}{\partial \nu_1}(x) &= -\nabla H(x)\cdot x|_{|x|=1} \\ &= -x_N h(|x|)|_{|x|=1} - x_N \begin{cases} 2\mu + N & \text{if} \quad \lambda > -\frac{N^2}{4}, \\ 1 & \text{if} \quad \lambda = -\frac{N^2}{4} \end{cases} \\ &= \begin{cases} -(2\mu+N)x_N & \text{if} \quad \lambda > -\frac{N^2}{4}, \\ -x_N & \text{if} \quad \lambda = -\frac{N^2}{4}, \end{cases} \end{aligned}$$

which proves property (v) (recall that $2\mu + N > 0$). This concludes the proof. □

Let $\xi, \vartheta, \iota \in C^\infty(\mathbb{R})$ be three cut-off functions satisfying the following requirements:

$$0 \le \xi \le 1, \quad \xi(s) = 1 \text{ if } |s| \le 1, \quad \xi(s) = 0 \text{ if } |s| \ge 2, \tag{2.2.9}$$

$$0 \le \vartheta \le 1, \quad \vartheta(s) = 1 \text{ if } s \le 0, \quad \vartheta(s) = 0 \text{ if } s \ge 1 \tag{2.2.10}$$

and

$$\iota \ge 0, \quad \operatorname{supp}(\iota) \subset\subset (0,1). \tag{2.2.11}$$

For $T, R, \ell \gg 1$, we consider the auxiliary functions:

$$\alpha_T(t) = \iota^\ell\left(\frac{t}{T}\right), \quad t > 0, \tag{2.2.12}$$

$$\beta_R(x) = H(x)\xi^\ell\left(\frac{|x|^2}{R^2}\right), \quad x \in \overline{\Omega}, \tag{2.2.13}$$

$$\gamma_R(x) = H(x)\vartheta^\ell\left(\frac{\ln\left(\frac{|x|}{\sqrt{R}}\right)}{\ln(\sqrt{R})}\right), \quad x \in \overline{\Omega}. \tag{2.2.14}$$

Combining the previous functions, we now introduce the following test functions:

$$\varphi(t,x) = \alpha_T(t)\beta_R(x), \quad (t,x) \in Q \tag{2.2.15}$$

and

$$\psi(t,x) = \alpha_T(t)\gamma_R(x), \quad (t,x) \in Q. \tag{2.2.16}$$

We note that test functions of the form (2.2.15) will be used in the proof of part (I) of Theorem 2.1.1. In the proof of Theorem 2.1.2, we will make use of test functions of the form (2.2.16). Next two preliminary results fix the regularities of our test functions.

Lemma 2.2.3. *The function φ defined by* (2.2.15) *belongs to* Φ*.*

Proof. By the definition of function φ, it can be easily seen that (A_1) and (A_2) are satisfied. On the other hand, by Lemma 2.2.2 (i), we have $\varphi_{|\Sigma_Q^i} = 0$, $i = 0, 1$, which shows that (A_3) is also satisfied. Furthermore, in view of (2.2.9) and (2.2.13), we have

$$\beta_R(x) = H(x), \quad x_N > 0,\ 1 < |x| \leq R,$$

which implies by Lemma 2.2.2 (v) and (2.2.15) that

$$\frac{\partial \beta_R}{\partial \nu_1}(x) = \frac{\partial H}{\partial \nu_1}(x) = -Cx_N, \quad x \in \Sigma_1$$

and

$$\frac{\partial \varphi}{\partial \nu_1}(t,x) = -Cx_N\alpha_T(t) \leq 0, \quad (t,x) \in \Sigma_Q^1. \tag{2.2.17}$$

This shows that φ satisfies (A_4) for $i = 1$. Finally, we have to show that (A_4) is satisfied for

$i = 0$. By (2.2.13), for all $x \in \Omega$, we have

$$\nabla \beta_R(x) = \xi^\ell\left(\frac{|x|^2}{R^2}\right)\nabla H(x) + H(x)\nabla\,\xi^\ell\left(\frac{|x|^2}{R^2}\right),$$

which implies by Lemma 2.2.2 (i), (iv) that

$$\frac{\partial \beta_R}{\partial \nu_0}(x) = \xi^\ell\left(\frac{|x|^2}{R^2}\right)\Big|_{x\in\Sigma_0}\frac{\partial H}{\partial \nu_0}(x) = -\left(h(|x|)\xi^\ell\left(\frac{|x|^2}{R^2}\right)\right)\Big|_{x\in\Sigma_0} \le 0.$$

Hence, by (2.2.15), we deduce that

$$\frac{\partial \varphi}{\partial \nu_0}(t,x) = \alpha_T(t)\frac{\partial \beta_R}{\partial \nu_0}(x) \le 0, \quad (t,x) \in \Sigma_Q^0,$$

which shows that φ satisfies (A_4) for $i = 0$. The proof of Lemma 2.2.3 is now completed. □

Lemma 2.2.4. *The function ψ defined by* (2.2.16) *belongs to* Φ.

Proof. By the definition of function ψ, and making use of Lemma 2.2.2 (i), it can be easily seen that (A_1), (A_2) and (A_3) are satisfied. On the other hand, by (2.2.10) and (2.2.14), we have

$$\gamma_R(x) = H(x), \quad x_N > 0,\ 1 < |x| \le \sqrt{R},$$

which implies by Lemma 2.2.2 (v) and (2.2.16) that

$$\frac{\partial \gamma_R}{\partial \nu_1}(x) = \frac{\partial H}{\partial \nu_1}(x) = -Cx_N, \quad x \in \Sigma_1$$

and

$$\frac{\partial \psi}{\partial \nu_1}(t,x) = -Cx_N\alpha_T(t) \le 0, \quad (t,x) \in \Sigma_Q^1. \tag{2.2.18}$$

This shows that ψ satisfies (A_4) for $i = 1$. Proceeding as in the proof of Lemma 2.2.3, we can show that (A_4) is also satisfied for $i = 0$. □

Turning to the function H defined by (2.2.7), we can provide some useful estimates too.

Lemma 2.2.5. *Let H be the function defined by* (2.2.7). *For $R < |x| < \sqrt{2}R$, $x_N > 0$, the following estimates hold:*

$$\left| H(x)\Delta\,\xi^{\ell}\left(\frac{|x|^2}{R^2}\right)\right| \leq CR^{-2}\ln R\, x_N|x|^{\mu}\xi^{\ell-2}\left(\frac{|x|^2}{R^2}\right), \tag{2.2.19}$$

$$\left| \nabla H(x)\cdot\nabla\,\xi^{\ell}\left(\frac{|x|^2}{R^2}\right)\right| \leq CR^{-2}\ln R\, x_N|x|^{\mu}\xi^{\ell-2}\left(\frac{|x|^2}{R^2}\right). \tag{2.2.20}$$

Proof. By (2.2.9) and invoking Lemma 2.2.2 (ii), for $R < |x| < \sqrt{2}R$, $x_N > 0$, we have

$$\left|\Delta\,\xi^{\ell}\left(\frac{|x|^2}{R^2}\right)\right| \leq CR^{-2}\xi^{\ell-2}\left(\frac{|x|^2}{R^2}\right),\;\; H(x) \leq Cx_N|x|^{\mu}\ln R,$$

which yields (2.2.19). On the other hand, by (2.2.8), for $R < |x| < \sqrt{2}R$, $x_N > 0$, and $\lambda > -\frac{N^2}{4}$, we have

$$\begin{aligned}
&\nabla H(x)\cdot\nabla\,\xi^{\ell}\left(\frac{|x|^2}{R^2}\right)\\
&= C\xi^{\ell-1}\left(\frac{|x|^2}{R^2}\right)\xi'\left(\frac{|x|^2}{R^2}\right)R^{-2}\left[h(|x|)e_N + x_N|x|^{\mu-2}\left(\mu + (N+\mu)|x|^{-N-2\mu}\right)x\right]\cdot x\\
&= C\xi^{\ell-1}\left(\frac{|x|^2}{R^2}\right)\xi'\left(\frac{|x|^2}{R^2}\right)R^{-2}\left[h(|x|)x_N + x_N|x|^{\mu}\left(\mu + (N+\mu)|x|^{-N-2\mu}\right)\right].
\end{aligned}$$

Then, by (2.2.9) and using that $N + 2\mu > 0$, we get

$$\begin{aligned}
&\left|\nabla H(x)\cdot\nabla\,\xi^{\ell}\left(\frac{|x|^2}{R^2}\right)\right| \\
&\le CR^{-2}\xi^{\ell-2}\left(\frac{|x|^2}{R^2}\right)x_N|x|^{\mu}\left[1+\left(|\mu|+(N+\mu)|x|^{-N-2\mu}\right)\right] \\
&\le CR^{-2}x_N|x|^{\mu}\xi^{\ell-2}\left(\frac{|x|^2}{R^2}\right) \\
&\le CR^{-2}x_N|x|^{\mu}\ln R\,\xi^{\ell-2}\left(\frac{|x|^2}{R^2}\right),
\end{aligned}$$

which proves (2.2.20). Similarly, by (2.2.8), for $R < |x| < \sqrt{2}R$, $x_N > 0$, and $\lambda = -\frac{N^2}{4}$, we have

$$\begin{aligned}
&\nabla H(x)\cdot\nabla\,\xi^{\ell}\left(\frac{|x|^2}{R^2}\right) \\
&= C\xi^{\ell-1}\left(\frac{|x|^2}{R^2}\right)\xi'\left(\frac{|x|^2}{R^2}\right)R^{-2}\left[h(|x|)e_N + x_N|x|^{\mu-2}\left(\mu\ln|x|+1\right)x\right]\cdot x \\
&= C\xi^{\ell-1}\left(\frac{|x|^2}{R^2}\right)\xi'\left(\frac{|x|^2}{R^2}\right)R^{-2}\left[h(|x|)x_N + x_N|x|^{\mu}\left(\mu\ln|x|+1\right)\right].
\end{aligned}$$

It follows by (2.2.9) that

$$\begin{aligned}
\left|\nabla H(x)\cdot\nabla\,\xi^{\ell}\left(\frac{|x|^2}{R^2}\right)\right| &\le CR^{-2}\xi^{\ell-2}\left(\frac{|x|^2}{R^2}\right)x_N|x|^{\mu}\left(2+|\mu|\ln|x|\right) \\
&\le CR^{-2}x_N|x|^{\mu}\ln R\,\xi^{\ell-2}\left(\frac{|x|^2}{R^2}\right),
\end{aligned}$$

which proves that the estimate (2.2.20) holds true. □

Using (2.2.8), (2.2.10) and similar calculations as above, we obtain the following estimates, to avoid repetition we omit the details.

Lemma 2.2.6. *Let H be the function defined by* (2.2.7). *For* $\sqrt{R} < |x| < R$, $x_N > 0$, *the following* *estimates hold:*

$$\begin{aligned}
\left| H(x)\Delta\,\vartheta^{\ell}\left(\frac{\ln\left(\frac{|x|}{\sqrt{R}}\right)}{\ln(\sqrt{R})}\right)\right| &\leq Cx_N(\ln R)^{-1}|x|^{\mu-2}\vartheta^{\ell-2}\left(\frac{\ln\left(\frac{|x|}{\sqrt{R}}\right)}{\ln(\sqrt{R})}\right),\\
\left| \nabla H(x)\cdot\nabla\,\vartheta^{\ell}\left(\frac{\ln\left(\frac{|x|}{\sqrt{R}}\right)}{\ln(\sqrt{R})}\right)\right| &\leq Cx_N(\ln R)^{-1}|x|^{\mu-2}\vartheta^{\ell-2}\left(\frac{\ln\left(\frac{|x|}{\sqrt{R}}\right)}{\ln(\sqrt{R})}\right).
\end{aligned}$$

2.2.3 Estimates of $J_i(\varphi)$

For $T, R, \ell \gg 1$, we shall estimate the terms $J_i(\varphi)$, $i = 1, 2$, defined by (2.2.1) and (2.2.2), where φ is the function given in (2.2.15).

Lemma 2.2.7. *The following estimate holds:*

$$\int_{supp\left(\frac{d^k\alpha_T}{dt^k}\right)} \alpha_T^{\frac{-1}{p-1}}(t)\left|\frac{d^k\alpha_T}{dt^k}(t)\right|^{\frac{p}{p-1}}\,dt \leq CT^{1-\frac{kp}{p-1}}. \tag{2.2.21}$$

Proof. By (2.2.11) and (2.2.12), we obtain

$$\begin{aligned}
\int_{\mathrm{supp}\left(\frac{d^k\alpha_T}{dt^k}\right)} \alpha_T^{\frac{-1}{p-1}}(t)\left|\frac{d^k\alpha_T}{dt^k}(t)\right|^{\frac{p}{p-1}}\,dt &= \int_0^T \iota^{\frac{-\ell}{p-1}}\left(\frac{t}{T}\right)\left|\frac{d^k}{dt^k}\left[\iota^{\ell}\left(\frac{t}{T}\right)\right]\right|^{\frac{p}{p-1}}\,dt\\
&\leq CT^{\frac{-kp}{p-1}}\int_0^T \iota^{\frac{-\ell}{p-1}}\left(\frac{t}{T}\right)\iota^{\frac{(\ell-k)p}{p-1}}\left(\frac{t}{T}\right)dt\\
&= CT^{1-\frac{kp}{p-1}}\int_0^1 \iota^{\ell-\frac{kp}{p-1}}(s)\,ds,
\end{aligned}$$

which yields (2.2.21). ☐

Lemma 2.2.8. *The following estimate holds:*

$$\int_{\boldsymbol{supp}(\beta_R)} |x|^{\frac{-\tau}{p-1}}\beta_R(x)\,dx \leq C\ln R\left(\ln R + R^{\frac{(\mu+N+1)p-\mu-N-1-\tau}{p-1}}\right). \tag{2.2.22}$$

Proof. By (2.2.9) and (2.2.13), we obtain

$$\begin{aligned}\int_{\mathrm{supp}(\beta_R)} |x|^{\frac{-\tau}{p-1}} \beta_R(x)\,dx &= \int_{1<|x|<\sqrt{2}R,\,x_N>0} |x|^{\frac{-\tau}{p-1}} H(x)\xi^{\ell}\left(\frac{|x|^2}{R^2}\right) dx\\ &\leq \int_{1<|x|<\sqrt{2}R,\,x_N>0} |x|^{\frac{-\tau}{p-1}} H(x)\,dx.\end{aligned}$$

Then, making use of Lemma 2.2.2 (ii), we deduce that

$$\begin{aligned}&\int_{\mathrm{supp}(\beta_R)} |x|^{\frac{-\tau}{p-1}} \beta_R(x)\,dx\\ &\leq \ln R \int_{1<|x|<\sqrt{2}R,\,x_N>0} |x|^{\mu-\frac{\tau}{p-1}} x_N\,dx\\ &\leq \ln R \int_{1<|x|<\sqrt{2}R} |x|^{\mu+1-\frac{\tau}{p-1}}\,dx\\ &= C\ln R \int_{r=1}^{\sqrt{2}R} r^{\mu+N-\frac{\tau}{p-1}}\,dr\\ &= C\ln R \begin{cases} \ln R & \text{if} \quad \tau = (\mu+N+1)(p-1),\\ 1 & \text{if} \quad \tau > (\mu+N+1)(p-1),\\ R^{\frac{(\mu+N+1)p-\mu-N-1-\tau}{p-1}} & \text{if} \quad \tau < (\mu+N+1)(p-1),\end{cases}\end{aligned}$$

which proves (2.2.22). □

From (2.2.1), (2.2.15), Lemmas 2.2.7 and 2.2.8, we deduce the following result.

Lemma 2.2.9. *The following estimate holds:*

$$J_1(\varphi) \leq CT^{1-\frac{kp}{p-1}} \ln R \left(\ln R + R^{\frac{(\mu+N+1)p-\mu-N-1-\tau}{p-1}}\right).$$

We now prove the following result.

Lemma 2.2.10. *The following estimate holds:*

$$\int_{supp(L_\lambda\beta_R)} |x|^{\frac{-\tau}{p-1}} \beta_R^{\frac{-1}{p-1}}(x) \left|L_\lambda\beta_R(x)\right|^{\frac{p}{p-1}} \, dx \le CR^{\frac{(N+\mu-1)p-N-1-\mu-\tau}{p-1}} (\ln R)^{\frac{p}{p-1}}. \tag{2.2.23}$$

Proof. By (2.2.13), for all $x \in \operatorname{supp}(\beta_R)$, we have

$$\begin{aligned}
L_\lambda\beta_R(x) &= -\Delta\left(H(x)\xi^\ell\left(\frac{|x|^2}{R^2}\right)\right) + \frac{\lambda}{|x|^2}H(x)\xi^\ell\left(\frac{|x|^2}{R^2}\right)\\
&= -\xi^\ell\left(\frac{|x|^2}{R^2}\right)\Delta H(x) - H(x)\Delta\,\xi^\ell\left(\frac{|x|^2}{R^2}\right) - 2\nabla H(x)\cdot\nabla\,\xi^\ell\left(\frac{|x|^2}{R^2}\right)\\
&\quad + \frac{\lambda}{|x|^2}H(x)\xi^\ell\left(\frac{|x|^2}{R^2}\right)\\
&= \xi^\ell\left(\frac{|x|^2}{R^2}\right)L_\lambda H(x) - H(x)\Delta\,\xi^\ell\left(\frac{|x|^2}{R^2}\right) - 2\nabla H(x)\cdot\nabla\,\xi^\ell\left(\frac{|x|^2}{R^2}\right),
\end{aligned}$$

which implies by Lemma 2.2.2 (i) and (2.2.9) that

$$L_\lambda\beta_R(x) = -H(x)\Delta\,\xi^\ell\left(\frac{|x|^2}{R^2}\right) - 2\nabla H(x)\cdot\nabla\,\xi^\ell\left(\frac{|x|^2}{R^2}\right) \tag{2.2.24}$$

and

$$\begin{aligned}
&\int_{\operatorname{supp}(L_\lambda\beta_R)} |x|^{\frac{-\tau}{p-1}} \beta_R^{\frac{-1}{p-1}}(x) \left|L_\lambda\beta_R(x)\right|^{\frac{p}{p-1}} \, dx\\
&= \int_{R<|x|<\sqrt{2}R,\, x_N>0} |x|^{\frac{-\tau}{p-1}} \beta_R^{\frac{-1}{p-1}}(x) \left|L_\lambda\beta_R(x)\right|^{\frac{p}{p-1}} \, dx.
\end{aligned} \tag{2.2.25}$$

On the other hand, by Lemma 2.2.5, for $R < |x| < \sqrt{2}R$, $x_N > 0$, we have

$$\left|L_\lambda\beta_R(x)\right| \le CR^{-2}\ln R\, x_N|x|^\mu\xi^{\ell-2}\left(\frac{|x|^2}{R^2}\right),$$

which yields

$$\left|L_\lambda\beta_R(x)\right|^{\frac{p}{p-1}} \le CR^{-\frac{2p}{p-1}}(\ln R)^{\frac{p}{p-1}} x_N^{\frac{p}{p-1}} |x|^{\frac{\mu p}{p-1}} \xi^{\frac{(\ell-2)p}{p-1}} \left(\frac{|x|^2}{R^2}\right). \tag{2.2.26}$$

Furthermore, by (2.2.13) and Lemma 2.2.2 (iii), for $R < |x| < \sqrt{2}R$, we have

$$\begin{aligned} \beta_R^{\frac{-1}{p-1}}(x) &= H^{\frac{-1}{p-1}}(x)\xi^{\frac{-\ell}{p-1}}\left(\frac{|x|^2}{R^2}\right) \\ &\leq Cx_N^{\frac{-1}{p-1}}|x|^{\frac{-\mu}{p-1}}\xi^{\frac{-\ell}{p-1}}\left(\frac{|x|^2}{R^2}\right). \end{aligned} \tag{2.2.27}$$

Thus, in view of (2.2.25), (2.2.26) and (2.2.27), we obtain

$$\begin{aligned} &\int_{\mathrm{supp}(L_\lambda\beta_R)} |x|^{\frac{-\tau}{p-1}}\beta_R^{\frac{-1}{p-1}}(x)\left|L_\lambda\beta_R(x)\right|^{\frac{p}{p-1}}\,dx \\ &\leq CR^{-\frac{2p}{p-1}}(\ln R)^{\frac{p}{p-1}}\int_{R<|x|<\sqrt{2}R,\,x_N>0}|x|^{\frac{-(\tau+\mu)+\mu p}{p-1}}x_N\xi^{\ell-\frac{2p}{p-1}}\left(\frac{|x|^2}{R^2}\right)dx \\ &\leq CR^{-\frac{2p}{p-1}}(\ln R)^{\frac{p}{p-1}}\int_{R<|x|<\sqrt{2}R,\,x_N>0}|x|^{\frac{p-1-(\tau+\mu)+\mu p}{p-1}}\,dx \\ &\leq CR^{-\frac{2p}{p-1}}(\ln R)^{\frac{p}{p-1}}R^{\frac{p-1-(\tau+\mu)+\mu p}{p-1}}R^N \\ &= CR^{\frac{(N+\mu-1)p-N-1-\mu-\tau}{p-1}}(\ln R)^{\frac{p}{p-1}}, \end{aligned}$$

which proves (2.2.23). □

Lemma 2.2.11. *The following estimate holds:*

$$J_2(\varphi) \leq CTR^{\frac{(N+\mu-1)p-N-1-\mu-\tau}{p-1}}(\ln R)^{\frac{p}{p-1}}. \tag{2.2.28}$$

Proof. By (2.2.2), for the test function given in (2.2.15), we have

$$J_2(\varphi) = \left(\int_{\mathrm{supp}(\alpha_T)}\alpha_T(t)\,dt\right)\left(\int_{\mathrm{supp}(L_\lambda\beta_R)}|x|^{\frac{-\tau}{p-1}}\beta_R^{\frac{-1}{p-1}}\left|L_\lambda\beta_R\right|^{\frac{p}{p-1}}\,dx\right). \tag{2.2.29}$$

On the other hand, by (2.2.11) and (2.2.12), we have

$$\begin{aligned}\int_{\mathrm{supp}(\alpha_T)} \alpha_T(t)\,dt &= \int_0^T \iota\left(\frac{t}{T}\right)^{\ell}\,dt \\ &= T\int_0^1 \iota(s)^{\ell}\,ds. \end{aligned} \tag{2.2.30}$$

Hence, by Lemma 2.2.10, (2.2.29) and (2.2.30), we deduce the inequality (2.2.28). □

2.2.4 Estimates of $J_i(\psi)$ in the critical case

In this subsection, for $\lambda > \frac{-N^2}{4}$ and $T, R, \ell \gg 1$, we shall estimate the terms $J_i(\psi)$, $i = 1, 2$, defined by (2.2.1) and (2.2.2), in the critical case $(N+\mu-1)p = N+\mu+1+\tau$ (see Theorem 2.1.2), where ψ is the function defined by (2.2.16).

The proof of the following lemma is similar to that of Lemma 2.2.8, so we omit the details.

Lemma 2.2.12. *Let $\lambda > -\frac{N^2}{4}$ and $(N+\mu-1)p = N+\mu+1+\tau$. The following estimate holds:*

$$\int_{supp(\gamma_R)} |x|^{\frac{-\tau}{p-1}}\gamma_R(x)\,dx \le C\left(\ln R + R^{\frac{2p}{p-1}}\right).$$

Using (2.2.1), (2.2.16), Lemmas 2.2.7 and 2.2.12, we deduce the following estimate.

Lemma 2.2.13. *Let $\lambda > -\frac{N^2}{4}$ and $(N+\mu-1)p = N+\mu+1+\tau$. The following estimate holds:*

$$J_1(\psi) \le C T^{1-\frac{kp}{p-1}}\left(\ln R + R^{\frac{2p}{p-1}}\right).$$

We now prove the following result.

Lemma 2.2.14. *Let $\lambda > -\frac{N^2}{4}$ and $(N+\mu-1)p = N+\mu+1+\tau$. The following estimate holds:*

$$\int_{supp(L_\lambda\gamma_R)} |x|^{\frac{-\tau}{p-1}}\gamma_R^{\frac{-1}{p-1}}(x)\left|L_\lambda\gamma_R(x)\right|^{\frac{p}{p-1}}\,dx \le C(\ln R)^{\frac{-1}{p-1}}. \tag{2.2.31}$$

Proof. By (2.2.14), Lemma 2.2.2 (i), and following the proof of Lemma 2.2.10, for all $x \in \mathrm{supp}(\gamma_R)$, we obtain

$$L_\lambda \gamma_R(x) = -H(x)\Delta\,\vartheta^\ell\left(\frac{\ln\left(\frac{|x|}{\sqrt{R}}\right)}{\ln(\sqrt{R})}\right) - 2\nabla H(x)\cdot\nabla\,\vartheta^\ell\left(\frac{\ln\left(\frac{|x|}{\sqrt{R}}\right)}{\ln(\sqrt{R})}\right), \tag{2.2.32}$$

which implies by (2.2.10) that

$$\begin{aligned}&\int_{\mathrm{supp}(L_\lambda\gamma_R)} |x|^{\frac{-\tau}{p-1}}\gamma_R^{\frac{-1}{p-1}}(x)\left|L_\lambda\gamma_R(x)\right|^{\frac{p}{p-1}}\,dx\\ &= \int_{\sqrt{R}<|x|<R,\,x_N>0} |x|^{\frac{-\tau}{p-1}}\gamma_R^{\frac{-1}{p-1}}(x)\left|L_\lambda\gamma_R(x)\right|^{\frac{p}{p-1}}\,dx.\end{aligned} \tag{2.2.33}$$

On the other hand, by (2.2.32) and Lemma 2.2.6, for $\sqrt{R} < |x| < R$, $x_N > 0$, we have

$$\left|L_\lambda\gamma_R(x)\right| \le Cx_N(\ln R)^{-1}|x|^{\mu-2}\vartheta^{\ell-2}\left(\frac{\ln\left(\frac{|x|}{\sqrt{R}}\right)}{\ln(\sqrt{R})}\right),$$

which yields

$$\left|L_\lambda\gamma_R(x)\right|^{\frac{p}{p-1}} \le Cx_N^{\frac{p}{p-1}}(\ln R)^{-\frac{p}{p-1}}|x|^{\frac{(\mu-2)p}{p-1}}\vartheta^{\frac{(\ell-2)p}{p-1}}\left(\frac{\ln\left(\frac{|x|}{\sqrt{R}}\right)}{\ln(\sqrt{R})}\right). \tag{2.2.34}$$

Furthermore, by (2.2.14) and Lemma 2.2.2 (iii), for $\sqrt{R} < |x| < R$, $x_N > 0$, we have

$$\gamma_R^{\frac{-1}{p-1}}(x) \le Cx_N^{-\frac{1}{p-1}}|x|^{-\frac{\mu}{p-1}}\vartheta^{-\frac{\ell}{p-1}}\left(\frac{\ln\left(\frac{|x|}{\sqrt{R}}\right)}{\ln(\sqrt{R})}\right). \tag{2.2.35}$$

Hence, in view of (2.2.10), (2.2.33), (2.2.34) and (2.2.35), we get

$$\begin{aligned}
&\int_{\mathrm{supp}(L_\lambda \gamma_R)} |x|^{\frac{-\tau}{p-1}} \gamma_R^{\frac{-1}{p-1}}(x) \left|L_\lambda \gamma_R(x)\right|^{\frac{p}{p-1}}\,dx \\
&\leq C(\ln R)^{\frac{-p}{p-1}} \int_{\sqrt{R}<|x|<R,\, x_N>0} |x|^{\frac{(\mu-2)p-\tau-\mu}{p-1}} x_N \vartheta^{\ell-\frac{2p}{p-1}} \left(\frac{\ln\left(\frac{|x|}{\sqrt{R}}\right)}{\ln(\sqrt{R})}\right) dx \\
&\leq C(\ln R)^{\frac{-p}{p-1}} \int_{\sqrt{R}<|x|<R} |x|^{\frac{(\mu-1)p-\tau-\mu-1}{p-1}}\,dx \\
&= C(\ln R)^{\frac{-p}{p-1}} \int_{r=\sqrt{R}}^{R} r^{\frac{(N+\mu-1)p-N-\mu-1-\tau}{p-1}} r^{-1}\,dr.
\end{aligned}$$

Since $(N+\mu-1)p = N+\mu+1+\tau$, the above estimate yields

$$\begin{aligned}
\int_{\mathrm{supp}(L_\lambda \gamma_R)} |x|^{\frac{-\tau}{p-1}} \gamma_R^{\frac{-1}{p-1}}(x) \left|L_\lambda \gamma_R(x)\right|^{\frac{p}{p-1}}\,dx &\leq C(\ln R)^{\frac{-p}{p-1}} \int_{r=\sqrt{R}}^{R} r^{-1}\,dr \\
&= C(\ln R)^{\frac{-1}{p-1}},
\end{aligned}$$

which proves (2.2.31). This concludes the proof. □

Using (2.2.2), (2.2.16), (2.2.30) and Lemma 2.2.14, we obtain the following estimate.

Lemma 2.2.15. *Let $\lambda > -\frac{N^2}{4}$ and $(N+\mu-1)p = N+\mu+1+\tau$. The following estimate holds:*

$$J_2(\psi) \leq CT(\ln R)^{\frac{-1}{p-1}}.$$

2.3 Proofs of the main results

We first establish the nonexistence results given by part (I) of Theorem 2.1.1, and Theorem 2.1.2. Next, we willestablish the existence result given by part (II) of Theorem 2.1.1.

2.3.1 Proof of Theorem 2.1.1 (I)

We use the contradiction argument. We suppose that $u \in L^p_{\text{loc}}(Q)$ is a weak solution to (2.1.1)–(2.1.2). Then, by Lemma 2.2.1, (2.2.3) holds for all $\varphi \in \Phi$ (with $J_i(\varphi) < \infty$, $i = 1, 2$). Hence, from Lemma 2.2.3, we deduce that for $T, R, \ell \gg 1$,

$$-\int_{\Sigma_Q^1} \frac{\partial \varphi}{\partial \nu_1} w(x)\, dS_x\, dt \le C \sum_{i=1}^{2} J_i(\varphi), \tag{2.3.1}$$

where φ is the function given by (2.2.15). On the other hand, by (2.2.17) and (2.2.30), we have

$$\begin{aligned}
-\int_{\Sigma_Q^1} \frac{\partial \varphi}{\partial \nu_1} w(x)\, dS_x\, dt &= C \int_0^\infty \int_{\Sigma_1} w(x) x_N \alpha_T(t)\, dS_x\, dt \\
&= C \left(\int_0^\infty \alpha_T(t)\, dt \right) \left(\int_{\Sigma_1} w(x) x_N\, dS_x \right) \\
&= CT \int_{\Sigma_1} w(x) x_N\, dS_x \\
&= CTI_w.
\end{aligned} \tag{2.3.2}$$

Hence, by (2.3.1), (2.3.2), Lemmas 2.2.9 and 2.2.11, we obtain

$$CTI_w \le T^{1-\frac{kp}{p-1}} \ln R \left(\ln R + R^{\frac{(\mu+N+1)p-\mu-N-1-\tau}{p-1}} \right) + TR^{\frac{(N+\mu-1)p-N-1-\mu-\tau}{p-1}} (\ln R)^{\frac{p}{p-1}},$$

that is,

$$I_w \le C \left(T^{-\frac{kp}{p-1}} (\ln R)^2 + T^{-\frac{kp}{p-1}} R^a \ln R + R^b (\ln R)^{\frac{p}{p-1}} \right), \tag{2.3.3}$$

where

$$a = \frac{(\mu+N+1)p - (\mu+N+1+\tau)}{p-1}$$

and

$$b = \frac{(N+\mu-1)p-(N+\mu+1+\tau)}{p-1}.$$

Taking $T = R^{\theta}$, where

$$\theta > \max\left\{\frac{a(p-1)}{kp}, 0\right\}, \tag{2.3.4}$$

the estimate (2.3.3) reduces to

$$I_w \le C\left(R^{\frac{-\theta kp}{p-1}}(\ln R)^2 + R^{a-\frac{\theta kp}{p-1}}\ln R + R^b(\ln R)^{\frac{p}{p-1}}\right). \tag{2.3.5}$$

Notice that from the choice (2.3.4) of the parameter θ, one has $a - \frac{\theta kp}{p-1} < 0$. Moreover, due to (2.1.10), one has $b < 0$. Hence, passing to the limit as $R \to \infty$ in (2.3.5), we obtain $I_w \le 0$, which contradicts the condition $w \in L^{1,+}(\Sigma_1)$. Consequently, (2.1.1)–(2.1.2) admits no weak solution. This completes the proof of Theorem 2.1.1 (I). □

2.3.2 Proof of Theorem 2.1.2

We also use the contradiction argument by supposing that $u \in L^p_{\text{loc}}(Q)$ is a weak solution to (2.1.1)–(2.1.2). Then, from Lemmas 2.2.1 and 2.2.4, we deduce that for $T, R, \ell \gg 1$,

$$-\int_{\Sigma_Q^1} \frac{\partial \psi}{\partial \nu_1} w(x)\, dS_x\, dt \le C \sum_{i=1}^{2} J_i(\psi), \tag{2.3.6}$$

where ψ is the function given by (2.2.16). On the other hand, by (2.2.18) and (2.2.30), we obtain

$$-\int_{\Sigma_Q^1} \frac{\partial \psi}{\partial \nu_1} w(x)\, dS_x\, dt = CTI_w. \tag{2.3.7}$$

Hence, making use of (2.3.6), (2.3.7), Lemmas 2.2.13 and 2.2.15, we obtain

$$TI_w \leq C\left[T^{1-\frac{kp}{p-1}}\left(\ln R + R^{\frac{2p}{p-1}}\right) + T(\ln R)^{\frac{-1}{p-1}}\right],$$

that is,

$$I_w \leq C\left(T^{-\frac{kp}{p-1}}\ln R + T^{-\frac{kp}{p-1}}R^{\frac{2p}{p-1}} + (\ln R)^{\frac{-1}{p-1}}\right). \tag{2.3.8}$$

Thus, taking $T = R^{\theta}$, where $\theta > \frac{2}{k}$, and passing to the limit as $R \to \infty$ in (2.3.8), we obtain a contradiction with $w \in L^{1,+}(\Sigma_1)$. □

We now prove our existence result.

2.3.3 Proof of Theorem 2.1.1 (II)

We consider separately two cases depending on the value of parameter λ.

(i) The case $\lambda > -\frac{N^2}{4}$.

For δ and ϵ satisfying respectively

$$\max\left\{-\mu, \frac{\tau + p + 1}{p - 1}\right\} < \delta < \mu + N \tag{2.3.9}$$

and

$$0 < \epsilon < \left(-\delta^2 + N\delta + \lambda\right)^{\frac{1}{p-1}}, \tag{2.3.10}$$

let

$$u_{\delta,\epsilon}(x) = \epsilon x_N |x|^{-\delta}, \quad x \in \Omega. \tag{2.3.11}$$

Notice that by (2.1.9), since $\lambda > -\frac{N^2}{4}$, one has $-\mu < \mu + N$. Moreover, due to (2.1.11), one has

$$\frac{\tau + p + 1}{p - 1} < \mu + N.$$

Hence, the set of values δ satisfying (2.3.9) is nonempty. On the other hand, observe that $-\mu$ and $\mu + N$ are the roots of the polynomial function

$$P(\delta) = -\delta^2 + N\delta + \lambda,$$

which implies that $P(\delta) > 0$ for any δ satisfying (2.3.9), so $P(\delta)^{\frac{1}{p-1}}$ is well-defined, and the set of ϵ satisfying (2.3.10) is nonempty. Elementary calculations show that

$$L_\lambda u_{\delta,\epsilon}(x) = \epsilon P(\delta) x_N |x|^{-\delta-2}, \quad x \in \Omega. \tag{2.3.12}$$

Then, in view of (2.3.9), (2.3.10), (2.3.11) and (2.3.12), for all $x \in \Omega$, we obtain

$$\begin{aligned} L_\lambda u_{\delta,\epsilon}(x) &\geq \epsilon \epsilon^{p-1} x_N |x|^{-\delta-2} \\ &= \left(|x|^\tau \epsilon^p x_N^p |x|^{-\delta p}\right)\left(x_N^{1-p} |x|^{-\delta-2+\delta p-\tau}\right) \\ &\geq |x|^\tau u_{\delta,\epsilon}^p(x) |x|^{\delta(p-1)-(\tau+p+1)} \\ &\geq |x|^\tau u_{\delta,\epsilon}^p(x), \end{aligned}$$

which shows that for any δ and ϵ satisfying respectively (2.3.9) and (2.3.10), functions of the form (2.3.11) are stationary solutions to (2.1.1)–(2.1.2) with

$$w(x) = \epsilon x_N, \quad x \in \Sigma_1.$$

We next study the second case.

(ii) The case $\lambda = -\frac{N^2}{4}$.

For

$$0 < \kappa < 1, \ \rho > 1, \ \varepsilon > 0, \tag{2.3.13}$$

we consider functions of the form

$$u_{\kappa,\rho,\varepsilon}(x) = \varepsilon x_N |x|^{\mu} [\ln(\rho|x|)]^{\kappa}, \quad x \in \Omega. \tag{2.3.14}$$

Taking into consideration that $\lambda = -\frac{N^2}{4}$ (so $\mu = -\frac{N}{2}$), elementary calculations show that

$$L_\lambda u_{\kappa,\rho,\varepsilon}(x) = \varepsilon\kappa(1-\kappa) x_N |x|^{\mu-2} [\ln(\rho|x|)]^{\kappa-2}, \quad x \in \Omega. \tag{2.3.15}$$

In view of (2.3.13), (2.3.14) and (2.3.15), for all $x \in \Omega$, we obtain

$$\begin{aligned} L_\lambda u_{\kappa,\rho,\varepsilon}(x) &= |x|^{\tau} u^p_{\kappa,\rho,\varepsilon}(x) \left(\varepsilon^{1-p} \kappa(1-\kappa) x_N^{1-p} |x|^{\mu-2-\mu p-\tau} [\ln(\rho|x|)]^{\kappa-2-\kappa p} \right) \\ &\geq |x|^{\tau} u^p_{\kappa,\rho,\varepsilon}(x) \left(\varepsilon^{1-p} \kappa(1-\kappa) |x|^{\zeta} [\ln(\rho|x|)]^{\kappa-2-\kappa p} \right), \end{aligned} \tag{2.3.16}$$

where

$$\zeta = (-\mu-1)p - (-\mu+1+\tau) = (N+\mu-1)p - (N+\mu+1+\tau).$$

Notice that due to (2.1.11), one has $\zeta > 0$, which yields

$$\lim_{s\to+\infty} \kappa(1-\kappa) s^{\zeta} [\ln(\rho s)]^{\kappa-2-\kappa p} = +\infty.$$

Consequently, there exists a constant $A > 0$ (independent on x) such that

$$\kappa(1-\kappa)|x|^{\zeta}[\ln(\rho|x|)]^{\kappa-2-\kappa p} \geq A, \quad x \in \Omega. \tag{2.3.17}$$

Thus, taking

$$0 < \varepsilon < A^{\frac{1}{p-1}}, \tag{2.3.18}$$

using (2.3.16) and (2.3.17), we obtain

$$L_\lambda u_{\kappa,\rho,\varepsilon}(x) \geq |x|^{\tau} u^{p}_{\kappa,\rho,\varepsilon}(x), \quad x \in \Omega,$$

which shows that for any κ, ρ and ε satisfying (2.3.13) and (2.3.18), functions of the form (2.3.14) are stationary solutions to (2.1.1)–(2.1.2) with

$$w(x) = \varepsilon x_N (\ln \rho)^{\kappa}, \quad x \in \Sigma_1.$$

This completes the proof of part (II) of Theorem 2.1.1. □

Chapter

3

A hyperbolic polyharmonic system in an exterior domain

The result of this chapter is published in: Manal Alfulaij, Mohamed Jleli and Bessem Samet. A hyperbolic polyharmonic system in an exterior domain. AIMS Mathematics 10.2 (2025): 2634-2651.

A nonlinear hyperbolic polyharmonic system in an exterior domain of $\mathbb{R}^N$ is considered under inhomogenous Navier-type boundary conditions. Using nonlinear estimates specifically adapted to the polyharmonic $(-\Delta)^m$, the geometry of the domains, and the boundary conditions, a sharp criterium for the nonexistence of weak solutions is obtained. Next, an optimal nonexistence result for the corresponding stationary problem is deduced.

3.1 Introduction

In this chapter, we study the questions of existence and nonexistence of weak solutions to the system of polyharmonic wave inequalities.

$$\begin{cases} u_{tt} + (-\Delta)^m u \geq |x|^a |v|^p, & (t,x) \in (0,\infty) \times \mathbb{R}^N \backslash \overline{B_1}, \\ v_{tt} + (-\Delta)^m v \geq |x|^b |u|^q, & (t,x) \in (0,\infty) \times \mathbb{R}^N \backslash \overline{B_1}. \end{cases} \tag{3.1.1}$$

Here, $(u,v) = (u(t,x), v(t,x))$, $N \geq 2$, B_1 is the open unit ball of $\mathbb{R}^N$, $m \geq 1$ is an integer, $a, b \geq -2m$, $(a,b) \neq (-2m,-2m)$, and $p, q > 1$. We will investigate (3.1.1) under the Navier-type boundary conditions

$$\begin{cases} (-\Delta)^i u \geq f_i(x), \ i = 0, \cdots, m-1, \ (t,x) \in (0,\infty) \times \partial B_1, \\ (-\Delta)^i v \geq g_i(x), \ i = 0, \cdots, m-1, \ (t,x) \in (0,\infty) \times \partial B_1, \end{cases} \tag{3.1.2}$$

where $f_i, g_i \in L^1(\partial B_1)$ and $(-\Delta)^0$ is the identity operator. Notice that no restriction on the signs of f_i or g_i is imposed.

The study of semilinear wave inequalities in $\mathbb{R}^N$ was firstly considered by Kato [10] and

Pohozaev & Véron [49]. It was shown that the problem

$$u_{tt} - \Delta u \geq |u|^p, \quad (t, x) \in (0, \infty) \times \mathbb{R}^N \tag{3.1.3}$$

possesses a critical exponent $p_K = \frac{N+1}{N-1}$ in the following sense:

(i) If $N \geq 2$ and $1 < p \leq p_K$, then (3.1.3) possesses no global weak solution, provided

$$\int_{\mathbb{R}^N} u_t(0, x)\, dx > 0. \tag{3.1.4}$$

(ii) If $p > p_K$, there are global positive solutions satisfying (3.1.4).

Caristi [53] studied the higher-order evolution polyharmonic inequality

$$\frac{\partial^j u}{\partial t^j} - |x|^\alpha \Delta^m u \geq |u|^p, \quad (t, x) \in (0, \infty) \times \mathbb{R}^N, \tag{3.1.5}$$

where $\alpha \leq 2m$. Caristi discussed separately the cases $\alpha = 2m$ and $\alpha < 2m$. For instance, in the hyperbolic case $j = 2$ and $\alpha = 0$, it was shown that, if $N \geq m + 1$ and $1 < p \leq \frac{N+m}{N-m}$, then (3.1.5) possesses no global weak solution, provided (3.1.4) holds. Other existence and nonexistence results for evolution inequalities involving the polyharmonic operator in the whole space can be found in [50, 48, 13].

The study of the blow-up for semilinear wave equations in exterior domains was firstly considered by Zhang [17]. Namely, among many other problems, Zhang investigated the equation

$$u_{tt} - \Delta u = |x|^a |u|^p, \quad (t, x) \in (0, \infty) \times \mathbb{R}^N \backslash D, \tag{3.1.6}$$

where $N \geq 3$, $a > -2$, and D is a smooth bounded subset of $\mathbb{R}^N$. It was shown that (3.1.6) under the Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = f(x) \geq 0, \quad (t,x) \in (0,\infty) \times \partial D,$$

admits a critical exponent $\frac{N+a}{N-2}$ in the following sense:

(i) If $1 < p < \frac{N+a}{N-2}$, then (3.1.6) admits no global solution, provided $f \not\equiv 0$.

(ii) If $p > \frac{N+a}{N-2}$, then (3.1.6) admits global solutions for some $f > 0$.

In [4, 5], it was shown that the critical value $p = \frac{N+a}{N-2}$ belongs to the case (i). Furthermore, the same result holds true, if (3.1.6) is considered under the Dirichlet boundary condition

$$u = f(x) \geq 0, \quad (t,x) \in (0,\infty) \times \partial D,$$

where $D = \overline{B_1}$.

In [8], the authors considered the system of wave inequalities (3.1.1) in the case $m = 1$. The system was studied under different types of inhomogeneous boundary conditions. In particular, under the boundary conditions (3.1.2) with $m = 1$ (Dirichlet-type boundary conditions), the authors obtained the following result: Assume that $a, b \geq -2$, $(a,b) \neq (-2,-2)$, $I_{f_0} := \int_{\partial B_1} f_0 \, dS_x \geq 0$, $I_{g_0} := \int_{\partial B_1} g_0 \, dS_x \geq 0$, $(I_{f_0}, I_{g_0}) \neq (0,0)$, and $p, q > 1$. If $N = 2$; or $N \geq 3$ and

$$N < \max\left\{\operatorname{sgn}(I_{f_0}) \times \frac{2p(q+1)+pb+a}{pq-1}, \operatorname{sgn}(I_{g_0}) \times \frac{2q(p+1)+qa+b}{pq-1}\right\},$$

then (3.1.1)-(3.1.2) (with $m = 1$) admits no weak solution. Moreover, the authors pointed out the sharpness of the above condition.

In the case $m = 2$, the system (3.1.1) was recently studied in [6] under different types of boundary conditions. In particular, under the boundary conditions (3.1.2) with $f_0 \equiv 0$ and

$g_0 \equiv 0$, i.e.,

$$\begin{cases} u \geq 0, \ -\Delta u \geq f_1(x), \quad (t,x) \in (0,\infty) \times \partial B_1, \\ v \geq 0, \ -\Delta v \geq g_1(x), \quad (t,x) \in (0,\infty) \times \partial B_1. \end{cases} \tag{3.1.7}$$

Namely, the following result was obtained: Let $N \geq 2$, $a, b \geq -4$, $(a,b) \neq (-4,-4)$, $\int_{\partial B_1} f_1 \, dS_x > 0$, $\int_{\partial B_1} g_1 \, dS_x > 0$, and $p, q > 1$. If $N \in \{2,3,4\}$; or

$$N \geq 5, \ N < \max\left\{ \frac{4p(q+1)+pb+a}{pq-1}, \frac{4q(p+1)+qa+b}{pq-1} \right\},$$

then (3.1.1) (with $m = 2$) under the boundary conditions (3.1.7) admits no weak solution. Moreover, it was shown that the above condition is sharp.

Further results related to the existence and nonexistence of solutions for evolution problems in exterior domains can be found in [46, 40, 51, 50, 52, 25].

The present work aims to extend the obtained results in [6, 8] from $m \in \{1,2\}$ to an arbitrary $m \geq 1$. Before presenting our main results, we need to define weak solutions to the considered problem.

Let

$$Q = (0,\infty) \times \mathbb{R}^N \backslash B_1, \quad \Sigma_Q = (0,\infty) \times \partial B_1.$$

Notice that $\Sigma_Q \subset Q$.

Definition 3.1.1. We say that φ is an admissible test function, if

(i) $\varphi \in C^{2,2m}_{t,x}(Q)$;

(ii) $\mathrm{supp}(\varphi) \subset\subset Q$ (φ is compactly supported in Q);

(iii) $\varphi \geq 0$;

(iv) For all $j = 0, 1, \cdots, m-1$,

$$\Delta^j \varphi|_{\Sigma_Q} = 0,\ (-1)^j \frac{\partial(\Delta^j \varphi)}{\partial \nu}|_{\Sigma_Q} \le 0,$$

where ν denotes the outward unit normal vector on ∂B_1, relative to $\mathbb{R}^N \backslash B_1$.

The set of all admissible test functions is denoted by Φ.

Definition 3.1.2. We say that the pair (u, v) is a weak solution to (3.1.1)-(3.1.2), if

$$(u, v) \in L^q_{\text{loc}}(Q) \times L^p_{\text{loc}}(Q),$$

$$\begin{aligned} &\int_Q |x|^a |v|^p \varphi \, dx \, dt - \sum_{i=0}^{m-1} \int_{\Sigma_Q} f_i(x) \frac{\partial((-\Delta)^{m-1-i}\varphi)}{\partial \nu} \, d\sigma \, dt \\ &\le \int_Q u(-\Delta)^m \varphi \, dx \, dt + \int_Q u \varphi_{tt} \, dx \, dt, \end{aligned} \tag{3.1.8}$$

and

$$\begin{aligned} &\int_Q |x|^b |u|^q \varphi \, dx \, dt - \sum_{i=0}^{m-1} \int_{\Sigma_Q} g_i \frac{\partial((-\Delta)^{m-1-i}\varphi)}{\partial \nu} \, d\sigma \, dt \\ &\le \int_Q v(-\Delta)^m \varphi \, dx \, dt + \int_Q v \varphi_{tt} \, dx \, dt \end{aligned} \tag{3.1.9}$$

for every $\varphi \in \Phi$.

Notice that, if (u, v) is a regular solution to (3.1.1)-(3.1.2), then (u, v) is a weak solution in the sense of Definition 3.1.2.

For every function $f \in L^1(\partial B_1)$, we set

$$I_f = \int_{\partial B_1} f(x) \, d\sigma.$$

Our first main result is stated in the following theorem.

Theorem 3.1.1. Let $p, q > 1$, $N \geq 2$, and $a, b \geq -2m$ with $(a, b) \neq (-2m, -2m)$. Let $f_i, g_i \in L^1(\partial B_1)$ for every $i = 0, \cdots, m-1$. Assume that $I_{f_{m-1}}, I_{g_{m-1}} \geq 0$ and $(I_{f_{m-1}}, I_{g_{m-1}}) \neq (0, 0)$. If $N \leq 2m$; or $N \geq 2m+1$ and

$$N < \max\left\{\operatorname{sgn}(I_{f_{m-1}}) \times \frac{2mp(q+1)+pb+a}{pq-1}, \operatorname{sgn}(I_{g_{m-1}}) \times \frac{2mq(p+1)+qa+b}{pq-1}\right\}, \tag{3.1.10}$$

then (3.1.1)-(3.1.2) possesses no weak solution.

Remark 3.1.1. Notice that (3.1.10) is equivalent to

$$N - 2m < \alpha,\ I_{f_{m-1}} > 0; \text{ or } \quad N - 2m < \beta,\ I_{g_{m-1}} > 0, \tag{3.1.11}$$

where

$$\alpha = \frac{a + 2m + p(b + 2m)}{pq - 1} \tag{3.1.12}$$

and

$$\beta = \frac{b + 2m + q(a + 2m)}{pq - 1}. \tag{3.1.13}$$

On the other hand, due to the condition $a, b \geq -2m$ and $(a, b) \neq (-2m, -2m)$, we have $\alpha, \beta > 0$, which shows that, if $N \leq 2m$, then (3.1.10) is always satisfied.

The proof of Theorem 3.1.1 is based on the construction of a suitable admissible test function and integral estimates. The construction of the admissible test function is specifically adapted to the polyharmonic operator $(-\Delta)^m$, the geometry of the domain, and the Navier-type boundary conditions (3.1.2).

Remark 3.1.2. By Theorem 3.1.1, we recover the nonexistence result obtained in [8] in the case $m = 1$. We also recover the nonexistence result obtained in [6] in the case $m = 2$.

Next, we are concerned with the existence of solutions to (3.1.1)-(3.1.2). Our second main result shows the sharpness of condition (3.1.10).

Theorem 3.1.2. Let $p, q > 1$ and $a, b \geq -2m$ with $(a, b) \neq (-2m, -2m)$. If

$$N - 2m > \max\{\alpha, \beta\}, \tag{3.1.14}$$

where α and β are given by (3.1.12) and (3.1.13), then (3.1.1)-(3.1.2) admits stationary solutions for some $f_i, g_i \in L^1(\partial B_1)$ $(i = 0, \cdots, m-1)$ with $I_{f_{m-1}}, I_{g_{m-1}} > 0$.

Theorem 3.1.2 will be proved by the construction of explicit stationary solutions to (3.1.1)-(3.1.2).

Remark 3.1.3. At this moment, we don't know wether there is existence or nonexistence in the critical case $N \geq 2m + 1$,

$$N = \max\left\{\operatorname{sgn}(I_{f_{m-1}}) \times \frac{2mp(q+1) + pb + a}{pq - 1}, \operatorname{sgn}(I_{g_{m-1}}) \times \frac{2mq(p+1) + qa + b}{pq - 1}\right\}.$$

This question is left open.

From Theorem 3.1.1, we deduce the following nonexistence result for the corresponding stationary polyharmonic system

$$\begin{cases} (-\Delta)^m u \geq |x|^a |v|^p, & x \in \mathbb{R}^N \backslash \overline{B_1}, \\ (-\Delta)^m v \geq |x|^b |u|^q, & x \in \mathbb{R}^N \backslash \overline{B_1} \end{cases} \tag{3.1.15}$$

under the Navier-type boundary conditions

$$\begin{cases} (-\Delta)^i u \geq f_i(x), \ i = 0, \cdots, m-1, \ x \in \partial B_1, \\ \\ (-\Delta)^i v \geq g_i(x), \ i = 0, \cdots, m-1, \ x \in \partial B_1. \end{cases} \tag{3.1.16}$$

Corollary 3.1.1. Let $p, q > 1$, $N \geq 2$, and $a, b \geq -2m$ with $(a, b) \neq (-2m, -2m)$. Let $f_i, g_i \in L^1(\partial B_1)$ for every $i = 0, \cdots, m-1$. Assume that $I_{f_{m-1}}, I_{g_{m-1}} \geq 0$ and $(I_{f_{m-1}}, I_{g_{m-1}}) \neq (0, 0)$. If $N \leq 2m$; or $N \geq 2m+1$ and (3.1.10) holds, then (3.1.15)-(3.1.16) possesses no weak solution.

The rest of this manuscript is organized as follows: Section 3.2 is devoted to some auxiliary results. Namely, we first construct an admissible test function in the sense of Definition 3.1.1. Next, we establish some useful integral estimates that involve the constructed test function. The proofs of Theorems 3.1.1 and 3.1.2 are provided in Section 3.3.

Throughout this paper, the letter C denotes a positive constant which is independent of the scaling parameters T, τ, and the solution (u, v). The value of C is not necessarily the same from one line to another.

3.2 Auxiliary results

In this section, we establish some auxiliary results that will be used later in the proof of our main result.

3.2.1 Admissible test function

Let us introduce the radial function H defined in $\mathbb{R}^N \backslash B_1$ by

$$H(x) = \begin{cases} \ln |x| & \text{if} \quad N = 2, \\ \\ 1 - |x|^{2-N} & \text{if} \quad N \geq 3. \end{cases} \tag{3.2.1}$$

We collect below some useful properties of the function H.

Lemma 3.2.1. The function H satisfies the following properties:

(i) $H \geq 0$;

(ii) $H \in C^{2m}(\mathbb{R}^N \backslash B_1)$;

(iii) $H|_{\partial B_1} = 0$;

(iv) $\Delta H = 0$ in $\mathbb{R}^N \backslash B_1$;

(v) For all $j \geq 1$,

$$\Delta^j H|_{\partial B_1} = \frac{\partial(\Delta^j H)}{\partial \nu}|_{\partial B_1} = 0;$$

(vi) $\dfrac{\partial H}{\partial \nu}|_{\partial B_1} = -C$.

Proof. (i)–(v) follow immediately form (3.2.1). On the other hand, we have

$$\frac{\partial H}{\partial \nu}|_{\partial B_1} = \begin{cases} -1 & \text{if} \quad N = 2, \\ -(N-2) & \text{if} \quad N \geq 3, \end{cases}$$

which proves (vi). □

We next consider a cut-off function $\xi \in C^\infty(\mathbb{R})$ satisfying the following properties:

$$0 \leq \xi \leq 1, \ \xi(s) = 1 \text{ if } |s| \leq 1, \ \xi(s) = 0 \text{ if } |s| \geq 2. \tag{3.2.2}$$

For all $\tau \gg 1$, let

$$\xi_\tau(x) = \xi\left(\frac{|x|}{\tau}\right), \quad x \in \mathbb{R}^N \backslash B_1,$$

that is (from (3.2.2)),

$$\xi_\tau(x) = \begin{cases} 1 & \text{if} \quad 1 \le |x| \le \tau, \\ \xi\left(\frac{|x|}{\tau}\right) & \text{if} \quad \tau \le |x| \le 2\tau, \\ 0 & \text{if} \quad |x| \ge 2\tau. \end{cases} \tag{3.2.3}$$

For $k \gg 1$, we introduce the function

$$\zeta_\tau(x) = H(x)\xi_\tau^k(x), \quad x \in \mathbb{R}^N \backslash B_1. \tag{3.2.4}$$

We now introduce a second cut-off function $G \in C^\infty(\mathbb{R})$ satisfying the following properties:

$$G \ge 0, \quad \operatorname{supp}(G) \subset\subset (0,1). \tag{3.2.5}$$

For $T > 0$ and $k \gg 1$, let

$$G_T(t) = G^k\left(\frac{t}{T}\right), \quad t \ge 0. \tag{3.2.6}$$

Let φ be the function defined by

$$\varphi(t,x) = G_T(t)\zeta_\tau(x), \quad (t,x) \in Q. \tag{3.2.7}$$

By Lemma 3.2.1, (3.2.3), (3.2.4), (3.2.5), (3.2.6), and (3.2.7), we obtain the following result.

Lemma 3.2.2. The function φ belongs to Φ.

3.2.2 A priori estimates

For all $\lambda > 1$, $\mu \ge -2m$, and $\varphi \in \Phi$, we consider the integral terms

$$J(\lambda,\mu,\varphi) = \int_Q |x|^{\frac{-\mu}{\lambda-1}} \varphi^{\frac{-1}{\lambda-1}} |(-\Delta)^m \varphi|^{\frac{\lambda}{\lambda-1}} \, dx\, dt \tag{3.2.8}$$

and

$$K(\lambda,\mu,\varphi)=\int_Q |x|^{\frac{-\mu}{\lambda-1}}\varphi^{\frac{-1}{\lambda-1}}|\varphi_{tt}|^{\frac{\lambda}{\lambda-1}}\,dx\,dt. \tag{3.2.9}$$

Lemma 3.2.3. Let φ be the admissible test function defined by (3.2.7). Assume that

(i) $J(p,a,\varphi), J(q,b,\varphi), K(p,a,\varphi), K(q,b,\varphi) < \infty$;

(ii) $I_{f_{m-1}}, I_{g_{m-1}} \geq 0$.

If (u,v) is a weak solution to (3.1.1)-(3.1.2), then

$$\begin{aligned} I_{f_{m-1}} &\leq CT^{-1}\left([J(p,a,\varphi)]^{\frac{p-1}{p}}+[K(p,a,\varphi)]^{\frac{p-1}{p}}\right)^{\frac{p}{pq-1}} \\ &\quad\cdot\left([J(q,b,\varphi)]^{\frac{q-1}{q}}+[K(q,b,\varphi)]^{\frac{q-1}{q}}\right)^{\frac{pq}{pq-1}} \end{aligned} \tag{3.2.10}$$

and

$$\begin{aligned} I_{g_{m-1}} &\leq CT^{-1}\left([J(q,b,\varphi)]^{\frac{q-1}{q}}+[K(q,b,\varphi)]^{\frac{q-1}{q}}\right)^{\frac{q}{pq-1}} \\ &\quad\cdot\left([J(p,a,\varphi)]^{\frac{p-1}{p}}+[K(p,a,\varphi)]^{\frac{p-1}{p}}\right)^{\frac{pq}{pq-1}}. \end{aligned} \tag{3.2.11}$$

Proof. Let (u,v) be a weak solution to (3.1.1)-(3.1.2) and φ be the admissible test function defined by (3.2.7). By (3.1.8), we have

$$\begin{aligned} &\int_Q |x|^a|v|^p\varphi\,dx\,dt-\sum_{i=0}^{m-1}\int_{\Sigma_Q} f_i(x)\frac{\partial((-\Delta)^{m-1-i}\varphi)}{\partial\nu}\,d\sigma\,dt \\ &\leq \int_Q u(-\Delta)^m\varphi\,dx\,dt+\int_Q u\varphi_{tt}\,dx\,dt. \end{aligned}$$

On the other hand, by Lemma 3.2.1: (v), (vi), (3.2.5), (3.2.6), and (3.2.7), we have

$$
\begin{aligned}
\sum_{i=0}^{m-1}\int_{\Sigma_Q} f_i(x)\frac{\partial((-\Delta)^{m-1-i}\varphi)}{\partial\nu}\,d\sigma\,dt &= \int_{\Sigma_Q} f_{m-1}(x)\frac{\partial\varphi}{\partial\nu}\,d\sigma\,dt \\
&= -C\int_{\Sigma_Q} f_{m-1}(x)G_T(t)\,d\sigma\,dt \\
&= -C\left(\int_0^\infty G_T(t)\,dt\right)\int_{\partial B_1} f_{m-1}(x)\,d\sigma \\
&= -C\left(\int_0^\infty G^k\left(\frac{t}{T}\right)dt\right)I_{f_{m-1}} \\
&= -CT\left(\int_0^1 G^k(s)\,ds\right)I_{f_{m-1}} \\
&= -CTI_{f_{m-1}}.
\end{aligned}
$$

Consequently, we obtain

$$
\int_Q |x|^a|v|^p\varphi\,dx\,dt + CTI_{f_{m-1}} \le \int_Q u(-\Delta)^m\varphi\,dx\,dt + \int_Q u\varphi_{tt}\,dx\,dt. \tag{3.2.12}
$$

Similarly, by (3.1.9), we obtain

$$
\int_Q |x|^b|u|^q\varphi\,dx\,dt + CTI_{g_{m-1}} \le \int_Q v(-\Delta)^m\varphi\,dx\,dt + \int_Q v\varphi_{tt}\,dx\,dt. \tag{3.2.13}
$$

Furthermore, by Hölder's inequality, we have

$$
\begin{aligned}
\int_Q u(-\Delta)^m\varphi\,dx\,dt &\le \int_Q |u|\left|(-\Delta)^m\varphi\right|\,dx\,dt \\
&= \int_Q\left(|x|^{\frac{b}{q}}|u|\varphi^{\frac{1}{q}}\right)\left(|x|^{\frac{-b}{q}}\left|(-\Delta)^m\varphi\right|\varphi^{\frac{-1}{q}}\right)dx\,dt \\
&\le \left(\int_Q |x|^b|u|^q\varphi\,dx\,dt\right)^{\frac{1}{q}}\left(\int_Q |x|^{\frac{-b}{q-1}}\left|(-\Delta)^m\varphi\right|^{\frac{q}{q-1}}\varphi^{\frac{-1}{q-1}}\,dx\,dt\right)^{\frac{q-1}{q}},
\end{aligned}
$$

that is,

$$\int_Q u(-\Delta)^m \varphi \, dx \, dt \leq \left(\int_Q |x|^b |u|^q \varphi \, dx \, dt \right)^{\frac{1}{q}} [J(q,b,\varphi)]^{\frac{q-1}{q}} . \tag{3.2.14}$$

Similarly, we obtain

$$\int_Q u\varphi_{tt} \, dx \, dt \leq \left(\int_Q |x|^b |u|^q \varphi \, dx \, dt \right)^{\frac{1}{q}} [K(q,b,\varphi)]^{\frac{q-1}{q}} . \tag{3.2.15}$$

Thus, it follows from (3.2.12), (3.2.14), and (3.2.15) that

$$\begin{aligned} &\int_Q |x|^a |v|^p \varphi \, dx \, dt + CTI_{f_{m-1}} \\ &\leq \left(\int_Q |x|^b |u|^q \varphi \, dx \, dt \right)^{\frac{1}{q}} \left([J(q,b,\varphi)]^{\frac{q-1}{q}} + [K(q,b,\varphi)]^{\frac{q-1}{q}} \right). \end{aligned} \tag{3.2.16}$$

Using (3.2.13) and proceeding as above, we also obtain

$$\begin{aligned} &\int_Q |x|^b |u|^q \varphi \, dx \, dt + CTI_{g_{m-1}} \\ &\leq \left(\int_Q |x|^a |v|^p \varphi \, dx \, dt \right)^{\frac{1}{p}} \left([J(p,a,\varphi)]^{\frac{p-1}{p}} + [K(p,a,\varphi)]^{\frac{p-1}{p}} \right). \end{aligned} \tag{3.2.17}$$

Using (3.2.16), (3.2.17), and taking into consideration that $I_{g_{m-1}} \geq 0$, we obtain

$$\begin{aligned} &\int_Q |x|^a |v|^p \varphi \, dx \, dt + CTI_{f_{m-1}} \\ &\leq \left(\int_Q |x|^a |v|^p \varphi \, dx \, dt \right)^{\frac{1}{pq}} \left([J(p,a,\varphi)]^{\frac{p-1}{p}} + [K(p,a,\varphi)]^{\frac{p-1}{p}} \right)^{\frac{1}{q}} \\ &\quad \cdot \left([J(q,b,\varphi)]^{\frac{q-1}{q}} + [K(q,b,\varphi)]^{\frac{q-1}{q}} \right). \end{aligned}$$

Then, by Young's inequality, it holds that

$$\begin{aligned}&\int_Q |x|^a|v|^p\varphi\,dx\,dt + CTI_{f_{m-1}}\\&\leq \frac{1}{pq}\int_Q |x|^a|v|^p\varphi\,dx\,dt\\&\quad+\frac{pq-1}{pq}\left([J(p,a,\varphi)]^{\frac{p-1}{p}}+[K(p,a,\varphi)]^{\frac{p-1}{p}}\right)^{\frac{pq}{q(pq-1)}}\left([J(q,b,\varphi)]^{\frac{q-1}{q}}+[K(q,b,\varphi)]^{\frac{q-1}{q}}\right)^{\frac{pq}{pq-1}}.\end{aligned}$$

Consequently, we have

$$\begin{aligned}&\left(1-\frac{1}{pq}\right)\int_Q |x|^a|v|^p\varphi\,dx\,dt + CTI_{f_{m-1}}\\&\leq \frac{pq-1}{pq}\left([J(p,a,\varphi)]^{\frac{p-1}{p}}+[K(p,a,\varphi)]^{\frac{p-1}{p}}\right)^{\frac{p}{pq-1}}\left([J(q,b,\varphi)]^{\frac{q-1}{q}}+[K(q,b,\varphi)]^{\frac{q-1}{q}}\right)^{\frac{pq}{pq-1}},\end{aligned}$$

which yields (3.2.10). Similarly, using (3.2.16), (3.2.17), and taking into consideration that $I_{f_{m-1}} \geq 0$, we obtain

$$\begin{aligned}&\int_Q |x|^b|u|^q\varphi\,dx\,dt + CTI_{g_{m-1}}\\&\leq \left(\int_Q |x|^b|u|^q\varphi\,dx\,dt\right)^{\frac{1}{pq}}\left([J(q,b,\varphi)]^{\frac{q-1}{q}}+[K(q,b,\varphi)]^{\frac{q-1}{q}}\right)^{\frac{1}{p}}\\&\quad\cdot\left([J(p,a,\varphi)]^{\frac{p-1}{p}}+[K(p,a,\varphi)]^{\frac{p-1}{p}}\right),\end{aligned}$$

which implies by Young's inequality that

$$\begin{aligned}&\int_Q |x|^b|u|^q\varphi\,dx\,dt + CTI_{g_{m-1}}\\&\leq \frac{1}{pq}\int_Q |x|^b|u|^q\varphi\,dx\,dt\\&\quad+\frac{pq-1}{pq}\left([J(q,b,\varphi)]^{\frac{q-1}{q}}+[K(q,b,\varphi)]^{\frac{q-1}{q}}\right)^{\frac{pq}{p(pq-1)}}\left([J(p,a,\varphi)]^{\frac{p-1}{p}}+[K(p,a,\varphi)]^{\frac{p-1}{p}}\right)^{\frac{pq}{pq-1}}.\end{aligned}$$

Thus, it holds that

$$\left(1-\frac{1}{pq}\right)\int_Q |x|^b|u|^q\varphi\,dx\,dt + CTI_{g_{m-1}}$$
$$\leq \frac{pq-1}{pq}\left([J(q,b,\varphi)]^{\frac{q-1}{q}}+[K(q,b,\varphi)]^{\frac{q-1}{q}}\right)^{\frac{q}{pq-1}}\left([J(p,a,\varphi)]^{\frac{p-1}{p}}+[K(p,a,\varphi)]^{\frac{p-1}{p}}\right)^{\frac{pq}{pq-1}},$$

which yields (3.2.11). □

3.2.3 Estimates of $J(\lambda,\mu,\varphi)$ and $K(\lambda,\mu,\varphi)$

The aim of this subsection is to estimate the integral terms $J(\lambda,\mu,\varphi)$ and $K(\lambda,\mu,\varphi)$, where $\lambda > 1$, $\mu \geq -2m$, and φ is the admissible test function defined by (3.2.7) with $\tau, k \gg 1$.

The following result follows immediately from (3.2.5) and (3.2.6).

Lemma 3.2.4. We have

$$\int_0^\infty G_T(t)\,dt = CT.$$

Lemma 3.2.5. We have

$$\int_0^\infty G_T^{\frac{-1}{\lambda-1}}\left|\frac{d^2G_T}{dt^2}\right|^{\frac{\lambda}{\lambda-1}}\,dt \leq CT^{1-\frac{2\lambda}{\lambda-1}}. \tag{3.2.18}$$

Proof. By (3.2.5) and (3.2.6), we have

$$\int_0^\infty G_T^{\frac{-1}{\lambda-1}}\left|\frac{d^2G_T}{dt^2}\right|^{\frac{\lambda}{\lambda-1}}\,dt = \int_0^T G_T^{\frac{-1}{\lambda-1}}\left|\frac{d^2G_T}{dt^2}\right|^{\frac{\lambda}{\lambda-1}}\,dt \tag{3.2.19}$$

and

$$\frac{d^2G_T}{dt^2}(t) = kT^{-2}G^{k-2}\left(\frac{t}{T}\right)\left((k-1)G'^2\left(\frac{t}{T}\right)+G\left(\frac{t}{T}\right)G''\left(\frac{t}{T}\right)\right)$$

for all $t \in (0,T)$. The above inequality yields

$$\left|\frac{d^2G_T}{dt^2}(t)\right| \leq CT^{-2}G^{k-2}\left(\frac{t}{T}\right), \quad t\in(0,T),$$

which implies that

$$G_T^{\frac{-1}{\lambda-1}} \left| \frac{d^2 G_T}{dt^2} \right|^{\frac{\lambda}{\lambda-1}} \leq C T^{\frac{-2\lambda}{\lambda-1}} G^{k-\frac{2\lambda}{\lambda-1}} \left(\frac{t}{T} \right), \quad t \in (0, T).$$

Then, by (3.2.19), it holds that

$$\begin{aligned} \int_0^\infty G_T^{\frac{-1}{\lambda-1}} \left| \frac{d^2 G_T}{dt^2} \right|^{\frac{\lambda}{\lambda-1}} dt &\leq C T^{\frac{-2\lambda}{\lambda-1}} \int_0^T G^{k-\frac{2\lambda}{\lambda-1}} \left(\frac{t}{T} \right) dt \\ &= C T^{1-\frac{2\lambda}{\lambda-1}} \int_0^1 G^{k-\frac{2\lambda}{\lambda-1}} (s)\, ds \\ &= C T^{1-\frac{2\lambda}{\lambda-1}}, \end{aligned}$$

which proves (3.2.18). □

To estimate $J(\lambda, \mu, \varphi)$ and $K(\lambda, \mu, \varphi)$, we consider separately the cases $N \geq 3$ and $N = 2$.

3.2.3.1 The case $N \geq 3$

Lemma 3.2.6. We have

$$\int_{\mathbb{R}^N \backslash B_1} |x|^{\frac{-\mu}{\lambda-1}} \zeta_\tau^{\frac{-1}{\lambda-1}} |(-\Delta)^m \zeta_\tau|^{\frac{\lambda}{\lambda-1}}\, dx \leq C \tau^{N-\frac{\mu+2m\lambda}{\lambda-1}}. \tag{3.2.20}$$

Proof. Since H and ξ_τ are radial functions (see (3.2.1) and (3.2.3)), to simplify writing, we set

$$H(x) = H(r), \;\; \xi_\tau(x) = \xi_\tau(r),$$

where $r = |x|$. By (3.2.4) and making use of Lemma 3.2.1 (iv), one can show that for all $x \in \mathbb{R}^N \backslash B_1$, we have

$$\begin{aligned} \Delta^m \zeta_\tau(x) &= \Delta^m \left(H(x) \xi_\tau^k(x) \right) \\ &= \sum_{i=0}^{2m-1} \frac{d^i H}{dr^i}(r) \sum_{j=1}^{2m-i} C_{i,j} \frac{d^j \xi_\tau^k}{dr^j}(r) r^{i+j-2m}, \end{aligned}$$

where $C_{i,j}$ are some constants, which implies by (3.2.3) that

$$\text{supp}\,(\Delta^m \zeta_\tau) \subset \left\{x \in \mathbb{R}^N : \tau \le |x| \le 2\tau\right\} \tag{3.2.21}$$

and

$$|\Delta^m \zeta_\tau(x)| \le C \sum_{i=0}^{2m-1} \left|\frac{d^i H}{dr^i}(r)\right| \sum_{j=1}^{2m-i} \left|\frac{d^j \xi_\tau^k}{dr^j}(r)\right| r^{i+j-2m}, \quad x \in \text{supp}\,(\Delta^m \zeta_\tau)\,. \tag{3.2.22}$$

On the other hand, for all $x \in \text{supp}\,(\Delta^m \zeta_\tau)$, we have by (3.2.1) and (3.2.3) that

$$\left|\frac{d^i H}{dr^i}(r)\right| = \begin{cases} H(r) & \text{if} \quad i = 0, \\ \\ Cr^{2-N-i} & \text{if} \quad i = 1, \cdots, 2m-1 \end{cases} \tag{3.2.23}$$

and (we recall that $0 \le \xi_\tau \le 1$)

$$\begin{aligned} \left|\frac{d^j \xi_\tau^k}{dr^j}(r)\right| &\le C\tau^{-j} \xi_\tau^{k-j}(r) \\ &\le C\tau^{-j} \xi_\tau^{k-2m}(r), \;\; j = 1, \cdots, 2m-i. \end{aligned} \tag{3.2.24}$$

Then, in view of (3.2.1), (3.2.21), (3.2.22), (3.2.23), and (3.2.24), we have

$$\begin{aligned} |\Delta^m \zeta_\tau(x)| &\le C\xi_\tau^{k-2m}(r) \left(H(r) \sum_{j=1}^{2m} \tau^{-j} r^{j-2m} + r^{2-N} \sum_{i=1}^{2m-1} \sum_{j=1}^{2m-i} \tau^{-j} r^{j-2m}\right) \\ &\le C\xi_\tau^{k-2m}(r) \left(\tau^{-2m} + \tau^{2-N-2m}\right) \\ &\le C\tau^{-2m} \xi_\tau^{k-2m}(x) \end{aligned}$$

for all $x \in \text{supp}\,(\Delta^m \zeta_\tau)$. Taking into consideration that $H \ge C$ for all $x \in \text{supp}\,(\Delta^m \zeta_\tau)$, the above estimate yields

$$|x|^{\frac{-\mu}{\lambda-1}} \zeta_\tau^{\frac{-1}{\lambda-1}} |(-\Delta)^m \zeta_\tau|^{\frac{\lambda}{\lambda-1}} \le C\tau^{\frac{-2m\lambda-\mu}{\lambda-1}} \xi_\tau^{k-\frac{2m\lambda}{\lambda-1}}(x), \quad x \in \text{supp}\,(\Delta^m \zeta_\tau)\,. \tag{3.2.25}$$

Finally, by (3.2.21) and (3.2.25), we obtain

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_1} |x|^{\frac{-\mu}{\lambda-1}} \zeta_\tau^{\frac{-1}{\lambda-1}} |(-\Delta)^m \zeta_\tau|^{\frac{\lambda}{\lambda-1}}\, dx &= \int_{\tau<|x|<2\tau} |x|^{\frac{-\mu}{\lambda-1}} \zeta_\tau^{\frac{-1}{\lambda-1}} |(-\Delta)^m \zeta_\tau|^{\frac{\lambda}{\lambda-1}}\, dx \\ &\le C\tau^{\frac{-2m\lambda-\mu}{\lambda-1}} \int_{\tau<|x|<2\tau} \xi_\tau^{k-\frac{2m\lambda}{\lambda-1}}(x)\, dx \\ &\le C\tau^{\frac{-2m\lambda-\mu}{\lambda-1}} \int_{r=\tau}^{2\tau} r^{N-1}\, dr \\ &= C\tau^{N-\frac{\mu+2m\lambda}{\lambda-1}},\end{aligned}$$

which proves (3.2.20). □

Lemma 3.2.7. We have

$$J(\lambda,\mu,\varphi) \le CT\tau^{N-\frac{\mu+2m\lambda}{\lambda-1}}.$$

Proof. By (3.2.7) and (3.2.8), we have

$$J(\lambda,\mu,\varphi) = \left(\int_0^\infty G_T(t)\, dt\right)\left(\int_{\mathbb{R}^N\setminus B_1} |x|^{\frac{-\mu}{\lambda-1}} \zeta_\tau^{\frac{-1}{\lambda-1}} |(-\Delta)^m \zeta_\tau|^{\frac{\lambda}{\lambda-1}}\, dx\right).$$

Then, using Lemmas 3.2.4 and 3.2.6, we obtain the desired estimate. □

Lemma 3.2.8. We have

$$\int_{\mathbb{R}^N\setminus B_1} |x|^{\frac{-\mu}{\lambda-1}} \zeta_\tau(x)\, dx \le C\left(\tau^{N-\frac{\mu}{\lambda-1}} + \ln\tau\right). \tag{3.2.26}$$

Proof. By (3.2.1), (3.2.2), (3.2.3), and (3.2.4), we have

$$\begin{aligned}
\int_{\mathbb{R}^N\setminus B_1} |x|^{\frac{-\mu}{\lambda-1}}\zeta_\tau(x)\,dx &= \int_{1<|x|<2\tau} |x|^{\frac{-\mu}{\lambda-1}}\left(1-|x|^{2-N}\right)\xi^\kappa\left(\frac{|x|}{\tau}\right)dx \\
&\le \int_{1<|x|<2\tau} |x|^{\frac{-\mu}{\lambda-1}}\,dx \\
&= C\int_{r=1}^{2\tau} r^{N-1-\frac{\mu}{\lambda-1}}\,dr \\
&\le \begin{cases} C\tau^{N-\frac{\mu}{\lambda-1}} & \text{if} \quad N-\frac{\mu}{\lambda-1}>0, \\ C\ln\tau & \text{if} \quad N-\frac{\mu}{\lambda-1}=0, \\ C & \text{if} \quad N-\frac{\mu}{\lambda-1}<0 \end{cases} \\
&\le C\left(\tau^{N-\frac{\mu}{\lambda-1}}+\ln\tau\right),
\end{aligned}$$

which proves (3.2.26). □

Lemma 3.2.9. We have

$$K(\lambda,\mu,\varphi) \le CT^{1-\frac{2\lambda}{\lambda-1}}\left(\tau^{N-\frac{\mu}{\lambda-1}}+\ln\tau\right).$$

Proof. By (3.2.7) and (3.2.9), we have

$$K(\lambda,\mu,\varphi) = \left(\int_0^\infty G_T^{\frac{-1}{\lambda-1}}\left|\frac{d^2G_T}{dt^2}\right|^{\frac{\lambda}{\lambda-1}}\,dt\right)\left(\int_{\mathbb{R}^N\setminus B_1} |x|^{\frac{-\mu}{\lambda-1}}\zeta_\tau(x)\,dx\right).$$

Then, using Lemmas 3.2.5 and 3.2.7, we obtain the desired estimate. □

3.2.3.2 The case $N = 2$

Lemma 3.2.10. We have

$$\int_{\mathbb{R}^2\setminus B_1} |x|^{\frac{-\mu}{\lambda-1}}\zeta_\tau^{\frac{-1}{\lambda-1}}|(-\Delta)^m\zeta_\tau|^{\frac{\lambda}{\lambda-1}}\,dx \le C\tau^{2-\frac{2m\lambda+\mu}{\lambda-1}}\ln\tau. \tag{3.2.27}$$

Proof. Proceeding as in the proof of Lemma 3.2.6, we obtain

$$\mathrm{supp}\,(\Delta^m \zeta_\tau) \subset \left\{x \in \mathbb{R}^2 : \tau \le |x| \le 2\tau\right\}$$

and

$$|\Delta^m \zeta_\tau(x)| \le C\tau^{-2m} \ln \tau \, \xi_\tau^{k-2m}(x), \quad x \in \mathrm{supp}\,(\Delta^m \zeta_\tau)\,.$$

The above estimate yields

$$|x|^{\frac{-\mu}{\lambda-1}} \zeta_\tau^{\frac{-1}{\lambda-1}} |(-\Delta)^m \zeta_\tau|^{\frac{\lambda}{\lambda-1}} \le C\tau^{\frac{-2m\lambda-\mu}{\lambda-1}} \ln \tau \, \xi_\tau^{k-\frac{2m\lambda}{\lambda-1}}(x), \quad x \in \mathrm{supp}\,(\Delta^m \zeta_\tau)\,.$$

Then, it holds that

$$\begin{aligned}
\int_{\mathbb{R}^2\setminus B_1} |x|^{\frac{-\mu}{\lambda-1}} \zeta_\tau^{\frac{-1}{\lambda-1}} |(-\Delta)^m \zeta_\tau|^{\frac{\lambda}{\lambda-1}} \, dx &\le C\tau^{\frac{-2m\lambda-\mu}{\lambda-1}} \ln \tau \int_{\tau<|x|<2\tau} \xi_\tau^{k-\frac{2m\lambda}{\lambda-1}}(x)\, dx \\
&\le C\tau^{\frac{-2m\lambda-\mu}{\lambda-1}} \ln \tau \int_{r=\tau}^{2\tau} r\, dr \\
&\le C\tau^{2-\frac{2m\lambda+\mu}{\lambda-1}} \ln \tau,
\end{aligned}$$

which proves (3.2.27). □

Using (3.2.7), (3.2.8), Lemmas 3.2.4 and 3.2.10, we obtain the following estimate of $J(\lambda, \mu, \varphi)$.

Lemma 3.2.11. We have

$$J(\lambda, \mu, \varphi) \le CT\tau^{2-\frac{2m\lambda+\mu}{\lambda-1}} \ln \tau.$$

Lemma 3.2.12. We have

$$\int_{\mathbb{R}^2\setminus B_1} |x|^{\frac{-\mu}{\lambda-1}} \zeta_\tau(x)\, dx \le C \ln \tau \left(\tau^{2-\frac{\mu}{\lambda-1}} + \ln \tau\right). \tag{3.2.28}$$

Proof. By (3.2.1), (3.2.2), (3.2.3), and (3.2.4), we have

$$\begin{aligned}
\int_{\mathbb{R}^2\setminus B_1} |x|^{\frac{-\mu}{\lambda-1}} \zeta_\tau(x)\,dx &= \int_{1<|x|<2\tau} |x|^{\frac{-\mu}{\lambda-1}} \ln|x|\, \xi^\kappa\left(\frac{|x|}{\tau}\right) dx \\
&\leq \int_{1<|x|<2\tau} |x|^{\frac{-\mu}{\lambda-1}} \ln|x|\,dx \\
&= C\int_{r=1}^{2\tau} r^{1-\frac{\mu}{\lambda-1}} \ln r\,dr \\
&\leq \begin{cases} C\tau^{2-\frac{\mu}{\lambda-1}} \ln\tau & \text{if} \quad 2-\frac{\mu}{\lambda-1} > 0, \\ C(\ln\tau)^2 & \text{if} \quad 2-\frac{\mu}{\lambda-1} = 0, \\ C\ln\tau & \text{if} \quad 2-\frac{\mu}{\lambda-1} < 0 \end{cases} \\
&\leq C\ln\tau\left(\tau^{2-\frac{\mu}{\lambda-1}} + \ln\tau\right),
\end{aligned}$$

which proves (3.2.28). □

Using (3.2.7), (3.2.9), Lemmas 3.2.5 and 3.2.12, we obtain the following estimate of $K(\lambda,\mu,\varphi)$.

Lemma 3.2.13. We have

$$K(\lambda,\mu,\varphi) \leq CT^{1-\frac{2\lambda}{\lambda-1}} \ln\tau\left(\tau^{2-\frac{\mu}{\lambda-1}} + \ln\tau\right).$$

3.3 Proofs of the main results

This section is devoted to the proofs of Theorems 3.1.1 and 3.1.2.

3.3.1 Proof of Theorem 3.1.1

By Remark 3.1.1, (3.1.10) is equivalent to (3.1.11). Without restriction of the generality, we assume that

$$N-2m < \alpha, \quad I_{f_{m-1}} > 0. \tag{3.3.1}$$

Indeed, exchanging the roles of $(I_{f_{m-1}}, a, p)$ and $(I_{g_{m-1}}, b, q)$, the case

$$N - 2m < \beta, \quad I_{g_{m-1}} > 0$$

reduces to (3.3.1).

We use the contradiction argument. Namely, let us suppose that (u, v) is a weak solution to (3.1.1)-(3.1.2) (in the sense of Definition 3.1.2). For $k, T, \tau \gg 1$, let φ be the admissible test function defined by (3.2.7). Then, by Lemma 3.2.3, we have

$$\begin{aligned} I_{f_{m-1}}^{\frac{pq-1}{p}} \leq CT^{-\frac{pq-1}{p}} &\left([J(p, a, \varphi)]^{\frac{p-1}{p}} + [K(p, a, \varphi)]^{\frac{p-1}{p}} \right) \\ &\cdot \left([J(q, b, \varphi)]^{\frac{q-1}{q}} + [K(q, b, \varphi)]^{\frac{q-1}{q}} \right)^q . \end{aligned} \tag{3.3.2}$$

Making use of Lemmas 3.2.7 and 3.2.12, we obtain that for all $N \geq 2$,

$$J(\lambda, \mu, \varphi) \leq CT\tau^{N - \frac{\mu + 2m\lambda}{\lambda - 1}} \ln \tau, \quad \lambda > 1,\ \mu \geq -2m. \tag{3.3.3}$$

Similarly, by Lemmas 3.2.9 and 3.2.13, we obtain that for all $N \geq 2$,

$$K(\lambda, \mu, \varphi) \leq CT^{1 - \frac{2\lambda}{\lambda - 1}} \left(\tau^{N - \frac{\mu}{\lambda - 1}} + \ln \tau \right) \ln \tau, \quad \lambda > 1,\ \mu \geq -2m. \tag{3.3.4}$$

In particular, for $(\lambda, \mu) = (p, a)$, we obtain by (3.3.3) and (3.3.4) that

$$\begin{aligned} &[J(p, a, \varphi)]^{\frac{p-1}{p}} + [K(p, a, \varphi)]^{\frac{p-1}{p}} \\ &\leq C \left[T^{\frac{p-1}{p}} \tau^{\left(N - \frac{a+2mp}{p-1}\right)\frac{p-1}{p}} (\ln \tau)^{\frac{p-1}{p}} + T^{\left(1 - \frac{2p}{p-1}\right)\frac{p-1}{p}} \left(\tau^{N - \frac{a}{p-1}} + \ln \tau \right)^{\frac{p-1}{p}} (\ln \tau)^{\frac{p-1}{p}} \right] \\ &= CT^{\frac{p-1}{p}} \tau^{\left(N - \frac{a+2mp}{p-1}\right)\frac{p-1}{p}} (\ln \tau)^{\frac{p-1}{p}} \left[1 + T^{-2} \left(\tau^{\frac{2mp}{p-1}} + \tau^{-\left(N - \frac{a+2mp}{p-1}\right)} \ln \tau \right)^{\frac{p-1}{p}} \right]. \end{aligned} \tag{3.3.5}$$

Furthermore, taking $T=\tau^\theta$, where

$$\theta > \max\left\{m, \left(\frac{a+2mp}{p-1}-N\right)\frac{p-1}{p}\right\}, \tag{3.3.6}$$

we obtain

$$1+T^{-2}\left(\tau^{\frac{2mp}{p-1}}+\tau^{-\left(N-\frac{a+2mp}{p-1}\right)}\ln\tau\right)^{\frac{p-1}{p}} \leq C.$$

Then, from (3.3.5), we deduce that

$$[J(p,a,\varphi)]^{\frac{p-1}{p}}+[K(p,a,\varphi)]^{\frac{p-1}{p}} \leq C\left[\tau^{\theta+N-\frac{a+2mp}{p-1}}\ln\tau\right]^{\frac{p-1}{p}}. \tag{3.3.7}$$

Similarly, for

$$\theta > \max\left\{m, \left(\frac{b+2mq}{q-1}-N\right)\frac{q-1}{q}\right\}, \tag{3.3.8}$$

we obtain

$$\left([J(q,b,\varphi)]^{\frac{q-1}{q}}+[K(q,b,\varphi)]^{\frac{q-1}{q}}\right)^q \leq C\left[\tau^{\theta+N-\frac{b+2mq}{q-1}}\ln\tau\right]^{q-1}. \tag{3.3.9}$$

Thus, for $T=\tau^\theta$, where θ satisfies (3.3.6) and (3.3.8), we obtain by (3.3.2), (3.3.7), and (3.3.9) that

$$I_{f_{m-1}}^{\frac{pq-1}{p}} \leq C\tau^{-\frac{\theta(pq-1)}{p}}\left[\tau^{\theta+N-\frac{a+2mp}{p-1}}\ln\tau\right]^{\frac{p-1}{p}}\left[\tau^{\theta+N-\frac{b+2mq}{q-1}}\ln\tau\right]^{q-1},$$

that is,

$$I_{f_{m-1}}^{\frac{pq-1}{p}} \leq C\tau^{\delta}(\ln\tau)^{\frac{pq-1}{p}}, \tag{3.3.10}$$

where

$$\begin{aligned}\delta &= \frac{pq-1}{p}\left[N-\frac{(b+2mq)p+a+2mp}{pq-1}\right]\\ &= \frac{pq-1}{p}(N-2m-\alpha).\end{aligned}$$

Since $N-2m<\alpha$, we have $\delta<0$. Then, by the Monotone Convergence Theorem, we are allowed to pass to the limit as $\tau\to\infty$ in the integral(3.3.10), we reach a contradiction with $I_{f_{m-1}}>0$. This completes the proof of Theorem 3.1.1. □

3.3.2 Proof of Theorem 3.1.2

Let us introduce the family of polynomial functions $\{P_i\}_{0\le i\le m}$, where

$$P_i(z)=\begin{cases}1 & \text{if} \quad i=0,\\ \displaystyle\prod_{j=0}^{i-1}(z+2j)\prod_{j=1}^{i}(N-2j-z) & \text{if} \quad i=1,\cdots,m.\end{cases}$$

From (3.1.14), we deduce that

$$N-2j>\max\{\alpha,\beta\},\quad j=1,\cdots,m.$$

Furthermore, because $a,b\ge-2m$ and $(a,b)\neq(-2m,-2m)$, we have $\alpha,\beta>0$. Then,

$$P_i(z)>0,\quad i=0,1,\cdots,m,\quad z\in\{\alpha,\beta\}. \tag{3.3.11}$$

For all

$$0<\varepsilon\le\min\left\{[P_m(\alpha)]^{\frac{1}{p-1}},[P_m(\beta)]^{\frac{1}{q-1}}\right\}, \tag{3.3.12}$$

we consider functions of the forms

$$u_\varepsilon(x)=\varepsilon|x|^{-\alpha},\quad x\in\mathbb{R}^N\backslash B_1 \tag{3.3.13}$$

and

$$v_\varepsilon(x) = \varepsilon |x|^{-\beta}, \quad x \in \mathbb{R}^N \backslash B_1. \tag{3.3.14}$$

Since u_ε and v_ε are radial functions, elementary calculations show that

$$(-\Delta)^i u_\varepsilon(x) = \varepsilon P_i(\alpha) |x|^{-\alpha-2i}, \quad i = 0, 1, \cdots, m, \quad x \in \mathbb{R}^N \backslash B_1 \tag{3.3.15}$$

and

$$(-\Delta)^i v_\varepsilon(x) = \varepsilon P_i(\beta) |x|^{-\beta-2i}, \quad i = 0, 1, \cdots, m, \quad x \in \mathbb{R}^N \backslash B_1. \tag{3.3.16}$$

Taking $i = m$ in (3.3.15), using (3.3.11), (3.3.12), (3.3.13), and (3.3.14), we obtain

$$\begin{aligned}
(-\Delta)^m u_\varepsilon(x) &= \varepsilon P_m(\alpha) |x|^{-\alpha-2m} \\
&= |x|^a \varepsilon^p |x|^{-\beta p} \left(\varepsilon^{1-p} P_m(\alpha) |x|^{-\alpha-2m-a+\beta p} \right) \\
&\geq |x|^a v_\varepsilon^p(x) |x|^{-\alpha-2m-a+\beta p}.
\end{aligned}$$

On the other hand, by (3.1.12) and (3.1.13), one can show that

$$-\alpha - 2m - a + \beta p = 0.$$

Then, we obtain

$$(-\Delta)^m u_\varepsilon(x) \geq |x|^a v_\varepsilon^p(x), \quad x \in \mathbb{R}^N \backslash B_1. \tag{3.3.17}$$

Similarly, taking $m = i$ in (3.3.16), using (3.3.11), (3.3.12), (3.3.13), and (3.3.14), we obtain

$$\begin{aligned}(-\Delta)^m v_\varepsilon(x) &= \varepsilon P_m(\beta)|x|^{-\beta-2m} \\ &= |x|^b \varepsilon^q |x|^{-\alpha q}\left(\varepsilon^{1-q} P_m(\beta)|x|^{-\beta-2m-b+\alpha q}\right) \\ &\geq |x|^b u_\varepsilon^q(x)|x|^{-\beta-2m-b+\alpha q}.\end{aligned}$$

Using that

$$-\beta - 2m - b + \alpha q = 0,$$

we obtain

$$(-\Delta)^m v_\varepsilon(x) \geq |x|^b u_\varepsilon^q(x), \quad x \in \mathbb{R}^N \backslash B_1. \tag{3.3.18}$$

Furthermore, by (3.3.11) and (3.3.15), for all $i = 0, \cdots, m-1$, we have

$$(-\Delta)^i u_\varepsilon(x) = \varepsilon P_i(\alpha) > 0, \quad x \in \partial B_1. \tag{3.3.19}$$

Similarly, by (3.3.11) and (3.3.16), for all $i = 0, \cdots, m-1$, we have

$$(-\Delta)^i v_\varepsilon(x) = \varepsilon P_i(\beta) > 0, \quad x \in \partial B_1. \tag{3.3.20}$$

Finally, (3.3.17), (3.3.18), (3.3.19), and (3.3.20) show that for all ε satisfying (3.3.12), the pair of functions $(u_\varepsilon, v_\varepsilon)$ given by (3.3.13) and (3.3.14) is a stationary solution to (3.1.1)-(3.1.2) with $f_i \equiv \varepsilon P_i(\alpha)$ and $g_i \equiv \varepsilon P_i(\beta)$ for all $i = 0, \cdots, m-1$. The proof of Theorem 3.1.2 is then completed.

□

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Summary

College:	**Science**
Department:	**Mathematics**
Specialization:	**Applied Mathematics-Nonlinear Partial Differential Equations**
Title:	**Blow up Results for Evolution Inequalities in Some Specific Domains**
Student Name:	**Manal Abdulaziz AlFulaij**
Supervisor Name:	**Prof. Mohamed Jleli**

Degree: **Ph.D** Convocation: **11-December** Year: **2025**

Tagged to search: The study of nonlinear evolution equations represents an active and dynamic field of scientific research due to their appearance as models for many physical phenomena. In this thesis, we primarily focus on investigating the existence and nonexistence of weak solutions for certain nonlinear evolution inequalities within specific domains in $\mathbb{R}^N$. A technique involving nonlinear capacity estimates was used, relying on the selection of appropriate test functions to minimize bounds in finite time to prove the nonexistence of solutions, while explicit stationary (time-independent) functions were constructed to satisfy the conditions in cases where the existence of solutions was established.

In the second chapter, the existence and nonexistence of weak solutions for semilinear higher-order evolution inequalities (in time) with Hardy potential were studied under inhomogeneous Dirichlet-type boundary conditions posed outside a half-ball in $\mathbb{R}^N$.

It was shown that the dividing line between the existence and nonexistence of solutions is determined by a Fujita-type critical exponent that depends on a suitable parameter λ, but does not depend on the order of the time derivative.

In the following chapter, a nonlinear polyharmonic system in an exterior domain of $\mathbb{R}^N$ was studied under inhomogeneous Navier-type boundary conditions. Using nonlinear estimates specifically adapted to the polyharmonic operator$(\Delta)^m$, the geometry of the domains, and the boundary conditions, a sharp criterion for the nonexistence of weak solutions was obtained. Subsequently, an optimal nonexistence result for the corresponding stationary problem was derived.

ملخص الرسالة

الكلية: العلوم

القسم: الرياضيات

التخصص: الرياضيات التطبيقية -معادلات تفاضليه غير خطيه

عنوان الرسالة: نتائج التفّجر في متباينات التطور في مجالات محدده

اسم الطالبة: منال عبد العزيز فهد الفليج

اسم المشرف: أ.د. محمد البوصيري جلالي

الدرجة العلمية: دكتوراه **تاريخ المناقشة** : 11/12/2025

الكلمات الدلالية للبحث: الحلول الضعيفة-داله اختبار-سعه التقديرات غير الخطية-وجود وعدم وجود الحلول-مجالات خارجيه-شروط حدودية-متباينات التطور-الأس الحرج-جهد هاردي-نظام متعدد التوافقات.

الملخص:

دراسة المعادلات التطورية غير الخطية تمثل مجالا نشطا وديناميكيا في البحث العلمي نظرا لظهورها كنماذج للعديد من الظواهر الفيزيائية.

في هذه الأطروحة نركز بشكل أساسي على دراسة وجود وعدم وجود حلول ضعيفة لبعض متباينات التطور غير الخطية ضمن مجالات معينه في $\mathbb{R}^N$, حيث تم استخدام تقنيه سعه التقديرات غير الخطية التي تعتمد على اختيار دوال اختبار مناسبه من اجل تصغير الحدود في زمن منتهي لإثبات عدم وجود الحلول, بينما تم بناء دوال صريحه ساكنه (مستقله عن الزمن) تحقق الشروط في حاله اثبات وجود الحلول.

في الفصل الثاني, تم دراسة وجود وعدم وجود حلول ضعيفة للمتباينات التطورية شبه الخطية ذات الرتب العالية (في الزمن) مع وجود جهد هاردي تحت شروط حدودية غير متجانسه من نوع ديريشلي التي تعرض خارج نص الكره في $\mathbb{R}^N$. تبين ان الخط الفاصل فيما يتعلق بوجود حل من عدمه تم تحديده من خلال أس حرج من نوع فوجيتا يعتمد على معامل مناسب λ ولكنه لا يعتمد على رتبه مشتقه الزمن.

في الفصل الذي يليه, تم دراسة نظام متعدد التوافقات غير الخطي في مجال خارجي في $\mathbb{R}^N$, تحت شروط حدودية غير متجانسه من نوع نافييه. باستخدام تقديرات غير خطيه تتكيف بشكل خاص مع متعدد التوافقات

$(-\Delta)^m$ وهندسه المجالات والشروط الحدودية, تم الحصول على معيار دقيق لعدم وجود الحلول الضعيفة, بعد ذلك تم استنتاج نتيجة مثلى لعدم وجود الحل للمسألة الساكنه المقابلة.