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Attributes of Subordination of a Specific Subclass of p-Valent Meromorphic Functions Connected to a Linear Operator

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Abstract: This work examines subordination conclusions for a specific subclass of p-valent meromorphic functions on the punctured unit disc of the complex plane where the function has a pole of order p. A new linear operator is used to define the subclass that is being studied. Furthermore, we present several corollaries with intriguing specific situations of the results.

Keywords: meromorphic; p-valent; subordination; starlike; convex

MSC: 30C45; 30C80; 30D30

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1. Introduction

Let the family of all functions that have the form

$$f(\zeta)=\zeta^{-p}+\sum_{j=1-p}^{\infty}d_j\zeta^j \quad (p\in\mathbb{N}), \tag{1}$$

be Σ_p, which is analytic in the punctured unit disc $\Delta^*=\Delta\backslash\{0\}$ $(\Delta=\{\zeta:\zeta\in\mathbb{C}:|\zeta|<1\})$. For two functions f and g, analytic in Δ, it is known that f is subordinate to g in Δ, written $f(\zeta)\prec g(\zeta)$ $(\zeta\in\Delta)$, if a Schwarz function $\omega(\zeta)$ exists, which is analytic in Δ, satisfying the following conditions (see [1,2]) $\omega(0)=0$ and $|\omega(\zeta)|<1$; $(\zeta\in\Delta)$ such that $f(\zeta)=g(\omega(\zeta))$ $(\zeta\in\Delta)$.

Inspired by El-Ashwah's paper [3], the operator $L_p^m(\lambda,\ell)$, where $\lambda>0$, $\ell>0$, and $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, is defined as follows for a function $f\in\Sigma_p$, provided by (1):

$$L_p^m(\lambda,\ell)f(\zeta)=\begin{cases} f(\zeta); & m=0 \\ \frac{\ell}{\lambda}\zeta^{\left(-p-\frac{\ell}{\lambda}\right)}\int\limits_0^{\zeta}t^{\left(\frac{\ell}{\lambda}+p-1\right)}L_p^{m-1}(\lambda,\ell)f(t)dt; & m=1,2,\ldots. \end{cases} \tag{2}$$

Additionally, in accordance with El-Ashwah and Hassan's most recent work [4], for a function $f\in\Sigma_p$, provided by (1), and also for $\mu>0$, $a,c\in\mathbb{C}$ and $Re(c-a)\geq 0$, the integral operator

$$J_{p,\mu}^{a,c}:\Sigma_p\longrightarrow\Sigma_p$$

is expressed for $Re(c-a) > 0$ as follows:

$$J^{a,c}_{p,\mu} f(\zeta) = \frac{\Gamma(c-p\mu)}{\Gamma(a-p\mu)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} f(\zeta t^{\mu})dt, \tag{3}$$

and for $a = c$ by

$$J^{a,a}_{p,\mu} f(\zeta) = f(\zeta). \tag{4}$$

For the purposes of this study, the operator $I^{p,m}_{\lambda,\ell}(a,c,\mu) : \Sigma_p \longrightarrow \Sigma_p$ is defined by iterations of the linear operators $L^m_p(\lambda,\ell)$ defined by (2), and $J^{a,c}_{p,\mu}$ defined by (3) and (4), as defined by the following:

$$I^{p,m}_{\lambda,\ell}(a,c,\mu) f(\zeta) = L^m_p(\lambda,\ell)\left(J^{a,c}_{p,\mu} f(\zeta)\right) = J^{a,c}_{p,\mu}\left(L^m_p(\lambda,\ell) f(\zeta)\right). \tag{5}$$

It is now evident that the generalized operator $I^{p,m}_{\lambda,\ell}(a,c,\mu)$ has the following expression:

$$I^{p,m}_{\lambda,\ell}(a,c,\mu) f(\zeta) = \zeta^{-p} + \frac{\Gamma(c-p\mu)}{\Gamma(a-p\mu)} \sum_{j=1-p}^{\infty} \frac{\Gamma(a+\mu j)}{\Gamma(c+\mu j)} \left[\frac{\ell}{\ell+\lambda(j+p)}\right]^m d_j \zeta^j, \tag{6}$$

$$(\mu>0; a, c\in\mathbb{C}, Re(a)>p\mu, Re(c-a)\geq 0; \ell>0; \lambda>0; m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}; p\in\mathbb{N}).$$

It is obvious that

$$I^{p,0}_{\lambda,\ell}(a,c,\mu) f(\zeta) = J^{a,c}_{p,\mu} f(\zeta) \quad \text{and} \quad I^{p,m}_{\lambda,\ell}(a,a,\mu) f(\zeta) = L^m_p(\lambda,\ell) f(\zeta). \tag{7}$$

The operator $I^{p,m}_{\lambda,\ell}(a,c,\mu)$ is a generalization of the following previously introduced operators:

(i) $I^{1,m}_{\nu,\lambda}(a+1,c+1,1) f(\zeta) = \Im^m_{\lambda,\nu}(a,c) f(\zeta)$ $\left(\lambda,\nu>0; a\in\mathbb{C}; c\in\mathbb{C}\backslash\mathbb{Z}_0^-; m\in\mathbb{N}_0\right)$ (see Raina and Sharma [5]);

(ii) $I^{p,0}_{\lambda,\ell}(a+p,c+p,1) f(\zeta) = \ell_p(a,c) f(\zeta)$ $\left(a\in\mathbb{R}; c\in\mathbb{R}\backslash\mathbb{Z}_0^-, \mathbb{Z}_0^- = \{0,1,2,\ldots\}; p\in\mathbb{N}\right)$ (see Liu and Srivastava [6] and Srivastava and Patel [7]);

(iii) $I^{1,\beta}_{1,\lambda}(\nu+1,2,1) f(\zeta) = I^{\beta}_{\lambda,\nu} f(\zeta)$ $(\beta\geq 0; \lambda>0; \nu>0)$ (see Piejko and Sokół [8]);

(iv) $I^{1,n}_{1,\lambda}(\nu+1,2,1) f(\zeta) = I^{n}_{\lambda,\nu} f(\zeta)$ $(n\in\mathbb{N}_0; \lambda>0; \nu>0)$ (see Cho et al. [9]);

(v) $I^{1,0}_{\lambda,\ell}(\nu+1,n+2,1) f(\zeta) = \ell_{n,\nu} f(\zeta)$ $(n>-1; \nu>0)$ (see Yuan et al. [10]);

(vi) $I^{p,0}_{\lambda,\ell}(n+2p,p+1,1) f(\zeta) = D^{n+p-1} f(\zeta)$ $(n \text{ is an integer}, n>-p,\ p\in\mathbb{N})$ (see Uralegaddi and Somanatha [11], Aouf [12] and Aouf and Srivastava [13]);

(vii) $I^{p,\alpha}_{1,1}(a,a,\mu) f(\zeta) = P^{\alpha}_p f(\zeta)$ $(\alpha\geq 0; p\in\mathbb{N})$ (see Aqlan et al. [14]);

(viii) $I^{1,\alpha}_{1,\beta}(a,a,\mu) f(\zeta) = P^{\alpha}_{\beta} f(\zeta)$ $(\alpha,\beta>0)$ (see Lashin [15]).

2. Preliminaries

We will need the following lemmas.

Lemma 1. *Using Equation (6), we can find the following recurrence relations:*

$$\zeta\left(I^{p,m}_{\lambda,\ell}(a,c,\mu) f(\zeta)\right)' = \frac{a-p\mu}{\mu} I^{p,m}_{\lambda,\ell}(a+1,c,\mu) f(\zeta) - \frac{a}{\mu} I^{p,m}_{\lambda,\ell}(a,c,\mu) f(\zeta). \tag{8}$$

and

$$\zeta\left(I^{p,m}_{\lambda,\ell}(a,c+1,\mu) f(\zeta)\right)' = \frac{c-p\mu}{\mu} I^{p,m}_{\lambda,\ell}(a,c,\mu) f(\zeta) - \frac{c}{\mu} I^{p,m}_{\lambda,\ell}(a,c+1,\mu) f(\zeta). \tag{9}$$

Also,

$$\zeta\left(I_{\lambda,\ell}^{p,m+1}(a,c,\mu)f(\zeta)\right)' = \frac{\ell}{\lambda}I_{\lambda,\ell}^{p,m}(a,c,\mu)f(\zeta) - \frac{\ell+\lambda p}{\lambda}I_{\lambda,\ell}^{p,m+1}(a,c,\mu)f(\zeta). \quad (10)$$

Lemma 2 ([2])**.** *Let the function $q(\zeta)$ be univalent in the unit disc Δ and let θ and φ be analytic in a domain D containing $q(\Delta)$ with $q(w) \neq 0$ for all $w \in q(\Delta)$. Set $Q(\zeta) = \zeta q'(\zeta)\varphi(q(\zeta))$ and $h(\zeta) = \theta(q(\zeta)) + Q(\zeta)$. Suppose that (i) $Q(\zeta)$ is starlike and univalent in Δ;*

(ii) $Re\left\{\frac{\zeta h'(\zeta)}{Q(\zeta)}\right\} > 0$ for $\zeta \in \Delta$. If p is analytic with $p(0) = q(0)$, $p(\Delta) \subseteq D$ and

$$\theta(p(\zeta)) + \zeta p'(\zeta)\varphi(p(\zeta)) \prec \theta(q(\zeta)) + \zeta q'(\zeta)\varphi(q(\zeta)), \quad (11)$$

then

$$p(\zeta) \prec q(\zeta) \;\; (\zeta \in \Delta), \quad (12)$$

and $q(\zeta)$ is the best dominant.

Lemma 3 ([16])**.** *Let q be a convex univalent function in Δ and let $\delta \in \mathbb{C}$, $\gamma \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ with*

$$Re\left\{1 + \frac{\zeta q''(\zeta)}{q'(\zeta)}\right\} > \max\left\{0, -Re\left\{\frac{\delta}{\gamma}\right\}\right\}. \quad (13)$$

If $p(\zeta)$ is analytic in Δ with $p(0) = q(0)$ and

$$\delta p(\zeta) + \gamma\zeta p'(\zeta) \prec \delta q(\zeta) + \gamma\zeta q'(\zeta), \quad (14)$$

then

$$p(\zeta) \prec q(\zeta) \;\; (\zeta \in \Delta), \quad (15)$$

and $q(\zeta)$ is the best dominant.

In recent years, there has been an increase in interest in research concerning meromorphic function classes. Ali et al. [17] extended the concept of subordination from fuzzy set theory to the geometry theory of analytic functions, clarifying the concept and demonstrating its basic properties. Furthermore, Kota and El-Ashwah [18] demonstrated various subordination features for meromorphic functions analytic in the punctured unit disc with a simple pole at the origin. Their research was coupled with two integral operators, from which conclusions and numerical examples were derived. Moreover Ali et al. [19] used the q-binomial theorem to introduce and study two subclasses of meromorphic functions. They provided inclusion relations and investigated an integral operator that preserves functions from these function classes. They also established a strict inequality involving a specific linear convolution operator.

Symmetry plays a fundamental role in computational science, especially in the geometric function theory of complex analysis. In order to highlight this role, we recall the function

$$\Phi(z) = \frac{1+Az}{1+Bz},$$

where $-1 \leq B < A \leq 1$. The function Φ is a convex function, and also Φ maps the open unit U conformally onto a disc symmetrical with respect to the real axis, which is centered at the point $\frac{1-AB}{1-B^2}$ $(B \neq \pm 1)$, and with a radius equal to $\frac{A-B}{1-B^2}$ $(B \neq \pm 1)$. Furthermore, the boundary circle of the disc intersects the real axis at the points $\frac{1-A}{1-B}$ and $\frac{1+A}{1+B}$ provided $B \neq \pm 1$. This symmetric function opened the door for great points of research on the topic of geometric function theory. We refer to the well-known starlike and convex functions conditions, which were introduced in 1973 by Janowski [20]

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz},$$

and
$$1+\frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz}.$$

There are many studies dealing with symmetric functions, cosine function [21], secant function [22], Balloon function [23], and many others. In this paper, we applied the symmetry of the function Φ to obtain several corollaries.

The essential idea is to find many adequate conditions for the function $f \in \Sigma_p$ and for a suitable univalent function q in Δ, under which various subordination conclusions hold. In many corollaries, we also presented a novel set of special instances based on those results.

3. Subordination Results

For brevity, assume throughout the remainder of the paper that $-1 \leq B < A \leq 1$, $0 \leq \alpha < p$, $\lambda > 0$, $\ell > 0$, $\mu > 0$, $a, c \in \mathbb{C}$, $Re\{a\} > p\mu$, $Re\{c-a\} \geq 0$, $p \in \mathbb{N}$, $m \in \mathbb{N}_0$, $\zeta \in \Delta$ and the powers are principal. The first result is found by investigating some sharp subordination results related to the operator $I^{p,m}_{\lambda,\ell}(a,c,\mu)$.

Theorem 1. *Let $\xi \in \mathbb{C}^*$, $f \in \Sigma_p$, and the function q be univalent and convex in Δ with $q(0) = 1$. Suppose f and q satisfy any one of the following pairs of inequalities:*
$$Re\left\{1+\frac{\zeta q''(\zeta)}{q'(\zeta)}\right\} > \max\left\{0, -\frac{p}{\mu}Re\left\{\frac{a-p\mu}{\xi}\right\}\right\}, \tag{16}$$
$$\frac{\xi}{p}\left(\zeta^p I^{p,m}_{\lambda,\ell}(a+1,c,\mu)f(\zeta)\right)+\frac{p-\xi}{p}\left(\zeta^p I^{p,m}_{\lambda,\ell}(a,c,\mu)f(\zeta)\right) \prec q(\zeta)+\frac{\mu\xi}{p(a-p\mu)}\zeta q'(\zeta), \tag{17}$$
or
$$Re\left\{1+\frac{\zeta q''(\zeta)}{q'(\zeta)}\right\} > \max\left\{0, -\frac{p}{\mu}Re\left\{\frac{c-p\mu-1}{\xi}\right\}\right\}, \tag{18}$$
$$\frac{\xi}{p}\left(\zeta^p I^{p,m}_{\lambda,\ell}(a,c-1,\mu)f(\zeta)\right)+\frac{p-\xi}{p}\left(\zeta^p I^{p,m}_{\lambda,\ell}(a,c,\mu)f(\zeta)\right) \prec q(\zeta)+\frac{\mu\xi}{p(c-p\mu-1)}\zeta q'(\zeta), \tag{19}$$
or
$$Re\left\{1+\frac{\zeta q''(\zeta)}{q'(\zeta)}\right\} > \max\left\{0, -\frac{p\ell}{\lambda}Re\left\{\frac{1}{\xi}\right\}\right\}, \tag{20}$$
$$\frac{\xi}{p}\left(\zeta^p I^{p,m-1}_{\lambda,\ell}(a,c,\mu)f(\zeta)\right)+\frac{p-\xi}{p}\left(\zeta^p I^{p,m}_{\lambda,\ell}(a,c,\mu)f(\zeta)\right) \prec q(\zeta)+\frac{\lambda\xi}{\ell p}\zeta q'(\zeta). \tag{21}$$
Then,
$$\zeta^p I^{p,m}_{\lambda,\ell}(a,c,\mu)f(\zeta) \prec q(\zeta), \tag{22}$$
and $q(\zeta)$ is the best dominant of (22).

Proof. Let
$$k(\zeta) = \zeta^p I^{p,m}_{\lambda,\ell}(a,c,\mu)f(\zeta), \tag{23}$$
and then it is easy to show that k is analytic in Δ and $k(0) = 1$. Differentiating both sides of (23) with respect to ζ, followed by applications of the identities (8), (9) and (10), will yield, respectively,
$$\zeta^p I^{p,m}_{\lambda,\ell}(a+1,c,\mu)f(\zeta) = k(\zeta)+\frac{\mu}{a-p\mu}\zeta k'(\zeta), \tag{24}$$
$$\zeta^p I^{p,m}_{\lambda,\ell}(a,c-1,\mu)f(\zeta) = k(\zeta)+\frac{\mu}{c-p\mu-1}\zeta k'(\zeta), \tag{25}$$
and
$$\zeta^p I^{p,m-1}_{\lambda,\ell}(a,c,\mu)f(\zeta) = k(\zeta)+\frac{\lambda}{\ell}\zeta k'(\zeta). \tag{26}$$

Now, the subordination conditions (17), (19) and (21) are respectively equivalent to

$$k(\zeta)+\frac{\mu\xi}{p(a-p\mu)}\zeta k'(\zeta)\prec q(\zeta)+\frac{\mu\xi}{p(a-p\mu)}\zeta q'(\zeta), \tag{27}$$

$$k(\zeta)+\frac{\mu\xi}{p(c-p\mu-1)}\zeta k'(\zeta)\prec q(\zeta)+\frac{\mu\xi}{p(c-p\mu-1)}\zeta q'(\zeta), \tag{28}$$

and

$$k(\zeta)+\frac{\xi\lambda}{p\ell}\zeta k'(\zeta)\prec q(\zeta)+\frac{\xi\lambda}{p\ell}\zeta q'(\zeta). \tag{29}$$

Therefore, by applying Lemma 3 to each of the subordination conditions, (27), (28) and (29), with appropriate choices of δ and γ, we obtain assertion (22) of Theorem 1. Then, the proof of Theorem 1 can be achieved. □

Putting $q(\zeta)=\frac{1+A\zeta}{1+B\zeta}$ into Theorem 1, we obtain the following corollary:

Corollary 1. *Let $\xi\in\mathbb{C}^*$. Let the function $f\in\Sigma_p$. Suppose any one of the following pairs of conditions are satisfied:*

$$\tfrac{|B|-1}{|B|+1}<\frac{p}{\mu}Re\Big\{\tfrac{a-p\mu}{\xi}\Big\}, \tag{30}$$

$$\tfrac{\xi}{p}\Big(\zeta^p I^{p,m}_{\lambda,\ell}(a+1,c,\mu)f(\zeta)\Big)+\tfrac{p-\xi}{p}\Big(\zeta^p I^{p,m}_{\lambda,\ell}(a,c,\mu)f(\zeta)\Big)\prec\tfrac{1+A\zeta}{1+B\zeta}+\tfrac{\mu\xi}{p(a-p\mu)}\tfrac{(A-B)\zeta}{(1+B\zeta)^2}, \tag{31}$$

or

$$\tfrac{|B|-1}{|B|+1}<\frac{p}{\mu}Re\Big\{\tfrac{c-p\mu-1}{\xi}\Big\}, \tag{32}$$

$$\tfrac{\xi}{p}\Big(\zeta^p I^{p,m}_{\lambda,\ell}(a,c-1,\mu)f(\zeta)\Big)+\tfrac{p-\xi}{p}\Big(\zeta^p I^{p,m}_{\lambda,\ell}(a,c,\mu)f(\zeta)\Big)\prec\tfrac{1+A\zeta}{1+B\zeta}+\tfrac{\mu\xi}{p(c-p\mu-1)}\tfrac{(A-B)\zeta}{(1+B\zeta)^2}, \tag{33}$$

or

$$\tfrac{|B|-1}{|B|+1}<\frac{p\ell}{\lambda}Re\Big\{\frac{1}{\xi}\Big\}, \tag{34}$$

$$\tfrac{\xi}{p}\Big(\zeta^p I^{p,m-1}_{\lambda,\ell}(a,c,\mu)f(\zeta)\Big)+\tfrac{p-\xi}{p}\Big(\zeta^p I^{p,m}_{\lambda,\ell}(a,c,\mu)f(\zeta)\Big)\prec\tfrac{1+A\zeta}{1+B\zeta}+\tfrac{\lambda\xi}{\ell p}\tfrac{(A-B)\zeta}{(1+B\zeta)^2}. \tag{35}$$

Then,

$$\zeta^p I^{p,m}_{\lambda,\ell}(a,c,\mu)f(\zeta)\prec\tfrac{1+A\zeta}{1+B\zeta}, \tag{36}$$

and $\frac{1+A\zeta}{1+B\zeta}$ is the best dominant of (36).

Proof. Upon setting $q(\zeta)=\dfrac{1+A\zeta}{1+B\zeta}$, we see that

$$1+\frac{\zeta q''(\zeta)}{q'(\zeta)}=\frac{1-B\zeta}{1+B\zeta},$$

then, we obtain

$$Re\Big\{1+\frac{\zeta q''(\zeta)}{q'(\zeta)}\Big\}>\frac{1-|B|}{1+|B|}\quad(\zeta\in\Delta).$$

Consequently, the hypotheses (30), (32) and (34) imply the conditions (16), (18), and (20), respectively, of Theorem 1. Therefore, assertion (36) follows from Theorem 1. The proof of Corollary 1 is complete. □

Taking $p=A=1$ and $B=-1$ in Corollary 1, we can obtain the following corollary.

Corollary 2. *Let $\xi \in \mathbb{C}^*$. Let the function $f \in \Sigma$. Suppose any one of the following pairs of conditions are satisfied:*

$$Re\left\{\frac{a-\mu}{\xi}\right\} > 0, \tag{37}$$

$$\xi\left(\zeta I^m_{\lambda,\ell}(a+1,c,\mu)f(\zeta)\right)+(1-\xi)\left(\zeta I^m_{\lambda,\ell}(a,c,\mu)f(\zeta)\right) \prec \frac{1+\zeta}{1-\zeta}+\frac{\mu\xi}{a-\mu}\frac{2\zeta}{(1-\zeta)^2}, \tag{38}$$

or

$$Re\left\{\frac{c-\mu-1}{\xi}\right\} > 0, \tag{39}$$

$$\xi\left(\zeta I^m_{\lambda,\ell}(a,c-1,\mu)f(\zeta)\right)+(1-\xi)\left(\zeta I^m_{\lambda,\ell}(a,c,\mu)f(\zeta)\right) \prec \frac{1+\zeta}{1-\zeta}+\frac{\mu\xi}{c-\mu-1}\frac{2\zeta}{(1-\zeta)^2}, \tag{40}$$

or

$$Re\left\{\frac{1}{\xi}\right\} > 0, \tag{41}$$

$$\xi\left(\zeta I^{m-1}_{\lambda,\ell}(a,c,\mu)f(\zeta)\right)+(1-\xi)\left(\zeta I^m_{\lambda,\ell}(a,c,\mu)f(\zeta)\right) \prec \frac{1+\zeta}{1-\zeta}+\frac{\lambda\xi}{\ell}\frac{2\zeta}{(1-\zeta)^2}. \tag{42}$$

Then,

$$\zeta I^m_{\lambda,\ell}(a,c,\mu)f(\zeta) \prec \frac{1+\zeta}{1-\zeta}, \tag{43}$$

and $\frac{1+\zeta}{1-\zeta}$ is the best dominant of (43).

Taking $a = c$ and $m = 0$ in Corollary 2, we can obtain the following corollary.

Corollary 3. *Let $\xi \in \mathbb{C}^*$. Let the function $f \in \Sigma$. Suppose any one of the following pairs of conditions are satisfied:*

$$Re\left\{\frac{a-\mu}{\xi}\right\} > 0, \tag{44}$$

$$\frac{\mu\xi}{a-\mu}\zeta(\zeta f(\zeta))' + \zeta f(\zeta) \prec \frac{1+\zeta}{1-\zeta}+\frac{\mu\xi}{a-\mu}\frac{2\zeta}{(1-\zeta)^2}, \tag{45}$$

or

$$Re\left\{\frac{c-\mu-1}{\xi}\right\} > 0, \tag{46}$$

$$\frac{\mu\xi}{c-\mu-1}\zeta(\zeta f(\zeta))' + \zeta f(\zeta) \prec \frac{1+\zeta}{1-\zeta}+\frac{\mu\xi}{c-\mu-1}\frac{2\zeta}{(1-\zeta)^2}, \tag{47}$$

or

$$Re\left\{\frac{1}{\xi}\right\} > 0, \tag{48}$$

$$\frac{\lambda\xi}{\ell}\zeta(\zeta f(\zeta))' + \zeta f(\zeta) \prec \frac{1+\zeta}{1-\zeta}+\frac{\lambda\xi}{\ell}\frac{2\zeta}{(1-\zeta)^2}. \tag{49}$$

Then,

$$\zeta f(\zeta) \prec \frac{1+\zeta}{1-\zeta}, \tag{50}$$

and $\frac{1+\zeta}{1-\zeta}$ is the best dominant of (50).

Also, we can introduce another subordination theorem, as follows.

Theorem 2. *Let $q(\zeta)$ be a non-zero univalent function in Δ with $q(0) = 1$. Let $\eta \in \mathbb{C}^*$ and $\tau, \varkappa \in \mathbb{C}$ with $\tau + \varkappa \neq 0$. Let $f \in \Sigma_p$ and suppose that f and q satisfy the conditions*

$$\frac{\tau\zeta^p I_{\lambda,\ell}^{p,m}(a+1,c,\mu)f(\zeta)+\varkappa\zeta^p I_{\lambda,\ell}^{p,m}(a,c,\mu)f(\zeta)}{\tau+\varkappa} \neq 0 \ \ (\zeta \in \Delta),$$

and

$$Re\left\{1+\frac{\zeta q''(\zeta)}{q'(\zeta)}-\frac{\zeta q'(\zeta)}{q(\zeta)}\right\} > 0 \ \ (\zeta \in \Delta). \tag{51}$$

If

$$\eta\left[p+\frac{\tau\zeta\left(I_{\lambda,\ell}^{p,m}(a+1,c,\mu)f(\zeta)\right)'+\varkappa\zeta\left(I_{\lambda,\ell}^{p,m}(a,c,\mu)f(\zeta)\right)'}{\tau I_{\lambda,\ell}^{p,m}(a+1,c,\mu)f(\zeta)+\varkappa I_{\lambda,\ell}^{p,m}(a,c,\mu)f(\zeta)}\right] \prec \frac{\zeta q'(\zeta)}{q(\zeta)}, \tag{52}$$

then

$$\left[\frac{\tau\zeta^p I_{\lambda,\ell}^{p,m}(a+1,c,\mu)f(\zeta)+\varkappa\zeta^p I_{\lambda,\ell}^{p,m}(a,c,\mu)f(\zeta)}{\tau+\varkappa}\right]^\eta \prec q(\zeta), \tag{53}$$

and $q(\zeta)$ is the best dominant of (53).

Proof. In view of Lemma 2, we set

$$\theta(w) = 0 \ \text{ and } \ \varphi(w) = \frac{1}{w}.$$

Thus,

$$Q(\zeta) = \zeta q'(\zeta)\varphi(q(\zeta)) = \frac{\zeta q'(\zeta)}{q(\zeta)} \ \text{ and } \ h(\zeta) = Q(\zeta).$$

According to hypothesis (51), we note that $Q(\zeta)$ is univalent; moreover,

$$Re\left\{\frac{\zeta Q'(\zeta)}{Q(\zeta)}\right\} = Re\left\{\frac{\zeta\left(\frac{\zeta q'(\zeta)}{q(\zeta)}\right)'}{\frac{\zeta q'(\zeta)}{q(\zeta)}}\right\} = Re\left\{1+\frac{\zeta q''(\zeta)}{q'(\zeta)}-\frac{\zeta q'(\zeta)}{q(\zeta)}\right\} > 0 \ \ (\zeta \in \Delta),$$

and then function $Q(\zeta)$ is also starlike in Δ. We can furthermore find that

$$Re\left\{\frac{\zeta h'(\zeta)}{Q(\zeta)}\right\} > 0 \ \ (\zeta \in \Delta).$$

Next, let the function p be defined by

$$p(\zeta) = \left[\frac{\tau\zeta^p I_{\lambda,\ell}^{p,m}(a+1,c,\mu)f(\zeta)+\varkappa\zeta^p I_{\lambda,\ell}^{p,m}(a,c,\mu)f(\zeta)}{\tau+\varkappa}\right]^\eta \ \ (\zeta \in \Delta). \tag{54}$$

Then, p is analytic in Δ, $p(0) = q(0) = 1$ and

$$\frac{\zeta p'(\zeta)}{p(\zeta)} = \eta\left[p+\frac{\tau\zeta\left(I_{\lambda,\ell}^{p,m}(a+1,c,\mu)f(\zeta)\right)'+\varkappa\zeta\left(I_{\lambda,\ell}^{p,m}(a,c,\mu)f(\zeta)\right)'}{\tau I_{\lambda,\ell}^{p,m}(a+1,c,\mu)f(\zeta)+\varkappa I_{\lambda,\ell}^{p,m}(a,c,\mu)f(\zeta)}\right]. \tag{55}$$

Using (55) in (52), we have

$$\frac{\zeta p'(\zeta)}{p(\zeta)} \prec \frac{\zeta q'(\zeta)}{q(\zeta)},$$

which is also equivalent to

$$\zeta p'(\zeta)\varphi(p(\zeta)) \prec \zeta q'(\zeta)\varphi(q(\zeta)),$$

or

$$\theta(p(\zeta)) + \zeta p'(\zeta)\varphi(p(\zeta)) \prec \theta(q(\zeta)) + \zeta q'(\zeta)\varphi(q(\zeta)).$$

Therefore, according to Lemma 2, we have

$$p(\zeta) \prec q(\zeta),$$

and $q(\zeta)$ is the best dominant. This is precisely the assertion in (53). The proof of Theorem 2 is complete. □

Taking $\tau = 0$, $\varkappa = 1$ and $q(\zeta) = \frac{1+A\zeta}{1+B\zeta}$ in Theorem 2, we can obtain the following corollary.

Corollary 4. *Let $\eta \in \mathbb{C}^*$. Let $f \in \Sigma_p$ and suppose that f satisfies the following conditions:*

$$\zeta^p I_{\lambda,\ell}^{p,m}(a,c,\mu)f(\zeta) \neq 0 \ \ (\zeta \in \Delta),$$

if

$$\eta\left[p + \frac{\zeta\left(I_{\lambda,\ell}^{p,m}(a,c,\mu)f(\zeta)\right)'}{I_{\lambda,\ell}^{p,m}(a,c,\mu)f(\zeta)}\right] \prec \frac{(A-B)\zeta}{(1+A\zeta)(1+B\zeta)}, \tag{56}$$

then

$$\left[\zeta^p I_{\lambda,\ell}^{p,m}(a,c,\mu)f(\zeta)\right]^\eta \prec \frac{1+A\zeta}{1+B\zeta}, \tag{57}$$

and $\frac{1+A\zeta}{1+B\zeta}$ is the best dominant of (57).

Taking $p = A = 1$ and $B = -1$ in Corollary 4, we can obtain the following corollary:

Corollary 5. *Let $\eta \in \mathbb{C}^*$. Let $f \in \Sigma$ and suppose that f satisfies the following conditions:*

$$\zeta I_{\lambda,\ell}^{m}(a,c,\mu)f(\zeta) \neq 0 \ \ (\zeta \in \Delta),$$

if

$$\eta\left[1 + \frac{\zeta\left(I_{\lambda,\ell}^{m}(a,c,\mu)f(\zeta)\right)'}{I_{\lambda,\ell}^{m}(a,c,\mu)f(\zeta)}\right] \prec \frac{2\zeta}{(1-\zeta^2)}, \tag{58}$$

then

$$\left[\zeta I_{\lambda,\ell}^{m}(a,c,\mu)f(\zeta)\right]^\eta \prec \frac{1+\zeta}{1-\zeta}, \tag{59}$$

and $\frac{1+\zeta}{1-\zeta}$ is the best dominant of (59).

Taking $a = c$, $\eta = 1$ and $m = 0$ in Corollary 5, we can obtain the following corollary:

Corollary 6. *Let $f \in \Sigma$ and suppose that f satisfies the following conditions:*

$$\zeta f(\zeta) \neq 0 \ \ (\zeta \in \Delta),$$

if

$$1 + \frac{\zeta f'(\zeta)}{f(\zeta)} \prec \frac{2\zeta}{(1-\zeta^2)}, \tag{60}$$

then

$$\zeta f(\zeta) \prec \frac{1+\zeta}{1-\zeta}, \tag{61}$$

and $\dfrac{1+\zeta}{1-\zeta}$ *is the best dominant of (61).*

Taking $\tau = 1$, $\varkappa = 0$ and $q(\zeta) = \dfrac{1+A\zeta}{1+B\zeta}$ in Theorem 2, we can obtain the following corollary.

Corollary 7. *Let $\eta \in \mathbb{C}^*$. Let $f \in \Sigma_p$ and suppose that f satisfies the following conditions:*

$$\zeta^p I_{\lambda,\ell}^{p,m}(a{+}1,c,\mu)f(\zeta) \neq 0 \ \ (\zeta \in \Delta),$$

if

$$\eta\left[p + \frac{\zeta\left(I_{\lambda,\ell}^{p,m}(a+1,c,\mu)f(\zeta)\right)'}{I_{\lambda,\ell}^{p,m}(a+1,c,\mu)f(\zeta)}\right] \prec \frac{(A-B)\zeta}{(1+A\zeta)(1+B\zeta)}, \tag{62}$$

then

$$\left[\zeta^p I_{\lambda,\ell}^{p,m}(a+1,c,\mu)f(\zeta)\right]^\eta \prec \frac{1+A\zeta}{1+B\zeta}, \tag{63}$$

and $\frac{1+A\zeta}{1+B\zeta}$ *is the best dominant of (63).*

Taking $A = p = 1$ and $B = -1$ in Corollary 7, we can obtain the following corollary.

Corollary 8. *Let $\eta \in \mathbb{C}^*$. Let $f \in \Sigma$ and suppose that f satisfies the following conditions:*

$$\zeta I_{\lambda,\ell}^{m}(a{+}1,c,\mu)f(\zeta) \neq 0 \ \ (\zeta \in \Delta),$$

if

$$\eta\left[1 + \frac{\zeta\left(I_{\lambda,\ell}^{m}(a+1,c,\mu)f(\zeta)\right)'}{I_{\lambda,\ell}^{m}(a+1,c,\mu)f(\zeta)}\right] \prec \frac{2\zeta}{(1-\zeta^2)}, \tag{64}$$

then

$$\left[\zeta I_{\lambda,\ell}^{m}(a+1,c,\mu)f(\zeta)\right]^\eta \prec \frac{1+\zeta}{1-\zeta}, \tag{65}$$

and $\frac{1+\zeta}{1-\zeta}$ *is the best dominant of (65).*

Taking $a = c$, $\eta = 1$ and $m = 0$ in Corollary 8, we can obtain the following corollary.

Corollary 9. *Let $f \in \Sigma$ and suppose that f satisfies the following conditions:*

$$\zeta^2 f'(\zeta) + \frac{a}{\mu}\zeta f(\zeta) \neq 0 \ \ (\zeta \in \Delta),$$

if

$$1 + \frac{\zeta\left(\zeta^2 f'(\zeta) + \frac{a}{\mu}\zeta f(\zeta)\right)'}{\zeta^2 f'(\zeta) + \frac{a}{\mu}\zeta f(\zeta)} \prec \frac{2\zeta}{(1-\zeta^2)}, \tag{66}$$

then

$$\frac{\mu}{a-\mu}\left(\zeta^2 f'(\zeta) + \frac{a}{\mu}\zeta f(\zeta)\right) \prec \frac{1+\zeta}{1-\zeta}, \tag{67}$$

and $\frac{1+\zeta}{1-\zeta}$ *is the best dominant of (67).*

Another theorem is introduced, as follows.

Theorem 3. *Let $\eta \in \mathbb{C}^*$ and $\upsilon, \tau, \varkappa \in \mathbb{C}$ with $\tau + \varkappa \neq 0$. Let $q(\zeta)$ be a univalent function in Δ with $q(0) = 1$ and*

$$Re\left\{1 + \frac{\zeta q''(\zeta)}{q'(\zeta)}\right\} > \max\{0, -Re\{\upsilon\}\} \ \ (\zeta \in \Delta). \tag{68}$$

Let $f \in \Sigma_p$, and suppose that f satisfies the condition

$$\frac{\tau\zeta^p I_{\lambda,\ell}^{p,m}(a+1,c,\mu)f(\zeta) + \varkappa\zeta^p I_{\lambda,\ell}^{p,m}(a,c,\mu)f(\zeta)}{\tau + \varkappa} \neq 0 \ \ (\zeta \in \Delta).$$

Set

$$\begin{aligned} \Omega(\zeta) \ &= \ \left[\frac{\tau\zeta^p I_{\lambda,\ell}^{p,m}(a+1,c,\mu)f(\zeta) + \varkappa\zeta^p I_{\lambda,\ell}^{p,m}(a,c,\mu)f(\zeta)}{\tau+\varkappa}\right]^\eta \\ &\cdot \left[\upsilon + \eta\left(\frac{\tau\zeta\left(I_{\lambda,\ell}^{p,m}(a+1,c,\mu)f(\zeta)\right)' + \varkappa\zeta\left(I_{\lambda,\ell}^{p,m}(a,c,\mu)f(\zeta)\right)'}{\tau I_{\lambda,\ell}^{p,m}(a+1,c,\mu)f(\zeta) + \varkappa I_{\lambda,\ell}^{p,m}(a,c,\mu)f(\zeta)} + p\right)\right]. \end{aligned} \tag{69}$$

If

$$\Omega(\zeta) \prec \upsilon q(\zeta) + \zeta q'(\zeta), \tag{70}$$

then

$$\left[\frac{\tau\zeta^p I_{\lambda,\ell}^{p,m}(a+1,c,\mu)f(\zeta) + \varkappa\zeta^p I_{\lambda,\ell}^{p,m}(a,c,\mu)f(\zeta)}{\tau + \varkappa}\right]^\eta \prec q(\zeta), \tag{71}$$

and $q(\zeta)$ is the best dominant of (71).

Proof. In view of Lemma 2, we set

$$\theta(w) = \upsilon w \ \text{ and } \ \varphi(w) = 1 \ \ (w \in \mathbb{C}),$$

and thus

$$Q(\zeta) = \zeta q'(\zeta)\varphi(q(\zeta)) = \zeta q'(\zeta) \ \text{ and } \ h(\zeta) = \upsilon q(\zeta) + \zeta q'(\zeta).$$

Then, we note that $Q(\zeta)$ is univalent. Moreover, using (68), we find that

$$Re\left\{\frac{\zeta Q'(\zeta)}{Q(\zeta)}\right\} = Re\left\{\frac{\zeta(\zeta q'(\zeta))'}{\zeta q'(\zeta)}\right\} = Re\left\{1 + \frac{\zeta q''(\zeta)}{q'(\zeta)}\right\} > 0 \ \ (\zeta \in \Delta),$$

and then function $Q(\zeta)$ is also starlike in Δ. Also, using (68), we find that

$$Re\left\{\frac{\zeta h'(\zeta)}{Q(\zeta)}\right\} = Re\left\{1 + \upsilon + \frac{\zeta q''(\zeta)}{q'(\zeta)}\right\} > 0 \ \ (\zeta \in \Delta).$$

Furthermore, by using the expression of $p(\zeta)$ defined by (54) and the expression of $\zeta p'(\zeta)$ defined by (55), we have

$$\begin{aligned} \theta(p(\zeta)) + \zeta p'(\zeta)\varphi(p(\zeta)) \ &= \ \upsilon p(\zeta) + \zeta p'(\zeta) \\ &= \left[\frac{\tau\zeta^p I_{\lambda,\ell}^{p,m}(a+1,c,\mu)f(\zeta) + \varkappa\zeta^p I_{\lambda,\ell}^{p,m}(a,c,\mu)f(\zeta)}{\tau+\varkappa}\right]^\eta \\ &\quad \cdot \left[\upsilon + \eta\left(\frac{\tau\zeta\left(I_{\lambda,\ell}^{p,m}(a+1,c,\mu)f(\zeta)\right)' + \varkappa\zeta\left(I_{\lambda,\ell}^{p,m}(a,c,\mu)f(\zeta)\right)'}{\tau I_{\lambda,\ell}^{p,m}(a+1,c,\mu)f(\zeta) + \varkappa I_{\lambda,\ell}^{p,m}(a,c,\mu)f(\zeta)} + p\right)\right] \\ &= \ \Omega(\zeta). \end{aligned}$$

Hypothesis (70) is now equivalent to

$$vp(\zeta)+\zeta p'(\zeta)\prec vq(\zeta)+\zeta q'(\zeta),$$

or

$$\theta(p(\zeta))+\zeta p'(\zeta)\varphi(p(\zeta))\prec\theta(q(\zeta))+\zeta q'(\zeta)\varphi(q(\zeta)).$$

Finally, an application of Lemma 2 yields

$$p(\zeta)\prec q(\zeta)$$

and $q(\zeta)$ is the best dominant. This is precisely the assertion in (71). The proof of Theorem 3 is complete. □

Taking $\tau{=}0$, $\varkappa{=}1$ and $q(\zeta){=}\dfrac{1{+}A\zeta}{1{+}B\zeta}$ in Theorem 3, we can obtain the following corollary.

Corollary 10. *Let $\eta\in\mathbb{C}^*$ and $v=\frac{|B|-1}{|B|+1}$. Let $f\in\Sigma_p$ and suppose that f satisfies the conditions*

$$\zeta^p I_{\lambda,\ell}^{p,m}(a,c,\mu)f(\zeta)\neq 0\ \ (\zeta\in\Delta),$$

and

$$\left[\zeta^p I_{\lambda,\ell}^{p,m}(a,c,\mu)f(\zeta)\right]^{\eta}\cdot\left[v{+}\eta\left(p{+}\frac{\zeta\left(I_{\lambda,\ell}^{p,m}(a,c,\mu)f(\zeta)\right)'}{I_{\lambda,\ell}^{p,m}(a,c,\mu)f(\zeta)}\right)\right]\prec v\frac{1{+}A\zeta}{1{+}B\zeta}+\frac{(A{-}B)\zeta}{(1{+}B\zeta)^2},\tag{72}$$

and then

$$\left[\zeta^p I_{\lambda,\ell}^{p,m}(a,c,\mu)f(\zeta)\right]^{\eta}\prec\frac{1{+}A\zeta}{1{+}B\zeta},\tag{73}$$

and $\dfrac{1{+}A\zeta}{1{+}B\zeta}$ is the best dominant of (73).

Taking $p=A=1$, $B=-1$ and $a=c$ in Corollary 10, we can obtain the following corollary.

Corollary 11. *Let $\eta\in\mathbb{C}^*$. Let $f\in\Sigma$ and suppose that f satisfies the conditions*

$$\zeta f(\zeta)\neq 0\ \ (\zeta\in\Delta),$$

and

$$\left[\zeta f(\zeta)\right]^{\eta}\cdot\left[\eta\left(1+\frac{\zeta f'(\zeta)}{f(\zeta)}\right)\right]\prec\frac{2\zeta}{(1-\zeta)^2},\tag{74}$$

and then

$$\left[\zeta f(\zeta)\right]^{\eta}\prec\frac{1+\zeta}{1-\zeta},\tag{75}$$

and $\dfrac{1{+}\zeta}{1{-}\zeta}$ is the best dominant of (75).

Remark 1. *The result obtained in Corollary 11 coincides with the recent result of Mishra et al. ([24], Corollary 4.9).*

Taking $\eta=1$ in Corollary 11, we can obtain the following corollary.

Corollary 12. *Let $f\in\Sigma$ and suppose that f satisfies the conditions*

$$\zeta f(\zeta)\neq 0\ \ (\zeta\in\Delta),\tag{76}$$

and

$$\zeta f(\zeta)+\zeta^2 f'(\zeta)\prec\frac{2\zeta}{(1-\zeta)^2}, \tag{77}$$

and then

$$\zeta f(\zeta)\prec\frac{1+\zeta}{1-\zeta}, \tag{78}$$

and $\frac{1+\zeta}{1-\zeta}$ *is the best dominant of (78).*

Taking $\tau=1$, $\varkappa=0$ and $q(\zeta)=\frac{1+A\zeta}{1+B\zeta}$ in Theorem 3, we can obtain the following corollary.

Corollary 13. *Let* $\eta\in\mathbb{C}^*$ *and* $\upsilon=\frac{|B|-1}{|B|+1}$*. Let* $f\in\Sigma_p$ *and suppose that* f *satisfies the conditions*

$$\zeta^p I^{p,m}_{\lambda,\ell}(a+1,c,\mu)f(\zeta)\neq 0\ \ (\zeta\in\Delta), \tag{79}$$

and

$$\left[\zeta^p I^{p,m}_{\lambda,\ell}(a+1,c,\mu)f(\zeta)\right]^{\eta}\cdot\left[\upsilon+\eta\left(\frac{\zeta\left(I^{p,m}_{\lambda,\ell}(a+1,c,\mu)f(\zeta)\right)'}{I^{p,m}_{\lambda,\ell}(a+1,c,\mu)f(\zeta)}+p\right)\right]\prec\upsilon\frac{1+A\zeta}{1+B\zeta}+\frac{(A-B)\zeta}{(1+B\zeta)^2}, \tag{80}$$

and then

$$\left[\zeta^p I^{p,m}_{\lambda,\ell}(a+1,c,\mu)f(\zeta)\right]^{\eta}\prec\frac{1+A\zeta}{1+B\zeta}, \tag{81}$$

and $q(\zeta)$ *is the best dominant of (81).*

Taking $p=A=\eta=1$, $B=-1$ and $a=c$ in Corollary 13, we obtain the following corollary.

Corollary 14. *Let* $f\in\Sigma$ *and suppose that* f *satisfies the conditions*

$$\zeta^2 f'(\zeta)+\frac{a}{\mu}\zeta f(\zeta)\neq 0\ \ (\zeta\in\Delta), \tag{82}$$

and

$$\frac{\mu\zeta}{a-\mu}\left(\zeta\left[\zeta f'(\zeta)+\frac{a}{\mu}f(\zeta)\right]\right)'\prec\frac{2\zeta}{(1-\zeta)^2} \tag{83}$$

then

$$\frac{\mu}{a-\mu}\left(\zeta^2 f'(\zeta)+\frac{a}{\mu}\zeta f(\zeta)\right)\prec\frac{1+\zeta}{1-\zeta}, \tag{84}$$

and $\frac{1+\zeta}{1-\zeta}$ *is the best dominant of (84).*

Remark 2. *Specializing the parameters in Theorems 1–3, as mentioned before, we can obtain the corresponding subordination properties of the Cho–Kwon–Srivastava operator [9], the Liu–Srivastava operator [6], the Uralegaddi–Somanatha operator [11], the Yuan–Liu–Srivastava operator [10], and others.*

4. Conclusions

This study investigates subordination results for p-valent meromorphic functions on the punctured unit disc of the complex plane. These functions have a p-pole. The subclass being explored is defined using a new linear operator. In addition, we gave a few corollaries with fascinating specific cases from the results. By specializing the parameters in Theorems 1–3, we could obtain the equivalent subordination bounds related to other operators in the space of meromorphic functions.

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