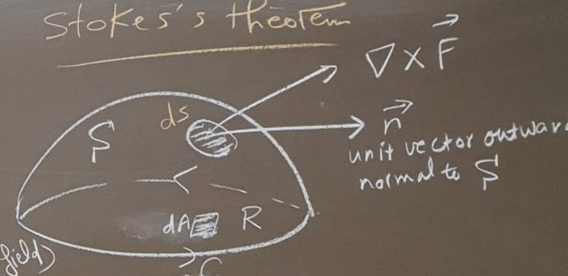
$$\iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_Q \text{DIV}\,\vec{F}\,dV$$

$\Rightarrow$ Div. th.

Stokes's theorem

$$\underbrace{\iint_S \overbrace{(\nabla\times\vec{F})}^{}_{\text{curl }\vec{F}}\cdot\vec{n}\,ds}_{} = \underbrace{\oint_C \vec{F}\cdot d\vec{r}}_{\downarrow}$$



the Flux of curl $\vec{F}$ over S = work done (if $\vec{F}$ is a force field)

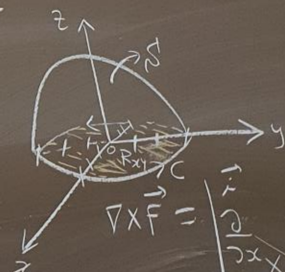
Ex let S be the part of the paraboloid $z = 9 - x^2 - y^2$, $z \geq 0$ and let C be the trace of S on xy-plane. Verify Stokes's th. for the vector field $\vec{F} = 3z\vec{i} + 4x\vec{j} + 2y\vec{k}$

Ans: S is the graph of Eqn $g(x,y,z)=0$

where $g(x,y,z) = z - (9 - x^2 - y^2)$
$= x^2 + y^2 + z - 9$

$$\Rightarrow \vec{n} = \frac{\nabla g(x,y,z)}{\|\nabla g\|}$$

$$\vec{n} = \frac{2x\vec{i} + 2y\vec{j} + \vec{k}}{\sqrt{4x^2+4y^2+1}}$$



$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3z & 4x & 2y \end{vmatrix}$$

rd

$$\nabla \times \vec{F} = 2\vec{i} + 3\vec{j} + 4\vec{k}$$

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS$$

$$= \iint_S \frac{4x + 6y + 4}{\sqrt{4x^2 + 4y^2 + 1}}\, dS$$

$$z = f(x,y)$$

$$\Rightarrow f(x,y) = 9 - x^2 - y^2$$

$$\Rightarrow f_x = -2x, \quad f_y = -2y$$

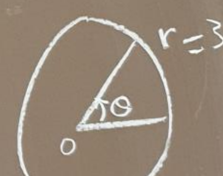
$$\Rightarrow \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS$$

$$= \iint_{R_{xy}} \frac{4x + 6y + 4}{\cancel{\sqrt{4x^2 + 4y^2 + 1}}} \cancel{\sqrt{4x^2 + 4y^2 + 1}}\, dA \quad \begin{array}{l} 0 \le r \le 3 \\ 0 \le \theta \le 2\pi \end{array}$$

$$= \iint_{R_{xy}} (4x + 6y + 4)\, dA$$

The projection R_{xy} is the circle $x^2 + y^2 = 9$, $\boxed{z = 0}$

$\Rightarrow$



$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, ds$$

$$= \int_0^{2\pi}\int_0^3 (4r\cos\theta + 6r\sin\theta + 4)\, r\,dr\,d\theta$$

(polar coords)

$$= \int_0^{2\pi}\int_0^3 (4r^2\cos\theta + 6r^2\sin\theta + 4r)\, dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{4r^3}{3}\cos\theta + \frac{6r^3}{3}\sin\theta + 2r^2\right]_0^3 d\theta$$

$$= \int_0^{2\pi} (36\cos\theta + 54\sin\theta + 18)\, d\theta$$

$$= \left[\underbrace{36\cancel{\sin\theta}}_{0} \underbrace{- 54\cos\theta}_{-\cancel{54}+\cancel{54}} + 18\theta\right]_0^{2\pi}$$

$$= 36\pi \quad (1)$$

$$\oint_C \vec{F} \cdot d\vec{r}$$

$\vec{F} = 3z\vec{i} + 4x\vec{j} + 2y\vec{k}$ | at $z=0$ $\Rightarrow$ $\vec{F} = 4x\vec{j} + 2y\vec{k}$

متجه الموضع position vector $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ | along C $\quad d\vec{r} = dx\vec{i} + dy\vec{j}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\Rightarrow \vec{F} \cdot d\vec{r} = 4x\,dy$ along the curve C

polar coords
$$\boxed{\begin{aligned} x &= 3\cos\theta \\ y &= 3\sin\theta \Rightarrow dy = 3\cos\theta\,d\theta \end{aligned}}$$

$$\Rightarrow \oint_C \vec{F} \cdot d\vec{r} = 4\oint_C x\,dy$$

$$= 4\int_0^{2\pi} 3\cos\theta\,(3\cos\theta)\,d\theta$$

$$= 36\int_0^{2\pi} \cos^2\theta\,d\theta$$

$$= \frac{36}{2}\int_0^{2\pi} (1+\cos 2\theta)\,d\theta$$

$$\oint_C \vec{F} \cdot d\vec{r} = 18\left[\theta + \cancel{\frac{\sin 2\theta}{2}}\right]_0^{2\pi} = 36\pi \quad ②$$

$$①, ② \Rightarrow \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\,ds = \oint_C \vec{F} \cdot d\vec{r}$$ Stokes's th.