7 Spectrum of linear operators

The concept of eigenvalues of matrices play fundamental role in linear algebra and is a starting point in finding canonical forms of matrices and developing functional calculus. As we saw similar theory can be developed on infinite-dimensional spaces for compact operators. However, the situation is rather more involved for general operators on infinite-dimensional spaces. In particular, many important exampples of operators have no eigenvalues at all. What is an analogue of eigenvalues for general normed spaces?

Definition 7.1. Let $A : X \to X$ be a bounded linear operator on a normed complex space X. The *spectrum* $\sigma(A)$ of A is the set of $\lambda \in \mathbb{C}$ such that the operator $A - \lambda I$ is not invertible.

In the finite-dimensional spaces, $\sigma(A)$ simply consists of the eigenvalues of A, but this notion is much richer in general In particular, one can show that $\sigma(A)$ is always non-empty.

Theorem 7.2. *The spectrum $\sigma(A)$ of any bounded linear operator A is a closed subset of contained in $|\lambda| \leq \|A\|$.*

Proof. Define the map $F : \mathbb{C} \to B(X)$ by $F(\lambda) = A - \lambda I$. We have $\|F(\lambda) - F(\mu)\| = |\lambda - \mu|$, so that F is continuous. We have $\sigma(A) = F^{-1}(B(X)\backslash G)$ where G denotes the set of invertible operators. Since G is open, we deduce that $B(X)\backslash G$ and $\sigma(A)$ are closed.

Suppose that $|\lambda| > \|A\|$. Then $\|\lambda^{-1}A\| < 1$, and it follows that $I - \lambda^{-1}A$ is invertible. Hence, $A - \lambda I = -\lambda(I - \lambda^{-1}A)$ is also invertible. This show that $\lambda \notin \sigma(A)$. □

Example 7.3.

1. Let $X = C([a, b])$, $\phi \in X$, and $A_\phi : X \to X$ be the multiplication operator: $A_\phi(f) = \phi f$. It is easy to check that if ϕ is monotone, A_ϕ has no eigenvalues. We claim that

$$\sigma(A_\phi) = \phi([a, b]).$$

Indeed, let $\lambda \in \phi([a, b])$. If B is the inverse of $A_\phi - \lambda I$, then $(A_\phi - \lambda I)B = I$, and for every $f \in X$, $(\phi - \lambda)B(f) = f$. Take $t_0 \in [a, b]$ such that $\phi(t_0) = \lambda$. Then $f(t_0) = 0$, but this cannot hold for for all $f \in X$. On the other hand, if $\lambda \notin \phi([a, b])$, then $(\phi - \lambda)^{-1}$ is continuous on $[a, b]$, and $A_{(\phi-\lambda)^{-1}}$ defines the inverse of $A_\phi - \lambda I = A_{\phi-\lambda}$.

2. Let $X = \ell^2$ and $S : (x_1, x_2, \ldots,) \mapsto (0, x_1, x_2, \ldots)$ be the shift operator on X. We claim that that S has no eigenvalues. Indeed, if $Sx = \lambda x$, then $\lambda x_1 = 0$ and $\lambda x_i - x_{i-1} = 0$ for all $i > 1$. For here we deduce that $x = 0$, so that S has no eigenvectors. Next we show that

$$\sigma(S) = \{|\lambda| \leq 1\}.$$

Since $\|S\| = 1$, it is clear that $\sigma(S)$ is contained in this set. To prove the opposite inclusion, we show that if $|\lambda| \leq 1$, then $S - \lambda I$ is not onto. When $\lambda = 0$, is easy to check that $y = (1, 0, \ldots)$ not in the image of $S - \lambda I$. Now let $\lambda \neq 0$. We suppose that there exists $x \in X$ such that $(S - \lambda I)x = y$. Then $-\lambda x_1 = 1$ and $x_{i-1} - \lambda x_i = 0$. We deduce that $x_i = -1/\lambda^i$. However, since $|\lambda| \geq 1$, this x is not in ℓ^2. Hence, $S - \lambda I$ is not onto as claimed.

Theorem 7.4. *If $A : H \to H$ is a self-adjoint operator on a Hilbert space H. Then $\sigma(H) \subset \mathbb{R}$.*

Proof. Take $\lambda = a + ib$ with $b \neq 0$. We note that since A is self-adjont $\langle Ax, x \rangle \in \mathbb{R}$. Then

$$|\langle (A - \lambda I)x, x \rangle| = |(\langle Ax, x \rangle + a\|x\|^2) + ib\|x\|^2| \geq |b|\|x\|^2.$$

Since $|\langle (A - \lambda I)x, x \rangle| \leq \|(A - \lambda I)x\|\|x\|$, we deduce that

$$\|(A - \lambda I)x\| \geq |b|\|x\| \quad \text{for all } x \in H. \tag{1}$$

In particular, $A - \lambda I$ is one-to-one.

We also claim that the image of $A - \lambda I$ is closed. Let $(A - \lambda I)x_n \to y$ for some $y \in H$. In particular, the sequence $(A - \lambda I)x_n$ is Cauchy. It follows from (1) that the sequence x_n is also Cauchy. Hence, $x_n \to x$ for some $x \in H$. Then by continuity, $Ax_n \to Ax$, so that $y = Ax$, and the image of $A - \lambda I$ is closed.

The same argument as above shows that

$$\ker(A - \bar{\lambda} I) = \ker((A - \lambda I)^*) = 0.$$

We will use that for every operator T, $\ker(T^*) = \operatorname{im}(T)^\perp$ (you can check this as an exercise). Since $\operatorname{im}(A - \lambda I)$ is closed,

$$H = \operatorname{im}(A - \lambda I) \oplus \operatorname{im}(A - \lambda I)^\perp = \operatorname{im}(A - \lambda I).$$

Hence, $A - \lambda I$ is onto.

We have shown that $A - \lambda I$ is a bijection, so that the linear map $(A - \lambda I)^{-1}$ is well defined. Moreover, it follows from (1) that it is bounded. This proves that $\lambda \notin \sigma(A)$. □

The spectral theory of linear operators plays central role in modern mathematics. In this course we are only able touch on it briefly. In conclusion we mention (without proof) that this theory can be used to prove that every bounded self-adjoint operator has a "canonical form" given by a multiplication operator.

Theorem 7.5 (spectral theorem for self-adjoint operators)**.** *Let H be a complex Hilbert space and $A : H \to H$ a bounded self-adjoint operator. Then there exist a measure space (Ω, μ) and an isomorphism $U : L^2(\Omega) \to H$ of Hilbert spaces such that*

$$A = UA_\phi U^{-1},$$

where A_ϕ is a multiplication operator $A_\phi : f \mapsto \phi f$ on $L^2(\Omega)$ for a bounded measurable function ϕ on Ω.

This is a far-reaching generalisation of the fact every self-adjoint matrix can be diagonalized.