Q1: Show that W=$\left\{\begin{bmatrix} a-b & b-a \\ c & d \end{bmatrix} | a,b,c,d \in \mathbb{R}\right\}$ is a subspace of M_{22} ($M_2(\mathbb{R})$). Then find a basis of W and dim(W).

S1: For all A,B∈W and k∈ℝ, we have that A=$\begin{bmatrix} a_1-b_1 & b_1-a_1 \\ c_1 & d_1 \end{bmatrix}$ and B=$\begin{bmatrix} a_2-b_2 & b_2-a_2 \\ c_2 & d_2 \end{bmatrix}$. So

1- 0=$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$=$\begin{bmatrix} 0-0 & 0-0 \\ 0 & 0 \end{bmatrix}$∈W and W is not empty.
2- A+B=$\begin{bmatrix} (a_1-b_1)+(a_2-b_2) & (b_1-a_1)+(b_2-a_2) \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$
=$\begin{bmatrix} (a_1+a_2)-(b_1+b_2) & (b_1+b_2)-(a_1+a_2) \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$∈W.
3- kA=$\begin{bmatrix} k(a_1-b_1) & k(b_1-a_1) \\ kc_1 & kd_1 \end{bmatrix} = \begin{bmatrix} ka_1-kb_1 & kb_1-ka_1 \\ kc_1 & kd_1 \end{bmatrix}$∈W.
1, 2 and 3 imply that W is a subspace of $M_2(\mathbb{R})$.

Now, $\begin{bmatrix} a-b & b-a \\ c & d \end{bmatrix} = \begin{bmatrix} a & -a \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -b & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$

=$a\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. So $\left\{\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ a span set for W. To check linear independence:

$a\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a-b & b-a \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. So a-b=0, b-a=0, c=0 and d=0. Hence, a=b, c=0 and d=0 which is infinitely many solutions. So the set is linearly dependent. So there is a vector which is linear combination of the others. Clearly that $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = -\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$, so we can eliminate one of them to get a new set that spans W. Suppose we eliminate the second one. Thus, we get the set $\left\{\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ which is linearly independent. Indeed, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are linearly independent since neither is a scalar multiple of the other. Also, we can't get $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ from a linear combination of the vectors , $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ since the entries that lie in first row are zeros whereas they are not in $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. Hence, $S = \left\{\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ is a basis of W and dim(W)=3.

Other way to prove that S is a basis is to convert each matrix to a row vector. So $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ will be $[1 \quad -1 \quad 0 \quad 0]$, $\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$ will be $[-1 \quad 1 \quad 0 \quad 0]$, will be $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ will be $[0 \quad 0 \quad 1 \quad 0]$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ will be $[0 \quad 0 \quad 0 \quad 1]$. Putting them as columns in a matrix and reducing them to REF as follows:

$$\begin{bmatrix}-1&1&0&0\\1&-1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\xrightarrow{1R_{12}}\begin{bmatrix}-1&1&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$\xrightarrow[R_{23}]{(-1)R_1}\begin{bmatrix}-1&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&1\end{bmatrix}\xrightarrow{R_{34}}\begin{bmatrix}-1&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

we find that the leading ones lie in the columns 1,2 and 4. So the vectors $[1 \quad -1 \quad 0 \quad 0]$, $[0 \quad 0 \quad 1 \quad 0]$ and $[0 \quad 0 \quad 0 \quad 1]$ form a basis of the column space and hence the set S forms a basis of W.

Q2: Find the values of m (if possible) such that the following system:

$$x_1 + x_2 - x_3 = 1$$
$$2x_1 + 3x_2 - 3x_3 = 1$$
$$x_1 + x_2 - mx_3 = m$$

has: (i)unique solution. (ii) infinitely many solutions. (iii) no solutions.

S2:

$$\left[\begin{array}{ccc|c}1&1&-1&1\\2&3&-3&1\\1&1&-m&m\end{array}\right]\xrightarrow[-1R13]{-2R_{12}}\left[\begin{array}{ccc|c}1&1&-1&1\\0&1&-1&-1\\0&0&-m+1&m-1\end{array}\right]$$

(i) m∈ℝ−{1} (ii) m=1. (iii) no values.

Q3: Let T: $\mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by T(x,y)=(x+y,2x+2y,3x+3y). Find bases of ker(T) and R(T) (R(T)=IM(T)). Also, find nullity(T) and rank(T).

S3: ker(T)={(x,y)∈$\mathbb{R}^2$|T(x,y)=(0,0,0)}= {(x,y)∈$\mathbb{R}^2$|x+y=0, 2x+2y=0, 3x+3y=0}. So

$$\begin{bmatrix}1&1\\2&2\\3&3\end{bmatrix}\xrightarrow[-3R13]{-2R_{12}}\begin{bmatrix}1&1\\0&0\\0&0\end{bmatrix}$$

$$y = t \in \mathbb{R}$$
$$x = -y = -t$$

⟹: ker(T)={(−t,t)|t∈ℝ}={t(−1,1)|t∈ℝ} and hence {(−1,1)} is a basis of ker(T). nullity(T)=dim(ker(T))=1.

R(T)={(x+y,2x+2y,3x+3y)|x,y∈ℝ}={(x,2x,3x)+(y,2y,3y)|x,y∈ℝ}={x(1,2,3)+y(1,2,3)|x,y∈ℝ}. So {(1,2,3)} generates R(T) and hence a basis since it is not zero. Thus, rank(T)=dim(R(T))=1.

Other way to find the basis:

The images of any basis of $\mathbb{R}^2$ will generate R(T). Take the standard basis of $\mathbb{R}^2$. So

T(1,0)=(1,2,3)=T(0,1). Now

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{bmatrix} \xrightarrow[-3R13]{-2R_{12}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So {(1,2,3)} is a basis of R(T).

Observe that: rank(T)+nullity(T)=1+1=2=dim($\mathbb{R}^2$).

Q4: Show that $A = \begin{bmatrix} -1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is diagonalizable and find a matrix P that diagonalizes A.

S4:

$$0 = |\lambda I - A| = \begin{vmatrix} \lambda+1 & -1 & +1 \\ 0 & \lambda-1 & 0 \\ 0 & -1 & \lambda \end{vmatrix} = (\lambda+1)(\lambda-1)\lambda$$

So the eigenvalues are 0, 1, −1. Since they are different, A is diagonalizable. Now, for the eigenvalue λ=0

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \xrightarrow[(-1)R_{23}]{(-1)R_{21}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{(-1)R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So x=−z=−t and y=0. Hence, {(−1,0,1)} is a basis of E_0. Now, for the eigenvalue λ=1

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{(-1)R_{31}} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{(-1)R_3} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

So x=0 and y=z=t. Hence, {(0,1,1)} is a basis of E_1. Now, for the eigenvalue λ=−1

$$\begin{bmatrix} 0 & -1 & 1 \\ 0 & -2 & 0 \\ 0 & -1 & -1 \end{bmatrix} \xrightarrow[(-1)R_{13}]{(-2)R_{12}} \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow[(-1)R_{23}]{(-1)R_1} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{(-\frac{1}{2}R_2)} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{(1)R_{21}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So x=t and y=z=0. Hence, {(1,0,0)} is a basis of E_{-1}. Therefore,

$$P = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Q5: Let $T:\mathbb{R}^2 \to \mathbb{R}^3$ be the transformation defined by:

$T(x_1,x_2)=(x_1,x_2,x_1+x_2)$.

Show that T is a linear transformation.

S5: For all real numbers x_1, x_2, y_1, y_2 and k, we have

$$T((x_1,x_2)+(y_1,y_2))=T(x_1+y_1,x_2+y_2)$$
$$=(x_1+y_1,x_2+y_2,(x_1+y_1)+(x_2+y_2))$$
$$=(x_1+y_1,x_2+y_2,(x_1+x_2)+(y_1+y_2))$$
$$=(x_1,x_2,x_1+x_2)+(y_1,y_2,y_1+y_2)$$
$$=T(x_1,x_2)+T(y_1,y_2)$$

$Also,$

$$T(k(x_1,x_2))=T((kx_1,kx_2))=(kx_1,kx_2,kx_1+kx_2)$$
$$=(kx_1,kx_2,k(x_1+x_2))=k(x_1,x_2,x_1+x_2)$$
$$=kT(x_1,x_2)$$

Q6: - Show that the following matrix is **not** diagonalizable.

$$A=\begin{bmatrix}1 & 1 & -1\\ 2 & 0 & 2\\ 3 & -3 & 5\end{bmatrix}$$

S6:

$$0=|\lambda I-A|=\begin{vmatrix}\lambda-1 & -1 & 1\\ -2 & \lambda & -2\\ -3 & 3 & \lambda-5\end{vmatrix}$$
$$=(\lambda-1)(\lambda^2-5\lambda+6)+(-2\lambda+10-6)+(-6+3\lambda)$$
$$=(\lambda-1)(\lambda-2)(\lambda-3)+(\lambda-2)$$
$$=(\lambda-2)((\lambda-1)(\lambda-3)+1)$$
$$=(\lambda-2)(\lambda^2-4\lambda+4)=(\lambda-2)(\lambda-2)^2=(\lambda-2)^3$$
$$\Rightarrow \lambda=2$$

If A is diagonalizable, then there exists an invertible matrix P such that $D=P^{-1}AP$ which is equivalent to $A=PDP^{-1}$ But $D=2I$, So $A= P(2I)P^{-1}=2PIP^{-1}=2PP^{-1}=2I$, a contradiction with the hypothesis, hence A is not diagonalizable.

Another solution: the algebraic multiplicity for $\lambda=2$ is 3. We need to find the geometric multiplicity=$\dim(E_2)$. But $\dim(E_2)=\text{nullity}(2I-A)$

$$\begin{bmatrix} 2-1 & -1 & 1 \\ -2 & 2 & -2 \\ -3 & 3 & 2-5 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 2 & -2 \\ -3 & 3 & -3 \end{bmatrix}$$

$$\xrightarrow[3R_{13}]{2R_{12}} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$nullity\,(2I - A) = 3 - 1 = 2 \neq 3$$

So the geometric multiplicity is not equal to the algebraic multiplicity, hence A is not diagonalizable.