Solutions - 1st Midterm Exam PHYS 404

1. • $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \underset{n=3}{\Longrightarrow} 4P_4(x) = 7xP_3(x) - 3P_2(x)$

$\Rightarrow P_4(x) = \frac{1}{4}\left\{7xP_3(x) - 3P_2(x)\right\} \Rightarrow P_4(x) = \frac{7x}{4}P_3(x) - \frac{3}{4}P_2(x)$

$\Rightarrow P_4(x) = \frac{7x}{4}\frac{1}{2}(5x^3 - 3x) - \frac{3}{4}\frac{1}{2}(3x^2 - 1) \Rightarrow P_4(x) = \frac{7x}{8}(5x^3 - 3x) - \frac{3}{8}(3x^2 - 1)$

$\Rightarrow P_4(x) = \frac{35x^4}{8} - \frac{21x^2}{8} - \frac{9x^2}{8} + \frac{3}{8} = \frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8}$

$= \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$

• $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \underset{n=4}{\Longrightarrow} 5P_5(x) = 9xP_4(x) - 4P_3(x)$

$\Rightarrow P_5(x) = \frac{9x}{5}P_4(x) - \frac{4}{5}P_3(x) \Rightarrow$

$$P_5(x) = \frac{9}{5}x \cdot \frac{1}{8}\left(35x^4 - 30x^2 + 3\right) - \frac{4}{5}\frac{1}{2}\left(5x^3 - 3x\right)$$

$$\Rightarrow P_5(x) = \frac{9}{40}x \cdot \left(35x^4 - 30x^2 + 3\right) - \frac{4}{10}\left(5x^3 - 3x\right)$$

$$\Rightarrow P_5(x) = \frac{315x^5}{40} - \frac{270x^3}{40} + \frac{27x}{40} - \frac{20x^3}{10} + \frac{12x}{10}$$

$$\Rightarrow P_5(x) = \frac{315x^5}{40} - \frac{350x^3}{40} + \frac{75}{40} = \frac{1}{40}\left(315x^5 - 350x^3 + 75\right)$$

$$= \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$$

4 Question 4 has been solved in the class

(2) $$f(x) = \sum_{k=0}^{\infty} A_k P_k(x) = A_0 P_0(x) + A_1 P_1(x) + A_2 P_2(x) + A_3 P_3(x) + \cdots$$

$$A_0 = \frac{1}{2}\int_{-1}^{1} P_0(x) f(x)\,dx = \frac{1}{2}\int_0^1 x\,dx = \frac{1}{2}\frac{x^2}{2}\Big|_0^1 = \frac{1}{4}$$

$$A_1 = \frac{3}{2}\int_{-1}^{1} P_1(x) f(x)\,dx = \frac{3}{2}\int_0^1 x^2\,dx = \frac{3}{2}\frac{x^3}{3}\Big|_0^1 = \frac{1}{2}$$

$$A_2 = \frac{5}{2}\int_{-1}^{1} P_2(x) f(x)\,dx = \frac{5}{2}\,\frac{1}{2}\int_0^1 x(3x^2-1)\,dx = \frac{5}{4}\int_0^1 (3x^3 - x)\,dx$$

$$= \frac{5}{4}\left\{3\int_0^1 x^3\,dx - \int_0^1 x\,dx\right\} = \frac{5}{4}\left\{\frac{3x^4}{4}\Big|_0^1 - \frac{x^2}{2}\Big|_0^1\right\} = \frac{5}{4}\left(\frac{3}{4} - \frac{1}{2}\right)$$

$$= \frac{5}{16}$$

$$A_3 = \frac{7}{2}\int_{-1}^{1} P_3(x) f(x)\,dx = \frac{7}{2}\,\frac{1}{2}\int_0^1 x(5x^3 - 3x)\,dx = \frac{7}{4}\int_0^1 (5x^4 - 3x^2)\,dx$$

$$= \frac{7}{4}\left\{ \int_0^1 5x^4\,dx - \int_0^1 3x^2\,dx \right\} = \frac{7}{4}\left\{ \frac{5x^5}{5}\Big|_0^1 - \frac{3x^3}{3}\Big|_0^1 \right\}$$

$$= \frac{7}{4}\left\{ 1 - 1 \right\} = 0$$

Thus $f(x) = \frac{1}{4} P_0(x) + \frac{1}{2} P_1(x) + \frac{5}{16} P_2(x) + \cdots$

③ α) We know that $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{n} J_{n+1}(x)$

Thus for $n=1$ we have: $\frac{d}{dx}\left[x^{-1} j_n(x)\right] = -x\, j_2(x)$

$$\Rightarrow j_2(x) = -\frac{1}{x}\frac{d}{dx}\left[\frac{J_n(x)}{x}\right] \Rightarrow j_2(x) = -\frac{1}{x}\frac{d}{dx}\frac{\left(\frac{\sin x}{x^2} - \frac{\cos x}{x}\right)}{x}$$

$$\Rightarrow j_2(x) = -\frac{1}{x}\frac{d}{dx}\left(\frac{\sin x}{x^3} - \frac{\cos x}{x^2}\right) = -\frac{1}{x}\left\{\frac{d}{dx}\left(\frac{\sin x}{x^3}\right) - \frac{d}{dx}\left(\frac{\cos x}{x^2}\right)\right\}$$

$$\Rightarrow j_2(x) = -\frac{1}{x}\left\{\frac{(\sin x)' x^3 - (x^3)'\sin x}{x^6} - \frac{(\cos x)' x^2 - (x^2)'\cos x}{x^4}\right\}$$

$$\Rightarrow j_2(x) = -\frac{1}{x}\left\{\frac{\cos x \cdot x^3 - 3x^2 \sin x}{x^6} - \frac{-\sin x \cdot x^2 - 2x\cos x}{x^4}\right\}$$

$$\Rightarrow j_2(x) = -\frac{1}{x}\left\{\frac{\cos x}{x^3} - \frac{3\sin x}{x^4} + \frac{\sin x}{x^2} + \frac{2\cos x}{x^3}\right\}$$

$$\Rightarrow j_2(x) = -\frac{1}{x}\left\{\frac{3\cos x}{x^3} - \frac{\sin x}{x^2}\left(\frac{3}{x^2} + 1\right)\right\}$$

b) $$\frac{d}{dx}\left[x^{-n} n_n(x)\right] = -x^{n} n_{n+1}(x) \overset{n=1}{\Longrightarrow}$$

$$\frac{d}{dx}\left[x^{-1} n_1(x)\right] = -x\, n_2(x) \Rightarrow n_2(x) = -\frac{1}{x}\frac{d}{dx}\left[x^{-1} n_1(x)\right]$$

$$\Rightarrow n_2(x) = -\frac{1}{x}\frac{d}{dx}\left(\frac{\frac{\cos x}{x^2} - \frac{\sin x}{x}}{x}\right) = -\frac{1}{x}\frac{d}{dx}\left(\frac{\cos x}{x^3} - \frac{\sin x}{x^2}\right)$$

$$\Rightarrow n_2(x) = -\frac{1}{x}\left\{\frac{d}{dx}\left(\frac{\cos x}{x^3}\right) - \frac{d}{dx}\left(\frac{\sin x}{x^2}\right)\right\} \Rightarrow n_2(x) = -\frac{1}{x}\left\{\frac{(\cos x)' x^3 - (x^3)' \cos x}{x^6} - \frac{(\sin x)' x^2 - (x^2)' \sin x}{x^4}\right\}$$

$$\Rightarrow n_2(x) = -\frac{1}{x}\left\{\frac{-\sin x\, x^3 - 3x^2 \cos x}{x^6} - \frac{\cos x \cdot x^2 - 2x \sin x}{x^4}\right\}$$

$$\Rightarrow n_2(x) = -\frac{1}{x}\left\{-\frac{\sin x}{x^3} - \frac{3\cos x}{x^4} - \frac{\cos x}{x^2} + \frac{2\sin x}{x^3}\right\}$$

$$\Rightarrow n_2(x) = -\frac{1}{x}\left\{\frac{\sin x}{x^3} - \frac{\cos x}{x^2}\left(\frac{3}{x^2} + 1\right)\right\}$$