PHYS 505
Final Exam
Thuesday 15th December 2015

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Student Grade: ……./40

SECTION A: ***Please answer all questions***

1. A system is at an eigenstate $\left|\ell,m\right\rangle$ of the operator ℓ_z. Show that the average values of the other two operators ℓ_x, ℓ_y are zero. You are given $l_\pm = l_x \pm i l_y$ and

$$l_+\left|\ell,m\right\rangle = \hbar\sqrt{\ell\left(\ell+1\right)-m\left(m+1\right)}\left|\ell,m+1\right\rangle$$
$$l_-\left|\ell,m\right\rangle = \hbar\sqrt{\ell\left(\ell+1\right)-m\left(m-1\right)}\left|\ell,m-1\right\rangle$$

Solution:

We know for the operators ℓ_x, ℓ_y that are related to the raising and lowering operator as follows: $l_\pm = l_x \pm i l_y$. Thus we can get the expressions by adding and subtracting

$$l_x = \frac{1}{2}\left(l_+ + l_-\right),\;\; l_y = \frac{1}{2i}\left(l_+ - l_-\right)$$

Thus:

$$\left\langle \ell,m\right|l_x\left|\ell,m\right\rangle = \frac{1}{2}\left\langle \ell,m\right|\left(l_+ + l_-\right)\left|\ell,m\right\rangle = \frac{1}{2}\left\{\left\langle \ell,m\right|l_+\left|\ell,m\right\rangle + \left\langle \ell,m\right|l_-\left|\ell,m\right\rangle\right\} =$$
$$\frac{\hbar}{2}\left\{\sqrt{\ell\left(\ell+1\right)-m\left(m+1\right)}\left\langle \ell,m\middle|\ell,m+1\right\rangle + \sqrt{\ell\left(\ell+1\right)-m\left(m-1\right)}\left\langle \ell,m\middle|\ell,m-1\right\rangle\right\} = 0$$

Similarly for the operator ℓ_y.

2. Two particles with spin $s_1 = 3/2$ interact with the Hamiltonian $H = A\mathbf{s}_1\cdot\mathbf{s}_2$ where A is a given constant. Calculate the energy eigenvalues of the system and the degree of degeneracy of the system. You are given that: $S = |s_1 - s_2| \ldots |s_1 + s_2|$. Also for any type of angular momentum $J^2 \left| j,m \right\rangle = \hbar^2 J(J+1) \left| j,m \right\rangle$

Solution:
We know that for the total spin we have:

$$\mathbf{s}^2 = \left(\mathbf{s}_1 + \mathbf{s}_2\right)^2 \Rightarrow \mathbf{s}^2 = \mathbf{s}_1^2 + \mathbf{s}_2^2 + 2\mathbf{s}_1 \cdot \mathbf{s}_2 \Rightarrow \mathbf{s}_1 \cdot \mathbf{s}_2 = \frac{1}{2}\left(\mathbf{s}^2 - \mathbf{s}_1^2 - \mathbf{s}_2^2\right)$$

Thus the Hamiltonian becomes

$$H = A\mathbf{s}_1 \cdot \mathbf{s}_2 = \frac{A}{2}\left(\mathbf{s}^2 - \mathbf{s}_1^2 - \mathbf{s}_2^2\right)$$

with eigenvalues

$$H = \frac{A\hbar^2}{2}\left[s\left(s+1\right) - s_1\left(s_1+1\right) - s_2\left(s_2+1\right)\right]$$

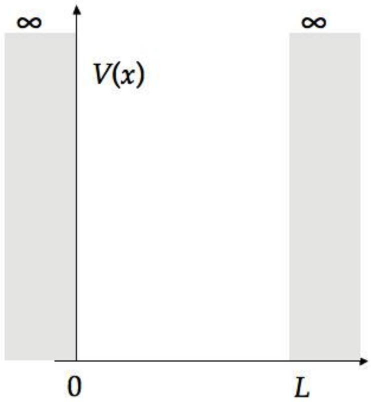
For the values of s we have: $s = \left|\frac{3}{2} - \frac{3}{2}\right| \ldots \left|\frac{3}{2} + \frac{3}{2}\right| = 0,\ 1,\ 2,\ 3$

Thus for the Hamiltonian we get the following 4 eigenvalues ($s_1 = s_2 = 3/2$):

$$E_0 = -15A\hbar^2/4,\ E_1 = -11A\hbar^2/4,\ E_1 = -3A\hbar^2/4,\ E_3 = 9A\hbar^2/4.$$

Each of these has a degeneracy d = 2s+1 so $d_0 = 1,\ d_1 = 3,\ d_2 = 5,\ d_3 = 7$.

3. A particle is inside an infinite square well of width L as shown in figure. We know that the unperturbed eigenfunctions and eigenenergies of the system are: $\psi_n^{(0)} = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$ and $E_n^{(0)} = \frac{n^2\pi^2\hbar^2}{2ma^2}$.



We add a small perturbation to the system given by:

$$V(x) = \begin{cases} 0 & 0 < x < L/4 \\ V_0 & L/4 < x < 3L/4 \\ 0 & 3L/4 < x < L \end{cases} .$$

Find the first order corrections of the energy eigenvalues. Find the first order corrections of the energy eigenvalues. You are given that $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$. Also $E_n^{(1)} = \langle \psi_n^0 | V \psi_n^o \rangle = \int_{-\infty}^{+\infty} (\psi_n^0)^* V \psi_n^o dx$

$$E_n^{(1)} = \langle \psi_n^0 | V \psi_n^o \rangle = \int_{-\infty}^{+\infty} (\psi_n^0)^* V \psi_n^o dx = \int_{L/4}^{3L/4} \left(\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)\right)^* V_0 \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) dx =$$

$$\frac{2V_0}{L} \int_{L/4}^{3L/4} \sin^2\left(\frac{n\pi x}{L}\right) dx = \frac{2V_0}{L} \int_{L/4}^{3L/4} \left[\frac{1}{2} - \frac{1}{2}\cos 2\left(\frac{n\pi x}{L}\right)\right] dx =$$

$$\frac{2V_0}{L}\left[\frac{1}{2}\int_{L/4}^{3L/4} dx - \frac{1}{2}\int_{L/4}^{3L/4} \cos\left(\frac{2n\pi x}{L}\right) dx\right] = \frac{2V_0}{L}\left[\frac{L}{4} - \frac{L}{4n\pi}\sin\left(\frac{2n\pi x}{L}\right)\Bigg|_{L/4}^{3L/4}\right] =$$

$$\frac{V_0}{2n\pi}\left[n\pi - \sin\left(\frac{3n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right)\right]$$

SECTION B: ***Please answer <u>ONLY ONE</u> question***

4. A non-polarized beam (that is, a beam which contains all possible spins) is made up of fermions with spin $s = ½$. The fermions interact with a potential $V(r) = V_0 e^{-a^2 r^2}$. Find the differential scattering cross section $d\sigma / d\Omega$.

You are given the following:

$$\int_0^\infty r e^{-a^2 r^2} \sin(qr) dr = \frac{q\sqrt{\pi}}{4a^3}\left(e^{-q^2/a^2}\right)^{1/4}, \quad \sin^2(\theta/2) = \frac{1}{2}(1-\cos\theta)$$

$$\frac{d\sigma}{d\Omega} = \left|f(\theta)\right|^2 \quad , \quad f(\theta) = -\frac{2m}{q\hbar^2}\int_0^\infty rV(r)\sin(qr)\,dr\,, \quad q = 2k\sin(\theta/2)$$

Solutions:

When we have an unpolarized beam we must take into account that we have a triplet state (with an antisymmetric spatial wavefunction) and one singlet state (with a symmetric spatial wavefunction). As we have shown in the class:

$$d\sigma/d\Omega = \frac{3}{4}\left(\frac{d\sigma}{d\Omega}\right)_A + \frac{1}{4}\left(\frac{d\sigma}{d\Omega}\right)_s = \frac{3}{4}\left|f(\theta) - f(\pi-\theta)\right|^2 + \frac{1}{4}\left|f(\theta) + f(\pi-\theta)\right|^2$$

$$= \left|f(\theta)\right|^2 + \left|f(\pi-\theta)\right|^2 - f(\theta)f(\pi-\theta)$$

The differential cross section is given by:

$$f(\theta) = -\frac{2m}{q\hbar^2}\int_0^\infty rV(r)\sin(qr)\,dr = -\frac{2m}{q\hbar^2}\int_0^\infty r\left(V_0 e^{-a^2r^2}\right)\sin(qr)\,dr =$$

$$= -\frac{2mV_0}{q\hbar^2}\int_0^\infty re^{-a^2r^2}\sin(qr)\,dr = \left(-\frac{2mV_0}{q\hbar^2}\right)\frac{q\sqrt{\pi}}{4a^3}\left(e^{-q^2/a^2}\right)^{1/4} = \left(-\frac{mV_0\sqrt{\pi}}{2\hbar^2a^3}\right)e^{-q^2/4a^2}$$

$$\underset{q=2k\sin(\theta/2)}{=} \left(-\frac{mV_0\sqrt{\pi}}{2\hbar^2a^3}\right)e^{-k^2\sin^2(\theta/2)/a^2}$$

So

$$f(\pi-\theta) = \left(-\frac{mV_0\sqrt{\pi}}{2\hbar^2a^3}\right)e^{-k^2\cos^2(\theta/2)/a^2}$$

$$\left|f(\theta)\right|^2 = \frac{m^2V_0^2\pi}{4\hbar^4a^6}e^{-q^2/2a^2} \underset{q=2k\sin(\theta/2)}{\Rightarrow} \left|f(\theta)\right|^2 = \frac{m^2V_0^2\pi}{4\hbar^4a^6}e^{-2k^2\sin^2(\theta/2)/a^2}$$

$$\left|f(\pi-\theta)\right|^2 = \frac{m^2V_0^2\pi}{4\hbar^4a^6}e^{-q^2/2a^2} \underset{q=2k\sin(\theta/2)}{\Rightarrow} \left|f(\pi-\theta)\right|^2 = \frac{m^2V_0^2\pi}{4\hbar^4a^6}e^{-2k^2\cos^2(\theta/2)/a^2}$$

$$f(\theta)f(\pi-\theta) = \frac{m^2V_0^2\pi}{4\hbar^4a^6}e^{-k^2\sin^2(\theta/2)/a^2}e^{-k^2\cos^2(\theta/2)/a^2} = \frac{m^2V_0^2\pi}{4\hbar^4a^6}e^{-k^2/a^2}$$

Thus for the differential cross-sectional area we have:

$$d\sigma / d\Omega = \left|f\left(\theta\right)\right|^2 + \left|f\left(\pi-\theta\right)\right|^2 - f\left(\theta\right)f\left(\pi-\theta\right) =$$

$$\frac{m^2V_0^2\pi}{4\hbar^4a^6}e^{-2k^2\sin^2\left(\theta/2\right)/a^2} + \frac{m^2V_0^2\pi}{4\hbar^4a^6}e^{-2k^2\cos^2\left(\theta/2\right)/a^2} - \frac{m^2V_0^2\pi}{4\hbar^4a^6}e^{-k^2/a^2} =$$

$$\frac{m^2V_0^2\pi}{4\hbar^4a^6}\left\{e^{-2k^2\sin^2\left(\theta/2\right)/a^2} + e^{-2k^2\cos^2\left(\theta/2\right)/a^2} - e^{-k^2/a^2}\right\}$$

.

5. For a particle in a potential given by $V(x) = F_0 x$ for $x \geq 0$ and $V(x) = \infty$ for $x < 0$, find the energy of the ground state by the variational method, using the trial functions of the form: $\psi\left(x\right)=Ax\exp(\text{-}\lambda x)$.

You are given:

$$\int_0^\infty x^2 e^{-2x}\,dx = \frac{1}{4},\quad \int_{-\infty}^\infty x^3 e^{-2\lambda x}\,dx = \frac{3}{8\lambda^4},\quad \int_{-\infty}^\infty x^4 e^{-\lambda x^2}\,dx = \frac{3}{4}\frac{\sqrt{\pi}}{\lambda^{5/2}},\quad \hat{p} = -i\hbar\frac{\partial}{\partial x}$$

Solution: First we normalize the wave-function:

$$\int_{-\infty}^\infty \left|\psi\left(x\right)\right|^2 dx = 1 \Rightarrow A^2\int_0^\infty x^2\exp(\text{-}2\lambda x)\,dx = 1 \Rightarrow \frac{A^2}{4\lambda^3} = 1 \Rightarrow A = 2\lambda^{3/2}$$

Thus the trial wavefunction is

$$\psi\left(x\right)=2\lambda^{3/2}x\exp(\text{-}x).$$

Then we calculate the average potential energy on the trial state

$$\bar{U} = \int_{-\infty}^\infty \left|\psi\left(x\right)\right|^2 V(x)\,dx = F_0\int_0^\infty \left|\psi\left(x\right)\right|^2 x\,dx = F_0\int_0^\infty \left(2\lambda^{3/2}x\exp(\text{-}\lambda x)\right)^2 x\,dx$$

$$= 4\lambda^3F_0\int_0^\infty x^3\exp(\text{-}2\lambda x)\,dx = 4\lambda^3F_0\frac{3}{8\lambda^4} = \frac{3F_0}{2\lambda}$$

Then we calculate the average kinetic on the trial state:

$$\bar{K} = \frac{1}{2m}\left\langle\psi\right|p^2\left|\psi\right\rangle = -\frac{\hbar^2}{2m}\left\langle\psi\right|\frac{\partial^2}{\partial x^2}\left|\psi\right\rangle = -\frac{\hbar^2}{2m}\int_0^\infty \psi^*\frac{\partial^2\psi}{\partial x^2}dx$$

But

$$\frac{\partial^2\psi}{\partial x^2} = 2\lambda^{3/2}\frac{\partial^2}{\partial x^2}\left(x\exp(\text{-}\lambda x)\right) = 2\lambda^{3/2}\exp(\text{-}\lambda x)\left(-2\lambda + x\lambda^2\right)$$
$$= 2\lambda^{5/2}\exp(\text{-}\lambda x)\left(x\lambda - 2\right)$$

So

$$\bar{K} = -\frac{\hbar^2}{2m}\int_0^\infty \psi^* \frac{\partial^2\psi}{\partial x^2}dx = -\frac{\hbar^2}{2m}2\lambda^{5/2}2\lambda^{3/2}\int_0^\infty x\exp(\text{-}\lambda x)\exp(\text{-}\lambda x)\left(x\lambda - 2\right)dx =$$
$$-\frac{2\hbar^2}{m}\lambda^{8/2}\int_0^\infty x\exp(\text{-}2\lambda x)\left(x\lambda - 2\right)dx = -\frac{2\hbar^2}{m}\lambda^{8/2}\left\{\lambda\int_0^\infty x^2\exp(\text{-}2\lambda x)dx - \right.$$
$$\left. -2\int_0^\infty x\exp(\text{-}2\lambda x)dx\right\} = -\frac{2\hbar^2}{m}\lambda^{8/2}\left\{\lambda\frac{1}{4\lambda^3} - 2\frac{1}{4\lambda^2}\right\} = \frac{2\hbar^2}{m}\lambda^{8/2}\frac{1}{4\lambda^2} = \frac{\hbar^2}{2m}\lambda^2$$

Thus the average total energy is:

$$\bar{E} = \bar{U} + \bar{K} = \frac{3F_0}{2\lambda} + \frac{\hbar^2}{2m}\lambda^2$$

This becomes minimum at the point where the first derivative is zero:

$$\frac{d\bar{E}}{d\lambda} = -\frac{3F_0}{2\lambda^2} + \frac{\hbar^2}{m}\lambda = 0 \Rightarrow \lambda = \left(\frac{3mF_0}{\hbar^2}\right)^{1/3}$$

And the energy of the ground state is:

$$\bar{E} = \bar{U} + \bar{K} = \frac{3F_0}{2\left(\frac{3mF_0}{\hbar^2}\right)^{1/3}} + \frac{\hbar^2}{2m}\left[\left(\frac{3mF_0}{\hbar^2}\right)^{1/3}\right]^2 \approx 1.966\left(\frac{\hbar^2F_0^2}{m}\right)^{1/3}$$

$$, \quad \psi_n^{(1)} = \sum_{m \neq n} \frac{\left\langle \psi_m^o \middle| V \middle| \psi_n^o \right\rangle}{\left(E_n^o - E_m^o\right)} \psi_m^{(o)}$$