PHYS 505
Final Exam
Thursday 26th December 2013

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Student Grade: ……./40

SECTION A: ***Please answer all questions***

1. Find in the Born approximation the differential and total cross-section for scattering in the field $V(r)=V_0 e^{-a^2r^2}$.
   You are given the following:
   $\int_0^\infty re^{-a^2r^2}\sin(qr)dr=\frac{q\sqrt{\pi}}{4a^3}\left(e^{-q^2/a^2}\right)^{1/4}$, $\sin^2(\theta/2)=\frac{1}{2}(1-\cos\theta)$

Solution:

The differential cross section is given by:

$$d\sigma/d\Omega=\left|f_B(\theta)\right|^2$$

$$f_B(\theta)=-\frac{2m}{q\hbar^2}\int_0^\infty rV(r)\sin(qr)dr=-\frac{2m}{q\hbar^2}\int_0^\infty r\left(V_0e^{-a^2r^2}\right)\sin(qr)dr=$$

$$=-\frac{2mV_0}{q\hbar^2}\int_0^\infty re^{-a^2r^2}\sin(qr)dr=\left(-\frac{2mV_0}{q\hbar^2}\right)\frac{q\sqrt{\pi}}{4a^3}\left(e^{-q^2/a^2}\right)^{1/4}=\left(-\frac{mV_0\sqrt{\pi}}{2\hbar^2a^3}\right)e^{-q^2/4a^2}$$

So

$$d\sigma/d\Omega=\left|f_B(\theta)\right|^2=\left|\left(-\frac{mV_0\sqrt{\pi}}{2\hbar^2a^3}\right)e^{-q^2/4a^2}\right|^2=\frac{m^2V_0^2\pi}{4\hbar^4a^6}e^{-q^2/2a^2}.$$

For the total cross-section we have:

We know that $q=2k\sin(\theta/2)$ *thus if we substitute in the formula for the differential cross-section we get:*

$$d\sigma/d\Omega=\frac{m^2V_0^2\pi}{4\hbar^4a^6}e^{-4k^2\sin^2(\theta/2)/2a^2}=\frac{m^2V_0^2\pi}{4\hbar^4a^6}e^{-2k^2\sin^2(\theta/2)/a^2}$$

$$\sigma = \int \left| f_B(\theta) \right|^2 d\Omega = \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \left| f_B(\theta) \right|^2 \sin\theta d\theta d\varphi = \frac{m^2 V_0^2 \pi}{4\hbar^4 a^6} 2\pi \int_{\theta=0}^{\theta=\pi} \sin\theta e^{-2k^2 \sin^2(\theta/2)/a^2} d\theta$$

$$\underset{\sin^2(\theta/2)=\frac{1}{2}(1-\cos\theta)}{=} \frac{m^2 V_0^2 \pi^2}{2\hbar^4 a^6} \int_{\theta=0}^{\theta=\pi} \sin\theta e^{-k^2/a^2} e^{-k^2 \cos(\theta)/a^2} d\theta \underset{u=\cos\theta}{=} \frac{m^2 V_0^2 \pi^2}{2\hbar^4 a^6} e^{-k^2/a^2} \int_{u=-1}^{u=1} e^{-k^2 u/a^2} du \Rightarrow$$

$$\sigma = -\frac{m^2 V_0^2 \pi^2}{2\hbar^4 a^4 k^2} \left(1 - e^{-2k^2/a^2}\right)$$

2. Two particles with spin $s_1 = 3/2$ and interact with the Hamiltonian $H = A\mathbf{s}_1 \cdot \mathbf{s}_2$ where A is a given constant. Calculate the energy eigenvalues of the system and the degree of degeneracy of the system.

Solution:

We know that for the total spin we have:

$$\mathbf{s}^2 = \left(\mathbf{s}_1 + \mathbf{s}_2\right)^2 \Rightarrow \mathbf{s}^2 = \mathbf{s}_1^2 + \mathbf{s}_2^2 + 2\mathbf{s}_1 \cdot \mathbf{s}_2 \Rightarrow \mathbf{s}_1 \cdot \mathbf{s}_2 = \frac{1}{2}\left(\mathbf{s}^2 - \mathbf{s}_1^2 - \mathbf{s}_2^2\right)$$

Thus the Hamiltonian becomes

$$H = A\mathbf{s}_1 \cdot \mathbf{s}_2 = \frac{A}{2}\left(\mathbf{s}^2 - \mathbf{s}_1^2 - \mathbf{s}_2^2\right)$$

with eigenvalues

$$H = \frac{A\hbar^2}{2}\left[s(s+1) - s_1(s_1+1) - s_2(s_2+1)\right]$$

For the values of s we have: $s = \left|\frac{5}{2} - \frac{3}{2}\right| ... \left|\frac{5}{2} + \frac{3}{2}\right| = 1,\ 2,\ 3,\ 4$

Thus for the Hamiltonian we get the following 4 eigenvalues ($s_1 = s_2 = 1/2$ *):*

$$E_1 = -21A\hbar^2/4,\ E_2 = -13A\hbar^2/4,\ E_3 = -A\hbar^2/4,\ E_4 = 15A\hbar^2/4.$$

Each of these has a degeneracy d = 2s+1 so $d_1 = 3,\ d_2 = 5,\ d_3 = 7,\ d_4 = 9$.

3. Two non-polarized beams (that is, beams which contain all possible spins) of two fermions with spin $s = 1/2$ interact with a Yukawa potential $V(r) = V_0 a e^{-r/a}/r$. Find the differential cross-section at an angle $\theta = \pi/2$ in the case where $k = a^{-1}$. You are given that:
$\int_0^\infty e^{-br}\sin(qr)dr = \frac{q}{b^2+q^2}$, $\sin(\pi/4) = \sqrt{2}/2$.

Solution:

The differential cross section is given by:

$$d\sigma / d\Omega = \left| f_B(\theta) \right|^2$$

$$f_B(\theta) = -\frac{2m}{q\hbar^2}\int_0^\infty rV(r)\sin(qr)\,dr = -\frac{2m}{q\hbar^2}\int_0^\infty r\left(V_0 a e^{-r/a}/r\right)\sin(qr)\,dr =$$

$$= -\frac{2mV_0 a}{q\hbar^2}\int_0^\infty e^{-r/a}\sin(qr)\,dr = \left(-\frac{2mV_0 a}{q\hbar^2}\right)\frac{a^2 q}{1+a^2q^2} = \left(-\frac{2mV_0}{\hbar^2}\right)\frac{a^3}{1+a^2q^2}$$

So

$$d\sigma / d\Omega = \left| f_B(\theta) \right|^2 = \left|\left(-\frac{2mV_0}{\hbar^2}\right)\frac{a^3}{1+a^2q^2}\right|^2 = \frac{4m^2V_0^2a^6}{\hbar^4}\frac{1}{\left(1+a^2q^2\right)^2}$$

Now if we consider that $q = 2k\sin(\theta/2)$ *we get*

$$d\sigma / d\Omega = \frac{4m^2V_0^2a^6}{\hbar^4}\frac{1}{\left(1+4a^2k^2\sin^2(\theta/2)\right)^2}$$

For $k = a^{-1}$ *we get*

$$d\sigma / d\Omega = \frac{4m^2V_0^2a^6}{\hbar^4}\frac{1}{\left(1+4\sin^2(\theta/2)\right)^2}.$$

Now since the beam is not polarized with $s = 1/2$ *we get*

$$\frac{d\sigma}{d\Omega} = \left|f(\theta)\right|^2 + \left|f(\pi-\theta)\right|^2 + \frac{(-1)^{2s}}{2s+1}2\,\mathrm{Re}\left[f(\theta)f^*(\pi-\theta)\right] \underset{s=1/2}{\Rightarrow}$$

$$\frac{d\sigma}{d\Omega} = \left|f(\theta)\right|^2 + \left|f(\pi-\theta)\right|^2 - \frac{1}{2}\mathrm{Re}\left[f(\theta)f^*(\pi-\theta)\right]$$

For $\theta = \pi/2$ *we get:*

$$\frac{d\sigma}{d\Omega}=\left|f\left(\pi/2\right)\right|^2+\left|f\left(\pi-\pi/2\right)\right|^2-\frac{1}{2}\mathrm{Re}\left[f\left(\pi/2\right)f^*\left(\pi-\pi/2\right)\right]\Rightarrow$$

$$\frac{d\sigma}{d\Omega}=\left|f\left(\pi/2\right)\right|^2+\left|f\left(\pi/2\right)\right|^2-\frac{1}{2}\mathrm{Re}\left[f\left(\pi/2\right)f^*\left(\pi/2\right)\right]\Rightarrow$$

$$\frac{d\sigma}{d\Omega}=\left|f\left(\pi/2\right)\right|^2+\left|f\left(\pi/2\right)\right|^2-\frac{1}{2}\left|f\left(\pi/2\right)\right|^2\Rightarrow$$

$$\frac{d\sigma}{d\Omega}=\frac{3}{2}\left|f\left(\pi/2\right)\right|^2$$

But for $\theta=\pi/2$ we have

$$\left|f\left(\pi/2\right)\right|^2=\frac{4m^2V_0^2a^6}{\hbar^4}\frac{1}{\left(1+4\sin^2\left(\pi/4\right)\right)^2}=\frac{4m^2V_0^2a^6}{\hbar^4}\frac{1}{\left(1+4\left(\sqrt{2}/2\right)^2\right)^2}=$$

$$\frac{4m^2V_0^2a^6}{\hbar^4}\frac{1}{\left(1+4\left(\sqrt{2}/2\right)^2\right)^2}=\frac{4m^2V_0^2a^6}{9\hbar^4}$$

Thus

$$\frac{d\sigma}{d\Omega}=\frac{3}{2}\left|f\left(\pi/2\right)\right|^2=\frac{3}{2}\frac{4m^2V_0^2a^6}{9\hbar^4}=\frac{2m^2V_0^2a^6}{3\hbar^4}.$$

SECTION B: ***Please answer ONLY ONE question***

4. A particle of mass m moves in a potential $V(x)=gx^4$ $(g>0)$. Try to estimate the energy of the ground state using the **method of variations**. Assume that the wavefunction of the ground state is a Gaussian function of the form: $\psi\left(x,\lambda\right)=Ne^{-\lambda x^2/2}$. You are given:

$$\int_{-\infty}^{\infty}e^{-\lambda x^2}dx=\sqrt{\frac{\pi}{\lambda}},\quad \int_{-\infty}^{\infty}x^2e^{-\lambda x^2}dx=\frac{1}{2}\frac{\sqrt{\pi}}{\lambda^{3/2}},\quad \int_{-\infty}^{\infty}x^4e^{-\lambda x^2}dx=\frac{3}{4}\frac{\sqrt{\pi}}{\lambda^{5/2}},\quad \widehat{p}=-i\hbar\frac{\partial}{\partial x}$$

Solution:

$$\int_{-\infty}^{+\infty} \left|\psi\left(x,\lambda\right)\right|^2 dx = 1 \Rightarrow \left|N\right|^2 \int_{-\infty}^{\infty} e^{-\lambda x^2} dx = 1 \Rightarrow \left|N\right|^2 \sqrt{\frac{\pi}{\lambda}} = 1$$

$$N = \left(\frac{\lambda}{\pi}\right)^{1/4}$$

$$\left\langle H\right\rangle = \frac{\left\langle p^2\right\rangle}{2m} + \left\langle V(x)\right\rangle = \frac{\left\langle p^2\right\rangle}{2m} + g\left\langle x^4\right\rangle$$

$$\left\langle x^4\right\rangle = \left|N\right|^2 \int_{-\infty}^{\infty} e^{-\lambda x^2} x^4 dx \Rightarrow \left\langle x^4\right\rangle = \left(\frac{\lambda}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\lambda x^2} x^4 dx = \frac{3}{4\lambda^2}$$

$$\widehat{p} = -i\hbar\frac{\partial}{\partial x} \Rightarrow \widehat{p}^2 = \left(-i\hbar\right)^2 \frac{\partial^2}{\partial x^2} \Rightarrow \widehat{p}^2 = -\hbar^2 \frac{\partial^2}{\partial x^2}$$

$$\left\langle \widehat{p}^2\right\rangle = \left\langle \psi\left|\widehat{p}^2\right|\psi\right\rangle = -\hbar^2\left(\frac{\lambda}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\lambda x^2/2} \frac{\partial^2 e^{-\lambda x^2/2}}{\partial x^2} dx \Rightarrow$$

$$\left\langle \widehat{p}^2\right\rangle = -\hbar^2\left(\frac{\lambda}{\pi}\right)^{1/2} \left(-\lambda\right)\int_{-\infty}^{\infty}\left(1-\lambda x^2\right)e^{-\lambda x^2} dx \Rightarrow$$

$$\left\langle \widehat{p}^2\right\rangle = \hbar^2\lambda\left(\frac{\lambda}{\pi}\right)^{1/2} \left\{\int_{-\infty}^{\infty} e^{-\lambda x^2} dx - \lambda\int_{-\infty}^{\infty} x^2 e^{-\lambda x^2} dx\right\} \Rightarrow$$

$$\left\langle \widehat{p}^2\right\rangle = \hbar^2\lambda\left(\frac{\lambda}{\pi}\right)^{1/2} \left\{\sqrt{\frac{\pi}{\lambda}} - \lambda\frac{\sqrt{\pi}}{2\lambda^{3/2}}\right\} \Rightarrow$$

$$\left\langle \widehat{p}^2\right\rangle = \hbar^2\lambda\left(\frac{\lambda}{\pi}\right)^{1/2} \left\{\frac{1}{2}\sqrt{\frac{\pi}{\lambda}}\right\} \Rightarrow \left\langle \widehat{p}^2\right\rangle = \frac{\hbar^2\lambda}{2}$$

Thus the average value for the energy becomes

$$\left\langle H\right\rangle = \frac{\left\langle p^2\right\rangle}{2m} + \left\langle V(x)\right\rangle = \frac{\left\langle p^2\right\rangle}{2m} + g\left\langle x^4\right\rangle \Rightarrow$$

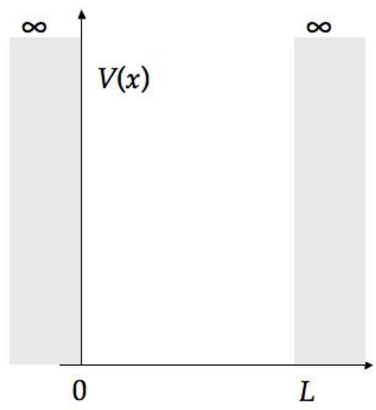
$$\left\langle E\left(\lambda\right)\right\rangle = \frac{\hbar^2\lambda}{4m} + \frac{3g}{4\lambda^2}$$

The energy has a minimum value at:

$$\frac{d\left\langle E\left(\lambda\right)\right\rangle}{d\lambda} = 0 \Rightarrow \lambda_0 = \left(\frac{6gm}{\hbar^2}\right)^{1/3}$$

$$\left\langle E\left(\lambda_0\right)\right\rangle = \frac{\hbar^2\lambda_0}{4m} + \frac{3g}{4\lambda_0^2} = \left(\frac{3}{4\cdot 6^{2/3}} + \frac{1}{4}\right)\frac{g^{1/3}\hbar^{4/3}}{m^{2/3}}$$

5. A particle is inside an infinite square well of width L as shown infgure. We know that the unperturbed eigenfunctions and eigenenergies of the system are: $\psi_n^{(0)} = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$ and $E_n^{(0)} = \frac{n^2\pi^2\hbar^2}{2ma^2}$.



We add a small perturbation to the system given by:

$$W = \begin{cases} 0 & 0 < x < L/2 \\ V_0 & L/2 \le x \le L \end{cases}.$$

Find the first order corrections of the energy eigenvalues. You are given that $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$.

Solution:

$$E_n^{(1)} = \left\langle \psi_n^0 \middle| W\psi_n^o \right\rangle = \int_{-\infty}^{+\infty} \left(\psi_n^0\right)^* W\psi_n^o dx = \int_{L/2}^{L} \left(\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\right)^* V_0\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) dx =$$

$$\frac{2V_0}{L}\int_{L/2}^{L} \sin^2\left(\frac{n\pi x}{L}\right)dx = \frac{2V_0}{L}\int_{L/2}^{L}\left[\frac{1}{2} - \frac{1}{2}\cos 2\left(\frac{n\pi x}{L}\right)\right]dx =$$

$$\frac{2V_0}{L}\left[\frac{1}{2}\int_{L/2}^{L} dx - \frac{1}{2}\int_{L/2}^{L}\cos 2\left(\frac{n\pi x}{L}\right)dx\right] = \frac{2V_0}{L}\left[\frac{L}{4} + \frac{L}{4n\pi}\sin\left(\frac{2n\pi x}{L}\right)\Big|_{L/2}^{L}\right] =$$

$$\frac{V_0}{2}\left[1 + \frac{1}{2n\pi}\sin\left(\frac{2n\pi L}{L}\right) - \frac{1}{2n\pi}\sin\left(\frac{n\pi L}{L}\right)\right] = V_0\left[1 - \frac{1}{2n\pi}\sin(n\pi)\right] = \frac{V_0}{2}$$

Mathematical Supplement:

- For any physical quantity A: $\Delta A = \sqrt{\left\langle A^2\right\rangle - \left\langle A\right\rangle^2}$
- For any type of angular momentum: $k = l, s$ or j.

$\mathbf{k}^2\left|k,m_k\right\rangle = k(k+1)\hbar^2\left|k,m_k\right\rangle$

$$k_+\left|k,m_k\right\rangle = \hbar\sqrt{k\left(k+1\right)-m_k\left(m_k+1\right)}\left|k,m_k+1\right\rangle$$
$$k_-\left|k,m_k\right\rangle = \hbar\sqrt{k\left(k+1\right)-m_k\left(m_k-1\right)}\left|k,m_k-1\right\rangle$$

- For a particle with spin 1/2

$$s_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad s_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad s_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

$$\underbrace{X_\uparrow = \begin{pmatrix}1\\0\end{pmatrix}}_{\text{spin up}},\quad \underbrace{X_\downarrow = \begin{pmatrix}0\\1\end{pmatrix}}_{\text{spin down}}$$

$$\frac{d\sigma}{d\Omega} = \left|f\left(\theta\right)\right|^2 \quad,\quad f_B\left(\theta\right) = -\frac{m}{2\pi\hbar^2}\tilde{V}\left(\mathbf{q}\right) \quad,\quad \tilde{V}\left(\mathbf{q}\right) = \int e^{-i\mathbf{q}\cdot\mathbf{r}}V(\mathbf{r})d^3\mathbf{r}$$

$$f_B\left(\theta\right) = -\frac{2m}{q\hbar^2}\int_0^\infty rV(r)\sin\left(qr\right)dr,\quad q = 2k\sin\left(\theta/2\right)$$

$$\psi \to e^{ikz} + f\left(\theta\right)\frac{e^{ikr}}{r},\qquad \psi\left(r,\theta\right) = \sum_{l=0}^{\infty} A_l\frac{\chi_l\left(r\right)P_l\left(\cos\theta\right)}{r}$$

$$\left[\frac{d}{dr^2} + k^2 - V(r) - \frac{l\left(l+1\right)}{r^2}\right]\chi_l(r) = 0$$

$$\chi_l(0) = 0$$

$$\chi_l(\infty) \to \left[A_l\underbrace{j_l\left(kr\right)}_{\substack{\textit{spherical}\\ \textit{Bessel function}}} + B_l\underbrace{n_l\left(kr\right)}_{\substack{\textit{spherical}\\ \textit{Neumann function}}}\right]r = \frac{1}{k}C_l\sin\left(kr - \frac{\pi l}{2} + \delta_l\right)$$

$$A_l = \left(2l+1\right)i^l e^{i\delta_l}$$
$$f_k\left(\theta\right) = \frac{1}{k}\sum_{l=0}^{\infty}\left(2l+1\right)e^{i\delta_l}\sin\delta_l P_l\left(\cos\theta\right)$$
$$d\sigma/d\Omega = \frac{1}{k^2}\left|\sum_{l=0}^{\infty}\left(2l+1\right)e^{i\delta_l}\sin\delta_l P_l\left(\cos\theta\right)\right|^2$$
$$\sigma_T = 2\pi\int_0^\pi\left|f\left(\theta\right)\right|^2\sin\theta d\theta = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}\left(2l+1\right)\sin^2\delta_l$$

$$\sigma_T = \frac{4\pi}{k}\operatorname{Im}\left(f_k\left(0\right)\right)$$

$$\tan\delta_l = \frac{kj_l'\left(ka\right)-\gamma_l j_l(ka)}{kn_l'\left(ka\right)-\gamma_l n_l(ka)}, \quad \gamma_l = \frac{1}{R_l}\frac{dR_l}{dr}\bigg|_{r=a} \qquad \left[R_l(r)=\chi_l(r)/r\right]$$

$$\frac{d\sigma}{d\Omega} = \left|f\left(\theta\right)\right|^2 + \left|f\left(\pi-\theta\right)\right|^2 + \frac{\left(-1\right)^{2s}}{2s+1}2\,\mathrm{Re}\left[f\left(\theta\right)f^*\left(\pi-\theta\right)\right]$$

$$E_n^{(1)} = \left\langle\psi_n^0\middle|V\psi_n^o\right\rangle, \quad \psi_n^{(1)} = \sum_{m\neq n}\frac{\left\langle\psi_m^o\middle|V\middle|\psi_n^o\right\rangle}{\left(E_n^o-E_m^o\right)}\psi_m^{(o)}$$