Q.3 Here $a_n = n\cdot\tan\left(\frac{1}{n}\right),\ n \geq 1$

(3)

$$\lim_{n\to\infty} a_n \overset{(1/2)}{=} \lim_{n\to\infty} n\cdot\tan\left(\frac{1}{n}\right) = \lim_{n\to\infty}\left(\frac{\sin(1/n)}{1/n}\cdot\frac{1}{\cos 1/n}\right)$$

let $\frac{1}{n} = k$, as $n\to\infty$, $k\to 0$,

$$\therefore \lim_{n\to\infty} a_n = \lim_{k\to 0}\left(\frac{\sin k}{k}\cdot\frac{1}{\cos k}\right) = \lim_{k\to 0}\frac{\sin k}{k}\cdot\lim_{k\to 0}\frac{1}{\cos k}$$

$$= 1\times\frac{1}{\cos 0} = 1 \neq 0. \quad (2)$$

Hence by the divergence (n^{th} term) test, the given series diverges. (1/2)

Q.4 Given $f(x) = \frac{1+x}{(1-x)^2}$

(6)

we know that, $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n,\ |x|<1$

Diff. both sides, we get. $+\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots = \sum_{n=0} nx^{n-1},\ |x|<1$

Multiplying both sides with $1+x$, we get $= \sum_{n=0}^{\infty}(2n+1)\cdot x^n,\ |x|<1$

$$f(x) = \frac{1+x}{(1-x)^2} = (1+x)(1+2x+3x^2+4x^3+\cdots) = 1+3x+5x^2+7x^3+\cdots$$

This is the power series representation of $f(x)$. (3)

Radius of Convergence $= 1$. For interval of convergence, we separately check convergence at $x = \pm 1$.

At $x = 1$; $\sum_{n=0}^{\infty}(2n+1)$ which is a divergent series b/c $\lim_{n\to\infty}(2n+1) = \infty \neq 0$

At $x = -1$; $\sum_{n=0}^{\infty}(-1)^n(2n+1)$, which is a divergent A.S.

$\therefore$ Interval of convergence $= (-1, 1)$. (2)

At $x = 1/2$; $\sum_{n=0}^{\infty}(2n+1)\cdot\left(\frac{1}{2}\right)^n = \frac{1+1/2}{(1-1/2)^2} \Rightarrow \sum_{n=0}^{\infty}\frac{(2n+1)}{2^n} = \frac{3}{2}\times\frac{4}{1}$

$\Rightarrow \sum_{n=0}^{\infty}\frac{(2n+1)}{2^n} = 6$ i.e., $\sum_{n=0}^{\infty}\frac{2n+1}{2^n}$ has sum equal to 6. (1)

Differential and Integral Calculus (MATH-205)

MT Exam/Semester I (2022-23) Time Allowed: 120 Minutes

Date: Monday, October 10, 2022 **Maximum Marks:** 30

Note: Attempt all SIX questions and give detailed solutions. Read statements of the questions carefully and make sure you have answered each question completely.

Question 1: (4°) Determine whether the following sequence converges or diverges. Find its limiting value as $n \to \infty$.

$$\left\{\left(1+\frac{7}{8n^3}\right)^{n^3}\right\}_{n=1}^{\infty}.$$

Question 2: (5°) Determine whether the series $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ converges or diverges. Find its sum, if it converges.

Question 3: (3°) Determine whether the infinite series $\sum_{n=1}^{\infty} n \tan\frac{1}{n}$ converges or diverges.

Question 4: (6°) Find the power series representation of $f(x) = \frac{1+x}{(1-x)^2}$. Find interval of convergence of this series. Hence, find the sum of the series $\sum_{n=0}^{\infty} \frac{2n+1}{2^n}$.

Question 5: (6°) Approximate $\int_0^{\frac{1}{2}} x^2 \cos x^3 \, dx$ using first four non-zero terms of the Maclaurin series. Find the exact value of the definite integral and the absolute error. Use 5 decimal point accuracy in your working.

Question 6: (6°) Given points $A(4,2,3)$, $B(8,1,8)$, $C(6,4,7)$, and $D(12,5,5)$. Find (i) the angle (in degrees) between $\overrightarrow{AB}$ and $\overrightarrow{CD}$, (ii) the component of $\overrightarrow{CD}$ along $\overrightarrow{AB}$ and vice versa, (iii) a vector of magnitude $\sqrt{2}$ in the direction of $\overrightarrow{AC}$, and (iv) find k if the magnitude of $\overrightarrow{AE}$ is $\sqrt{17}$, where $E(k,-1,\frac{1}{3})$.

— Good Luck —

Q.2 Given infinite Series is $\sum_{n=1}^{\infty} \ln\left(\frac{n}{1+n}\right)$. $\Rightarrow a_n = \frac{n}{n+1}, n \geq 1$

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n^{th} partial sum, $S_n = \ln\left(\frac{1}{2}\right) + \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3}{4}\right) + \dots + \ln\left(\frac{n}{n+1}\right)$ 2

$= \ln\left(\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \dots \times \frac{n-1}{n} \times \frac{n}{n+1}\right) = \ln\left(\frac{1}{n+1}\right) = -\ln(n+1)$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (-\ln(n+1)) = -\ln\left(\lim_{n\to\infty}(n+1)\right) = -\ln(\infty) = -\infty$ 2

$\Rightarrow \{S_n\}_{n=1}^{\infty}$ diverges. Hence, $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ also diverges 1

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Q.1 Sem I/2022-23/MT

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P# 8.1

Given Sequence : $\left\{\left(1+\frac{7}{8n^3}\right)^{n^3}\right\}_{n=1}^{\infty}$

Here, $a_n = \left(1+\frac{7}{8n^3}\right)^{n^3}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1+\frac{7}{8n^3}\right)^{n^3}$ (1½) let $n^3 = k$, then as $n\to\infty$, $k\to\infty$

$= \lim_{k\to\infty}\left(1+\frac{7}{8k}\right)^{k} = \lim_{k\to\infty}\left(1+\frac{1}{\frac{8}{7}k}\right)^{k}$

$= \left[\lim_{k\to\infty}\left(1+\frac{1}{\frac{8}{7}k}\right)^{\frac{8}{7}k}\right]^{7/8} = e^{7/8}$ (2)

$\Rightarrow$ The given sequence converges to $e^{7/8}$. (½)

Q.5 MT-Sem-I/2022-23

6

P# 8.8

Given definite integral, $\int_0^{1/2} x^2 . \cos x^3 . dx$.

let $f(x) = x^2 . \cos x^3$, To find Maclaurin's Series of $f(x)$, we proceed as follows. we know that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^n . \frac{x^{2n}}{(2n)!} + \dots = \sum_{n=0}^{\infty} (-1)^n . \frac{x^{2n}}{(2n)!}$$

$$\Rightarrow \cos x^3 = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \dots + (-1)^n . \frac{x^{6n}}{(6n)!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n . x^{6n}}{(6n)!}, \ -\infty < x < \infty$$

$$\Rightarrow f(x) = x^2 . \cos x^3 = x^2 - \frac{x^8}{2!} + \frac{x^{14}}{4!} - \frac{x^{20}}{6!} + \dots + (-1)^n . \frac{x^{6n+2}}{(6n)!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n . x^{6n+2}}{(6n)!}, \ -\infty < x < \infty \quad \text{—(1)}$$

Taking 1st four non-zero terms, we ~~got~~ have (3)

$$f(x) = x^2 \cos x^3 \approx 1 - \frac{x^8}{6!} + \frac{x^{14}}{12!} - \frac{x^{20}}{18!}$$

$$\therefore \int_0^{1/2} x^2 \cos x^3 . dx \approx \left| x - \frac{x^9}{6!9} + \frac{x^{15}}{12!15} - \frac{x^{21}}{18!21} \right|_0^{1/2}$$

$$= \frac{1}{2} - \frac{1}{6!\,9\times 2^9} + \frac{1}{12!\,15\times 2^{15}} - \frac{1}{18!\,21\times 2^{21}} = 0.5$$ (1½)

Exact value of the definite integral:

$$\int x^2 . \cos x^3 . dx = \frac{1}{3}\int \cos x^3 . (3x^2) dx = \frac{1}{3} . \sin x^3$$

$$\therefore \int_0^{1/2} x^2 . \cos x^3 . dx = \frac{1}{3}\left|\sin x^3\right|_0^{1/2} = \frac{1}{3} . \sin \frac{1}{8} \approx 0.04156$$ (1)

$\therefore$ Absolute Error $= |0.5 - 0.04156| = 0.45844$ (½)

Q.6/(6)

MT-Sem-I/2022-23

Given pts. are $A(4,2,3)$, $B(8,1,8)$, $C(6,4,7)$, & $D(12,5,5)$. and $E(k,-1,\frac{1}{3})$.

$\vec{AB} = \langle 8-4, 1-2, 8-3\rangle = \langle 4,-1,5\rangle$, $\vec{CD} = \langle 6,+1,-2\rangle$

(i) let θ be the angle b/w $\vec{AB}$ & $\vec{CD}$, then

$$\cos\theta = \frac{\vec{AB}\cdot\vec{CD}}{\|\vec{AB}\|\,\|\vec{CD}\|} = \frac{\langle 4,-1,5\rangle\cdot\langle 6,1,-2\rangle}{\sqrt{16+1+25}\;\sqrt{36+1+4}} = \frac{24-1-10}{\sqrt{42}\;\sqrt{41}}$$

$$= \frac{13}{\sqrt{1722}} \qquad \therefore \theta = \cos^{-1}\left(\frac{13}{\sqrt{1722}}\right) = 71.74^\circ, \text{ or } 1.25 \text{ rad}$$

1

(ii) $$\text{Comp}_{\vec{AB}}\vec{CD} = \frac{\vec{CD}\cdot\vec{AB}}{\|\vec{AB}\|} = \frac{13}{\sqrt{42}}\;;\quad \text{Comp}_{\vec{CD}}\vec{AB} = \frac{\vec{AB}\cdot\vec{CD}}{\|\vec{CD}\|} = \frac{13}{\sqrt{41}}$$

2

(iii) $\vec{AC} = \langle 6-4, 4-2, 7-3\rangle = \langle 2,2,4\rangle$

A unit vector in the direction of $\vec{AC} = \frac{\vec{AC}}{\|\vec{AC}\|} = \frac{1}{\sqrt{24}}\langle 2,2,4\rangle$

$\therefore$ vector of magnitude $\sqrt{2}$ in the direction of $\vec{AC}$

1

$$= \frac{\sqrt{2}}{\sqrt{24}}\langle 2,2,4\rangle = \frac{1}{\sqrt{12}}\langle 2,2,4\rangle = \frac{1}{\sqrt{6}}\langle 1,1,2\rangle$$

(iv). $\vec{AE} = \langle k-4, -1-2, \frac{1}{3}-3\rangle = \langle k-4, -3, -\frac{8}{3}\rangle$

$$\|\vec{AE}\| = \sqrt{17} \Rightarrow \sqrt{(k-4)^2 + (-3)^2 + \left(-\frac{8}{3}\right)^2} = \sqrt{17} \Rightarrow (k-4)^2 + 9 + \frac{64}{9} = 17$$

$$\Rightarrow (k-4)^2 = 17 - \frac{145}{9} = \frac{18}{9} \Rightarrow k = 4 \pm \sqrt{2}$$

2