KING SAUD UNIVERSITY,DEPARTMENT OF MATHEMATICS
TIME: 3H, FULL MARKS:40, SII(Summer) 19/09/1434 MATH 204

Important: Questions 1, 2, 3, and 4 are obligatory. You have the chioce of answering Q5 or the multiple choice questions. Please draw the answer table of multiple choice questions in your answer copy book.

Question 1[3,6]. a) Show that the following differential equation is exact, hence solve it

$$(e^x y + xe^x y)dx + (xe^x + 2)dy = 0.$$

b) Find the general solution of the following differential equations

$$\textbf{i)}\ \ y' = \frac{y}{x + \sqrt{xy}}, \qquad \textbf{ii)}\ dy = 16 \tan^{-1} y(1 + y^2)dx$$

Question 2[4,4]. a) Find the orthogonal trajectories of the family of curves:

$$y \sin x = c$$

b) Solve the initial value problem:

$$\begin{cases} (\sin x)y' + (\cos x)y - x = 0, \\ y(\pi/4) = 2 \end{cases}$$

Question 3[4,4]. a) Find only the form of the particular solution y_p of the differential equation

$$y'' + 8y' + 16y = x^2 e^{-4x}.$$

b) Solve the differential equation

$$y^{(4)} - 64y = 8$$

Question 4[4] By using the power series method, find the solution of the differential equation

$$y'' + y = -2xy',$$

about the ordinary point $x_0 = 0$.

Question 5[5,6]. a) Sketch the graph of the following function and find its Foutier series

$$f(x) = |\cos x|, \quad -\pi \le x \le \pi \quad \text{with} \quad f(x + 2\pi) = f(x),$$

b) Find the Fourier integral representation for the function:

$$f(x) = \begin{cases} 0, & x < -1 \\ -1, & -1 \le x < 0 \\ 2, & 0 \le x < 1 \\ 0, & x \ge 1 \end{cases}$$

Deduce that $\int_0^\infty \frac{\sin x}{x} = \frac{\pi}{2}$.

Multiple Choice questions

Mark: **a**,**b**,**c** or **d** for the correct answer in the space provided bellow for **Q1** to **Q5**

Q.No	1	2	3	4	5	6
Answer						

Q1. For the DE: $(xy^2)dx - Kx^2ydy = 0$ to be exact, the value of K must be:

a) -1, b) 1, c) 2 d) none of these

Q2. The value of the slope at $(\frac{\pi}{2}, 1)$ of the curve $y^2 sinx = \frac{2}{\pi}x$ is

a) $-\frac{2}{\pi}$, b) $\frac{2}{\pi}$, c) $\frac{1}{\pi}$, d) $\frac{1}{2\pi}$

Q3. The roots of the auxiliary equation of the DE: $x^2y'' + xy' + y = 0$ are:

a) $+i, -i$ b) $-1, +1$ c) $1 + i, 1 - i$ d) None of these

Q4. The simplest form of the particular solution of the DE: $y''' - y'' = x^3$ is

a) $y_p = Ax^3$, b) $y_p = Ax^6$, c) $y_p = Ax^4$, d) $y_p = Ax^5$

Q5. For solving the differential equation: $x^2y'' + xy' + y = 0$, the adequate substitution is

a) $y = lnx$, b) , $y = x^m$ c) $y = (lnx)^m$, d) $y = e^{mx}$

Q6. The functions: x, x^2, x^3 are

a) Linearly independent on IR,
b) Linearly dependent on IR
b) Linearly dependent on $(0, \infty)$
c) None of these

(1)

Q1 a) $\frac{\partial M}{\partial y} = e^x + x e^x$, $\frac{\partial N}{\partial x} = e^x + x e^x$ $\Big\} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ the DE is exact

$\Rightarrow \exists F(x,y) / \begin{cases} \frac{\partial F}{\partial x} = e^x y (1+x) \to (1) \\ \frac{\partial F}{\partial y} = x e^x + 2 \to (2) \end{cases}$ (1)

From (2), $F(x,y) = x e^x y + 2y + C(x) \to (3)$ (1)

From (3) $\frac{\partial F}{\partial x} = (e^x + x e^x) y + C'(x) \to (4)$

From (1) and (4) $C'(x) = 0 \Rightarrow C(x) = C$ (1)

Thus $F(x,y) = \boxed{x e^x y + 2y = C}$

b) $y' = \frac{y}{x + \sqrt{xy}}$ Homogeneous Eq

i) $y' = \frac{\frac{y}{x}}{1 + \sqrt{\frac{y}{x}}}$, let $v = \frac{y}{x} \Rightarrow y' = x v' + v$ (1)

Then $x v' + v = \frac{v}{1 + \sqrt{v}} \Rightarrow x v' = \frac{v}{1+\sqrt{v}} - \frac{v}{1} = \frac{\cancel{v} - \cancel{v} - v\sqrt{v}}{1+\sqrt{v}}$

Hence $\frac{1+\sqrt{v}}{v\sqrt{v}} dv = -\frac{dx}{x} \Rightarrow \int v^{-\frac{3}{2}} dv + \int \frac{dv}{v} = -\ln|x| + C$

(1)

$\Rightarrow -2 v^{-\frac{1}{2}} + \ln|v| + \ln|x| = C$

$\Rightarrow -\frac{2}{\sqrt{\frac{y}{x}}} + \ln\left|\frac{y}{x}\right| + \ln|x| = C$ (1)

$\Rightarrow -2\sqrt{\frac{y}{x}} + \ln|y| = C$

ii) $\frac{dy}{(1+y^2)\tan^{-1} y} = 16\, dx \Rightarrow \int \frac{d(\tan^{-1} y)}{\tan^{-1} y} = 16x + C$ (3)

$\Rightarrow \ln|\tan^{-1} y| = 16x + C$

Q_2 a) $y' \sin x + y \cos x = 0$ (DE for the F.T)

$\Rightarrow y' = -y \frac{\cos x}{\sin x}$ (1)

The DE for the F.O.T is $y' = \frac{\sin x}{y \cos x}$ (2)

$\Rightarrow y\,dy = \frac{\sin x}{\cos x} dx \Rightarrow \frac{y^2}{2} = -\ln|\cos x| + C$ (1)

b) $y' + \frac{\cos x}{\sin x} y = \frac{x}{\sin x}$ (linear Eq)

$\mu(x) = e^{\int \frac{\cos x}{\sin x} dx} = e^{\ln|\sin x|} = \sin x$ (2)

then $\frac{d}{dx}(y \sin x) = x \Rightarrow y \sin x = \frac{x^2}{2} + C$

$y(\frac{\pi}{4}) = 2 \Rightarrow 2 \frac{\sqrt{2}}{2} = \frac{\pi^2}{32} + C \Rightarrow C = \sqrt{2} - \frac{\pi^2}{32}$ (2)

Hence $y \sin x = \frac{x^2}{2} + \sqrt{2} - \frac{\pi^2}{32}$

Q_3 a) $y'' + 8y' + 16y = x^2 e^{-4x}$

Ch Eq: $m^2 + 8m + 16 = 0 \Rightarrow (m+4)^2 = 0$

$\Rightarrow m_1 = m_2 = -4$ (2)

$y_p = x^s [Ax^2 + Bx + C] e^{-4x}$

Since $r = -4$ is a double root for the Ch Equation (2)

then $s = 2$

Hence $y_p = x^2 [Ax^2 + Bx + C] e^{-4x}$

Q3. b) $y^{(4)} - 64y = 8$, $y_g = y_c + y_p$

Ch Eq: $m^4 - 64 = 0 \Rightarrow (m^2 - 8)(m^2 + 8) = 0$

$m_1 = 2\sqrt{2}$, $m_2 = -2\sqrt{2}$, $m_3 = 2\sqrt{2}\, i$, $m_4 = -2\sqrt{2}\, i$

$$y_c = c_1 e^{2\sqrt{2}x} + c_2 e^{-2\sqrt{2}x} + c_3 \cos(2\sqrt{2}x) + c_4 \sin(2\sqrt{2}x)$$

$y_p = A$, $y^{(4)} = 0 \Rightarrow -64A = 8 \Rightarrow A = -\frac{1}{8}$

Thus $y_p = -\frac{1}{8}$

$$y_g = c_1 e^{2\sqrt{2}x} + c_2 e^{-2\sqrt{2}x} + c_3 \cos(2\sqrt{2}x) + c_4 \sin(2\sqrt{2}x) - \frac{1}{8}$$

Q4 $y'' + y + 2xy' = 0$, $y = \sum_0^\infty a_n x^n$, $y' = \sum_1^\infty n a_n x^{n-1}$, $y'' = \sum_2^\infty n(n-1) a_n x^{n-2}$

Hence $\sum_2^\infty n(n-1) a_n x^{n-2} + \sum_0^\infty a_n x^n + 2 \sum_{n=1}^\infty n a_n x^n = 0$

$$\Rightarrow \sum_{n=0}^\infty (n+2)(n+1) a_{n+2} x^n + \sum_0^\infty a_n x^n + 2 \sum_1^\infty n a_n x^n = 0$$

$$\Rightarrow 2a_2 + a_0 + \sum_{n=1}^\infty \left[(n+2)(n+1) a_{n+2} + (1+2n) a_n\right] x^n = 0$$

$\Rightarrow a_2 = -\frac{a_0}{2}$, $(n+2)(n+1) a_{n+2} + (1+2n) a_n = 0$, $\forall n \geq 1$

$\Rightarrow a_{n+2} = -\frac{(1+2n)}{(n+2)(n+1)} a_n$, $\forall n \geq 1$

$n=1$: $a_3 = \frac{-3a_1}{6} = -\frac{a_1}{2}$

$n=2$: $a_4 = -\frac{5}{12} a_2 = -\frac{5}{12}\left(-\frac{a_0}{2}\right) = \frac{5a_0}{24}$

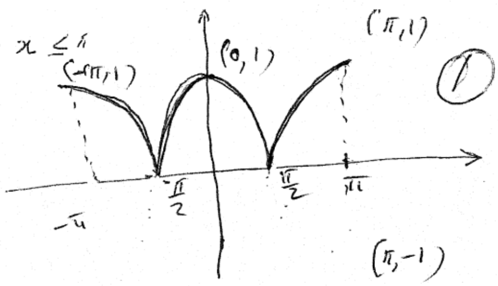
$n=3$: $a_5 = -\frac{7}{20} a_3 = \frac{-7}{20}\left(\frac{-a_1}{2}\right) = \frac{7a_1}{40}$

$$y = a_0 + a_1 x - \frac{a_0}{2} x^2 - \frac{a_1}{2} x^3 + \frac{5}{24} a_0 x^4 + \frac{7a_1}{40} x^5 + \cdots$$

$$= a_0\left[1 - \frac{x^2}{2} + \frac{5}{24} x^4 + \cdots\right] + a_1\left[x - \frac{x^3}{2} + \frac{7}{40} x^5 + \cdots\right] = a_0 y + a_1 y$$

4

Q5: a) $f(x) = |\cos x|, \ -\pi \le x \le \pi$



1

Since $f(x)$ is even on $[-\pi,\pi]$,
then $b_n = 0$

$$f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

1

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos x\,dx - \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \cos x\,dx$$

$$= \frac{2}{\pi}\Big[\sin x\Big]_0^{\frac{\pi}{2}} - \frac{2}{\pi}\Big[\sin x\Big]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{2}{\pi}\cdot 1 - \frac{2}{\pi}[-1] = \frac{2}{\pi} + \frac{2}{\pi} = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos x \cos nx\,dx - \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \cos x \cos nx\,dx \quad 1$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\cos(1+n)x + \cos(1-n)x}{2}\,dx - \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \frac{\cos(1+n)x + \cos(1-n)x}{2}\,dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(1+n)x}{1+n}\Big|_0^{\frac{\pi}{2}} + \frac{\sin(1-n)x}{1-n}\Big|_0^{\frac{\pi}{2}}\right] - \frac{1}{\pi}\left[\frac{\sin(1+n)x}{1+n}\Big|_{\frac{\pi}{2}}^{\pi} + \frac{\sin(1-n)x}{1-n}\Big|_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(1+n)\frac{\pi}{2}}{1+n} + \frac{\sin(1-n)\frac{\pi}{2}}{1-n}\right] - \frac{1}{\pi}\left[-\frac{\sin(1+n)\frac{\pi}{2}}{1+n} - \frac{\sin(1-n)\frac{\pi}{2}}{1-n}\right] \quad 1$$

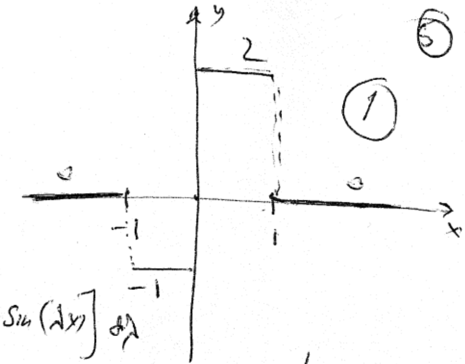
$$\frac{1}{\pi}\left[\frac{\cos\frac{n\pi}{2}}{1+n} + \frac{\cos(\frac{n\pi}{2})}{1-n}\right] - \frac{1}{\pi}\left[\frac{-\cos\frac{n\pi}{2}}{1+n} - \frac{\cos\frac{n\pi}{2}}{1-n}\right]$$

$$= \frac{4}{\pi(1-n^2)}\cos\frac{n\pi}{2} \qquad n \ne 1, \qquad a_1 = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos^2 x - \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \cos^2 x\,dx = 0$$

$$f(x) \simeq \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=2} \frac{\cos(\frac{n\pi}{2})}{1-n^2} \cos nx \quad 1$$

5

b) $f(x) = \begin{cases} 0, & x < -1 \\ -1, & -1 \le x < 0 \\ 2 & 0 \le x < 1 \\ 0, & x \ge 1 \end{cases}$



1

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x)\right] d\lambda$$

$$A(\lambda) = \int_{-\infty}^{\infty} f(x)\cos(\lambda x)\,dx = -\int_{-1}^{0}\cos(\lambda x)\,dx + 2\int_0^1 \cos(\lambda x)\,dx$$

$$= -\left.\frac{\sin(\lambda x)}{\lambda}\right|_{-1}^{0} + 2\left.\frac{\sin(\lambda x)}{\lambda}\right|_0^1$$ 1

$$= \frac{-\sin\lambda}{\lambda} + \frac{2\sin\lambda}{\lambda} = \frac{\sin\lambda}{\lambda}$$

$$B(\lambda) = \int_{-\infty}^{\infty} f(x)\sin(\lambda x)\,dx = -\int_{-1}^{0}\sin(\lambda x)\,dx + 2\int_0^1 \sin(\lambda x)\,dx$$

$$= \left.\frac{\cos(\lambda x)}{\lambda}\right|_{-1}^{0} - 2\left.\frac{\cos(\lambda x)}{\lambda}\right|_0^1$$ 2

$$= \frac{1}{\lambda} - \frac{\cos\lambda}{\lambda} - \frac{2\cos\lambda}{\lambda} + \frac{2}{\lambda}$$

$$= \frac{3}{\lambda} - \frac{3\cos\lambda}{\lambda} = \frac{3}{\lambda}(1-\cos\lambda)$$

Thus $f(x) = \frac{1}{\pi}\int_0^\infty \left[\frac{\sin\lambda}{\lambda}\cos\lambda x + \frac{3}{\lambda}(1-\cos\lambda)\sin(\lambda x)\right] d\lambda$

Let $x = 0$, then $f(0) = \frac{1}{\pi}\int_0^\infty \frac{\sin\lambda}{\lambda}\,d\lambda = \frac{1}{2}$

$$\Rightarrow \int_0^\infty \frac{\sin\lambda}{\lambda}\,d\lambda = \frac{\pi}{2}$$ 2

Multiple Choice:

1	2	3	4	5	6
a	c	a	d	b	a