Solutions of

OPER 441: Modeling and Simulation
Exercises Sheet #3

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Solution1:

a. The multiplicative linear congruential generator is a special case of the linear congruential generator with c = 0; therefore the LCG theorem can be applied to check if the generator will achieve its maximum period. The theorem states that an LCG has a full period if and only if the following three conditions hold:
   1. The only positive integer that (exactly) divides both *m* and *c* is 1 (i.e. *c* and *m* have no common factors other than 1).
   2. If *q* is a prime number that divides *m* then *q* should divide (*a*-1) (i.e. (*a*-1) is a multiple of every prime number that divides *m*).
   3. If 4 divides *m*, then 4 should divide (*a*-1) (i.e. (*a*-1) is a multiple of 4 if *m* is a multiple of 4).

   - Condition 1 does not hold because *c* = 0, meaning that *m* and *c* have multiple common factors.
   - Condition 2 holds because the prime numbers, *q*, that divide *m* = 64 are 1 and 2. (*a*-1) = 12, and both 1 and 2 divide 12.
   - Condition 3 holds because 4 divides both *m* = 64 and (*a*-1) = 12.

   Also, since *m* is a power of 2 ($m = 64 = 2^6$) and *c* = 0, the longest possible period is *m*/4 = 64/4 = 16.

b. Below is a period's worth of uniform random variables from each of the supplied seeds.

   For $X_o = 1$,

Table 1 – A Period's Worth of Uniform Random Variables

i	Ri	Ui
1	13	0.2031
2	41	0.6406
3	21	0.3281
4	17	0.2656
5	29	0.4531
6	57	0.8906
7	37	0.5781
8	33	0.5156
9	45	0.7031
10	9	0.1406
11	53	0.8281
12	49	0.7656
13	61	0.9531
14	25	0.3906
15	5	0.0781
16	1	0.0156

For $X_o = 2$,

Table 2 – A Period's Worth of Uniform Random Variables

i	Ri	Ui
1	26	0.4063
2	18	0.2813
3	42	0.6563
4	34	0.5313
5	58	0.9063
6	50	0.7813
7	10	0.1563
8	2	0.0313

For $X_o = 3$,

Table 3 – A Period's Worth of Uniform Random Variables

i	Ri	Ui
1	39	0.6094
2	59	0.9219
3	63	0.9844
4	51	0.7969
5	23	0.3594
6	43	0.6719
7	47	0.7344
8	35	0.5469
9	7	0.1094
10	27	0.4219
11	31	0.4844
12	19	0.2969
13	55	0.8594
14	11	0.1719
15	15	0.2344
16	3	0.0469

For $X_o = 4$,

Table 4 – A Period's Worth of Uniform Random Variables

i	Ri	Ui
1	52	0.8125
2	36	0.5625
3	20	0.3125
4	4	0.0625

*by excel

Solution2:

Condition 1 does not hold because $c = 0$, meaning that m and c have multiple common factors. Thus, it cannot reach its full period. Also, since m is a power of 2 ($m = 64 = 2^6$) and $c = 0$, the longest possible period is $m/4 = 64/4 = 16$.

A	B	C	D	E	F
a	11				
m	64				
c	0				
x0	1				
	i	x(i)	(a*x(i)+c)	x(i+1)	U(i+1)
	0	1	11	11	0.171875
	1	11	121	57	0.890625
	2	57	627	51	0.796875
	3	51	561	49	0.765625
	4	49	539	27	0.421875
	5	27	297	41	0.640625
	6	41	451	3	0.046875
	7	3	33	33	0.515625
	8	33	363	43	0.671875
	9	43	473	25	0.390625
	10	25	275	19	0.296875
	11	19	209	17	0.265625
	12	17	187	59	0.921875
	13	59	649	9	0.140625
	14	9	99	35	0.546875
	15	35	385	1	0.015625
	16	1	11	11	0.171875

A	B	C	D	E	F
a	11				
m	64				
c	0				
x0	2				
	i	x(i)	(a*x(i)+c)	x(i+1)	U(i+1)
	0	2	22	22	0.34375
	1	22	242	50	0.78125
	2	50	550	38	0.59375
	3	38	418	34	0.53125
	4	34	374	54	0.84375
	5	54	594	18	0.28125
	6	18	198	6	0.09375
	7	6	66	2	0.03125
	8	2	22	22	0.34375

A	B	C	D	E	F
a	11				
m	64				
c	0				
x0	3				
	i	x(i)	(a*x(i)+c)	x(i+1)	U(i+1)
	0	3	33	33	0.515625
	1	33	363	43	0.671875
	2	43	473	25	0.390625
	3	25	275	19	0.296875
	4	19	209	17	0.265625
	5	17	187	59	0.921875
	6	59	649	9	0.140625
	7	9	99	35	0.546875
	8	35	385	1	0.015625
	9	1	11	11	0.171875
	10	11	121	57	0.890625
	11	57	627	51	0.796875
	12	51	561	49	0.765625
	13	49	539	27	0.421875
	14	27	297	41	0.640625
	15	41	451	3	0.046875
	16	3	33	33	0.515625

A	B	C	D	E	F
a	11				
m	64				
c	0				
x0	4				
	i	x(i)	(a*x(i)+c)	x(i+1)	U(i+1)
	0	4	44	44	0.6875
	1	44	484	36	0.5625
	2	36	396	12	0.1875
	3	12	132	4	0.0625
	4	4	44	44	0.6875

Solution3:

a) A linear congruential generator has full period (cycle length is m) if and only if the following three conditions hold

- Condition 1 holds because the only positive integer that divides both m = 16 and $c = 5$ is 1
- The prime numbers, q, that divide m = 16 are q = 1 and 2. (a-1) = 11-1=10, and both 1 and 2 divide 10. Thus, condition 2 holds.
- Condition 3 does not hold because 4 divides m = 16 but not (a-1) = 10.

Therefore, this generator does not achieve the maximum possible period length.

b) To generate 2 pseudo-random number, using this parameters, $a = 11, c = 5, m = 16$

$x_0 = 1$

$x_1 = (11 \times 1 + 5) \bmod 16 = 0, \quad U_1 = \frac{0}{16} = 0$

$x_2 = (11 \times 0 + 5) \bmod 16 = 5, \quad U_2 = \frac{5}{16} = 0.3125$

$x_3 = (11 \times 5 + 5) \bmod 16 = 12, \quad U_3 = \frac{12}{16} = 0.75$

Thus, the LCG does not obtain full period.

A	B	C	D	E	F
a	11				
m	16				
c	5				
x0	1				
	i	x(i)	(a*x(i)+c)	x(i+1)	U(i+1)
	0	1	16	0	0
	1	0	5	5	0.3125
	2	5	60	12	0.75
	3	12	137	9	0.5625
	4	9	104	8	0.5
	5	8	93	13	0.8125
	6	13	148	4	0.25
	7	4	49	1	0.0625
	8	1	16	0	0
	9	0	5	5	0.3125

Solution4:

a) We apply the conditions on the generator and see whether it has full period or not,

- Condition 1: 1 is the only common factor of c =13 and m=16
- Condition 2: The prime numbers, q, that divide m = 16 are q = 1 and 2. (a-1) = 13-1=12, and both 1 and 2 divide 12. Thus, condition 2 holds.
- Condition 3: 4 divides m = 16 and 4 divides (a-1) =12. Thus, condition 3 holds.

Thus, LCG obtains the full period of 16

b) To generate 2 pseudo-random number, using this parameters, $a = 13, c = 13, m = 16$

$x_0 = 37$

$$x_1 = (13 \times 37 + 13)\,\text{mod}16 = 14, \quad U_1 = \frac{14}{16} = 0.875$$

$$x_2 = (13 \times 14 + 13)\,\text{mod}16 = 3, \quad U_2 = \frac{3}{16} = 0.1875$$

$$x_3 = (13 \times 3 + 13)\,\text{mod}16 = 4, \quad U_3 = \frac{4}{16} = 0.25$$

Solution5:

a) We apply the conditions on the generator and see whether it has full period or not,

- Condition 1: 1 is the only common factor of c=3 and m = 16, Condition 1 holds.
- Condition 2: The prime numbers, q, that divide m = 16 are q = 1 and 2. (a-1) = 4-1=3, since 2 does not divide 3, condition 3 does not hold. No need to check condition 3.

Thus, LCG does not obtain full period

b) To generate 2 pseudo-random number, using this parameters, $a = 4, c = 3, m = 16$

$x_0 = 11$

$$x_1 = (4 \times 11 + 3)\,\text{mod}16 = 15, \quad U_1 = \frac{15}{16} = 0.9375$$

$$x_2 = (4 \times 15 + 3)\,\text{mod}16 = 15, \quad U_2 = \frac{15}{16} = 0.9375$$

$$x_3 = (4 \times 15 + 3)\,\text{mod}16 = 15, \quad U_3 = \frac{15}{16} = 0.9375$$

This generator is degenerate at 15, that means $x_4 = x_5 = \cdots = x_i = 15$, and $U_4 = U_5 = \cdots = U_i = 0.9375, \forall i$, therefore, this generator is not good to use for simulation.

Solution6:

a) same as the previous questions, we need to check the conditions,

- Condition 1: 1 is the only common factor of c =1 and m=10
- Condition 2: The prime numbers, q, that divide m = 10 are q = 1, 2, 5. (a-1) = 8-1=7, only 1 divides 7, so condition 2 does not hold
- Condition 3: 4 does not divide 10, thus condition 3 does not hold

Thus, LCG does not reach full period

b) To generate 2 pseudo-random number, using this parameters, $a = 8, c = 1, m = 10$

$x_0 = 3$

$x_1 = (8 \times 3 + 1)\text{mod}10 = 5, \quad U_1 = \frac{5}{10} = 0.5$

$x_2 = (8 \times 5 + 1)\text{mod}10 = 1, \quad U_2 = \frac{1}{10} = 0.1$

$x_3 = (8 \times 1 + 1)\text{mod}10 = 9, \quad U_3 = \frac{9}{10} = 0.9$

Solution7:

a) If n = 100 is large enough to assume that the sample average is normally distributed, we have a 95% confidence interval based on the student-t statistic of:

$$\bar{x} \pm \left(t_{n-1,1-\frac{\alpha}{2}}\right) * \frac{s}{\sqrt{n}}$$

0.4615 ± 1.9842* 0.2915/10 = 0.4615 ± 0.0578, [0.4037, 0.5193]

b) Using

$d = \left(t_{n-1,1-\frac{\alpha}{2}}\right) * \frac{s}{\sqrt{n}} \rightarrow n = \left[\left(t_{n-1,1-\frac{\alpha}{2}}\right) * \frac{s}{d}\right]^2$ where $d = .01$ then,

approximately 3346

c) Dn = max(0.0907, 0.0807) = 0.0907, D(0.05) ≈ 1.63/10 = 0.163. Since Dn < D(0.05) do not reject the hypothesis of U(0,1).

Kolmogorov–Smirnov Test (K-S test)

TABLE B 7 Kolmogorov–Smirnov Test Critical Values

n	$D_{0.1}$	$D_{0.05}$	$D_{0.01}$
10	0.36866	0.40925	0.48893
11	0.35242	0.39122	0.46770
12	0.33815	0.37543	0.44905
13	0.32549	0.36143	0.43247
14	0.31417	0.34890	0.41762
15	0.30397	0.33760	0.40420
16	0.29472	0.32733	0.39201
17	0.28627	0.31796	0.38086
18	0.27851	0.30936	0.37062
19	0.27136	0.30143	0.36117
20	0.26473	0.29408	0.35241
25	0.23768	0.26404	0.31657
30	0.21756	0.24170	0.28987
35	0.20185	0.22425	0.26897
Over 35	$1.22/\sqrt{n}$	$1.36/\sqrt{n}$	$1.63/\sqrt{n}$

Solution8:

Test if the sequence is distributed U (0, 1) using both a K-S test and a Chi-Squared test (using 5 intervals) at a 95% confidence level?

do not reject the hypothesis of U(0,1)

by excel

Solution9:

$$H_0: U \sim Uniform(0,1) \qquad vs \qquad H_1: U\ not\ Uniform(0,1)$$

a) Chi-squared test for given probabilities

X-squared = 17.2, df = 9, p-value = 0.04567

Since the p-value = 0.04567 <= 0.5 ($\chi_0^2 = 17.2 > \chi_{9,0.95}^2 = 16.9$), the hypothesis should be rejected.

G7 | =COUNTIF($A$1:$A$50;"<"&F7)-COUNTIF($A$1:$A$50;"<"&E7)

	A	D	E	F	G	H	I	J
4	0.9804							
5	0.3356							
6	0.3807	i	start in	end out	Oi	pi=1/(#interval)	Ei=n*pi	((Oi-Ei)^2)/Ei
7	0.8364	1	0	0.1	0	0.1	5	5
8	0.6242	2	0.1	0.2	4	0.1	5	0.2
9	0.8589	3	0.2	0.3	5	0.1	5	0
10	0.5824	4	0.3	0.4	9	0.1	5	3.2
11	0.247	5	0.4	0.5	2	0.1	5	1.8
12	0.6146	6	0.5	0.6	5	0.1	5	0
13	0.9084	7	0.6	0.7	8	0.1	5	1.8
14	0.3387	8	0.7	0.8	2	0.1	5	1.8
15	0.9672	9	0.8	0.9	6	0.1	5	0.2
16	0.9583	10	0.9	1	9	0.1	5	3.2
17	0.9376			n=	50		ch0=	17.2
18	0.3795							
19	0.3258							
20	0.7864					n=	50	
21	0.2989					#interval=	10	
22	0.4566					alpha=	0.05	
23	0.8699					chi(#interval-1,1-alpha)=	16.9189776	
24	0.1709					p-value=P(chi(#interval-1)>chi0)=	0.045674574	
25	0.6557							
26	0.9462							

Chi-Square test						
i	b(i-1)	bi	Oi	pi	Ei = npi	$\frac{(Oi-Ei)^2}{Ei}$
1	0	0.1	0	0.1	5	5
2	0.1	0.2	4	0.1	5	0.2
3	0.2	0.3	5	0.1	5	0
4	0.3	0.4	9	0.1	5	3.2
5	0.4	0.5	2	0.1	5	1.8
6	0.5	0.6	5	0.1	5	0
7	0.6	0.7	8	0.1	5	1.8
8	0.7	0.8	2	0.1	5	1.8
9	0.8	0.9	6	0.1	5	0.2
10	0.9	1	9	0.1	5	3.2
k=10			n=50	1		$\chi_0^2 = \mathbf{17.2}$

n	50	Decision: Reject $\boldsymbol{H_0}$ Since $\chi_0^2 = \mathbf{17.2} > \chi_{9,0.95}^2 = \mathbf{16.9}$ i.e. U not Uniform(0,1)
k	10	
α	0.05	
$\boldsymbol{\chi_{9,0.95}^2}$	**16.9**	

b) One-sample Kolmogorov-Smirnov test

D(50) = 0.1572, D(.05) = 0.1923

Since the D(.05) > D(50), the hypothesis should not be rejected.

K-S Test

i	x_i	$\frac{i}{n}$	$\frac{i-1}{n}$	$F(x_i)$	$\frac{i}{n} - F(x_i)$	$F(x_i) - \frac{i-1}{n}$	i	x_i	$\frac{i}{n}$	$\frac{i-1}{n}$	$F(x_i)$	$\frac{i}{n} - F(x_i)$	$F(x_i) - \frac{i-1}{n}$
1	0.1117	0.02	0	0.1117	-0.0917	0.1117	26	0.6146	0.52	0.5	0.6146	-0.0946	0.1146
2	0.1246	0.04	0.02	0.1246	-0.0846	0.1046	27	0.6242	0.54	0.52	0.6242	-0.0842	0.1042
3	0.1489	0.06	0.04	0.1489	-0.0889	0.1089	28	0.6536	0.56	0.54	0.6536	-0.0936	0.1136
4	0.1709	0.08	0.06	0.1709	-0.0909	0.1109	29	0.6545	0.58	0.56	0.6545	-0.0745	0.0945
5	0.2268	0.1	0.08	0.2268	-0.1268	0.1468	30	0.6557	0.6	0.58	0.6557	-0.0557	0.0757
6	0.2379	0.12	0.1	0.2379	-0.1179	0.1379	31	0.6653	0.62	0.6	0.6653	-0.0453	0.0653
7	0.247	0.14	0.12	0.247	-0.107	0.127	32	0.6723	0.64	0.62	0.6723	-0.0323	0.0523
8	0.2972	0.16	0.14	0.2972	-0.1372	0.1572	33	0.6964	0.66	0.64	0.6964	-0.0364	0.0564
9	0.2989	0.18	0.16	0.2989	-0.1189	0.1389	34	0.7551	0.68	0.66	0.7551	-0.0751	0.0951
10	0.3045	0.2	0.18	0.3045	-0.1045	0.1245	35	0.7864	0.7	0.68	0.7864	-0.0864	0.1064
11	0.3237	0.22	0.2	0.3237	-0.1037	0.1237	36	0.8075	0.72	0.7	0.8075	-0.0875	0.1075
12	0.3258	0.24	0.22	0.3258	-0.0858	0.1058	37	0.8364	0.74	0.72	0.8364	-0.0964	0.1164
13	0.3356	0.26	0.24	0.3356	-0.0756	0.0956	38	0.842	0.76	0.74	0.842	-0.082	0.102
14	0.3387	0.28	0.26	0.3387	-0.0587	0.0787	39	0.8469	0.78	0.76	0.8469	-0.0669	0.0869
15	0.3427	0.3	0.28	0.3427	-0.0427	0.0627	40	0.8589	0.8	0.78	0.8589	-0.0589	0.0789
16	0.3525	0.32	0.3	0.3525	-0.0325	0.0525	41	0.8699	0.82	0.8	0.8699	-0.0499	0.0699
17	0.3795	0.34	0.32	0.3795	-0.0395	0.0595	42	0.9058	0.84	0.82	0.9058	-0.0658	0.0858
18	0.3807	0.36	0.34	0.3807	-0.0207	0.0407	43	0.9084	0.86	0.84	0.9084	-0.0484	0.0684
19	0.4047	0.38	0.36	0.4047	-0.0247	0.0447	44	0.9376	0.88	0.86	0.9376	-0.0576	0.0776
20	0.4566	0.4	0.38	0.4566	-0.0566	0.0766	45	0.9462	0.9	0.88	0.9462	-0.0462	0.0662
21	0.5095	0.42	0.4	0.5095	-0.0895	0.1095	46	0.9496	0.92	0.9	0.9496	-0.0296	0.0496
22	0.5195	0.44	0.42	0.5195	-0.0795	0.0995	47	0.9537	0.94	0.92	0.9537	-0.0137	0.0337
23	0.548	0.46	0.44	0.548	-0.088	0.108	48	0.9583	0.96	0.94	0.9583	0.0017	0.0183
24	0.5649	0.48	0.46	0.5649	-0.0849	0.1049	49	0.9672	0.98	0.96	0.9672	0.0128	0.0072
25	0.5824	0.5	0.48	0.5824	-0.0824	0.1024	50	0.9804	1	0.98	0.9804	0.0196	0.0004

n	50	Decision: Accept $\boldsymbol{H_0}$ Since $\boldsymbol{D_{50}} = \mathbf{0.1572} < \boldsymbol{D_{0.05}} = \mathbf{0.1923}$ i.e. U ~ Uniform(0,1)
α	0.05	
$\boldsymbol{D^+}$	0.0196	
$\boldsymbol{D^-}$	0.1572	
$\boldsymbol{D_{50}}$	**0.1572**	
$\boldsymbol{D_{0.05}}$	**0.1923**	

Solution10:

a)

x	0	1	2	3	4
P(X=x)	0.3	0.2	0.2	0.1	0.2
F(x)	0.3	0.5	0.7	0.8	1.0

b)

c)

$$F^{-1}(u)=\begin{cases}0 & ,0\le u\le .3\\ 1 & ,.3<u\le .5\\ 2 & ,.5<u\le .7\\ 3 & ,.7<u\le .8\\ 4 & ,.8<u\le 1\end{cases}$$

d)

U	0.2379	0.7551	0.2989	0.247	0.3237
X	0	3	0	0	1

*by excel

H10 | =IF(F10>0.8;4;IF(F10>0.7;3;IF(F10>0.5;2;IF(F10>0.3;1;0))))

	A	B	C	D	E	F	G	H	I
1	x	p(x)		F(x)					
2	0	0.3	0	0.3					
3	1	0.2	0.3	0.5					
4	2	0.2	0.5	0.7					
5	3	0.1	0.7	0.8					
6	4	0.2	0.8	1					
7									
8									
9									
10					U1=	0.2379	X1=	0	
11					U2=	0.7551	X2=	3	
12					U3=	0.2989	X3=	0	
13									
14									

Solution11:

a) X = -10 ln (1 – 0.9396) = 28.0677 (where $\lambda = \frac{1}{10}$)

Parameters	$\lambda > 0$ rate, or inverse scale
Support	$x \in [0, \infty)$
PDF	$\lambda e^{-\lambda x}$
CDF	$1 - e^{-\lambda x}$
Quantile	$-\ln(1 - F) / \lambda$
Mean	λ^{-1} $(= \beta)$
Median	$\lambda^{-1} \ln(2)$
Mode	0
Variance	λ^{-2} $(= \beta^2)$
Skewness	2
Ex. kurtosis	6

b) X = 12 + Floor((22-12+1)*0.1694) = 13

A random variable X follows the dicrete uniform distribution on the interval $[a, a+1, \ldots, b]$, if it may attain each of these values with equal probability. We have then:

$$P(X = x) = \begin{cases} \frac{1}{n} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise,} \end{cases}$$

$$F(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a+1}{n} & \text{for } a \leq x \leq b \\ 1 & \text{for } x > b, \end{cases}$$

$$E(X) = \frac{1}{2}(a + b) \text{ and } V(X) = \frac{n^2 - 1}{12},$$

where $n = b - a + 1$ denotes the number of values that X may take.

The inverse distribution function of X is

$$F^{-1}(u) = a + \lfloor u(b - a + 1) \rfloor \qquad 0 < u < 1.$$

where

$$\text{floor}(x) = \lfloor x \rfloor$$

is the floor function which is the function that takes as input a real number x and gives as output the greatest integer that is less than or equal to x