

Q1

توزيع الاحتمال في  $x > 0$ 

Q2

 $X \sim \chi^2_v$ 

$$f(x) = \left(\frac{1}{2}\right)^{\frac{v}{2}} \frac{1}{\Gamma\left(\frac{v}{2}\right)} e^{-\frac{1}{2}x} x^{\frac{v}{2}-1}, \quad x > 0$$

$$M_X(t) = E(e^{xt}) = \left(\frac{1}{2}\right)^{\frac{v}{2}} \frac{1}{\Gamma\left(\frac{v}{2}\right)} \int_0^{\infty} e^{xt} x^{-\frac{1}{2}x} x^{\frac{v}{2}-1} dx$$

$$= \left(\frac{1}{2}\right)^{\frac{v}{2}} \frac{1}{\Gamma\left(\frac{v}{2}\right)} \int_0^{\infty} e^{-x \frac{(1-2t)}{2}} x^{\frac{v}{2}-1} dx$$

$$= \left(\frac{1}{2}\right)^{\frac{v}{2}} \frac{1}{\Gamma\left(\frac{v}{2}\right)} \frac{\Gamma\left(\frac{v}{2}\right)}{\left(\frac{(1-2t)}{2}\right)^{\frac{v}{2}}} = \frac{1}{(1-2t)^{\frac{v}{2}}} = (1-2t)^{-\frac{v}{2}}$$

Q3

 $X_1 \sim \chi^2_{v_1}$  and  $X_2 \sim \chi^2_{v_2}$  are independent

$$(a) M_Z(t) = M_{X_1+X_2}(t) = M_{X_1}(t) M_{X_2}(t) = (1-2t)^{-\frac{v_1}{2}} (1-2t)^{-\frac{v_2}{2}} = (1-2t)^{-\frac{(v_1+v_2)}{2}}$$

$$(b) Z \sim \chi^2_{v_1+v_2}$$

Q4

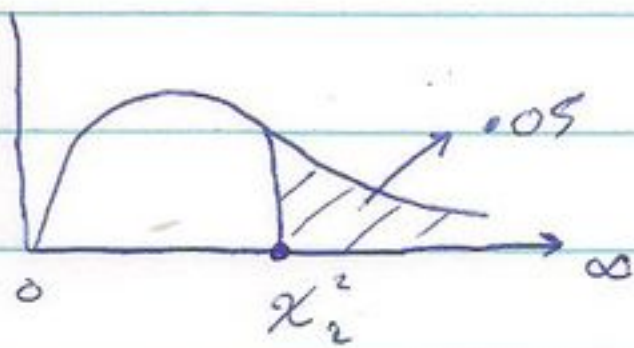
توزيع الاحتمال في  $x > 0$



Q5

$$X \sim \chi^2_{v=5}$$

(a)

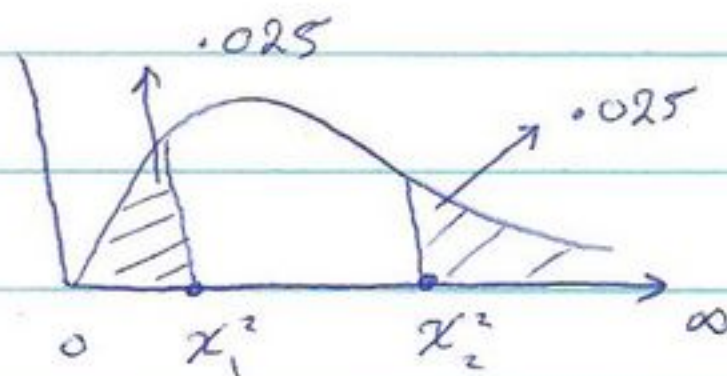


$$P(\chi^2_v > \chi^2_2) = 0.05 \Rightarrow 1 - P(\chi^2_v < \chi^2_2) = 0.05$$

$$\Rightarrow P(\chi^2_v < \chi^2_2) = 0.95 \Rightarrow \chi^2_{v,0.95} = \chi^2_2 = 11.07$$

(b)

$$\frac{0.05}{2} = 0.025$$



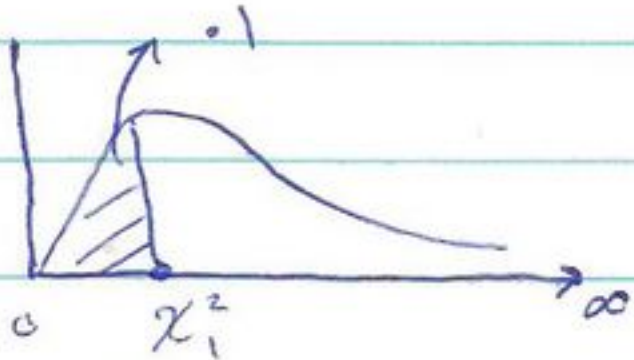
Since the distribution of  $\chi^2_v$  is not symmetric,  $\chi^2_1 \neq \chi^2_2$

$$* P(\chi^2_v < \chi^2_1) = 0.025 \Rightarrow \chi^2_{v,0.025} = \chi^2_1 = 0.831$$

$$* P(\chi^2_v > \chi^2_2) = 0.025$$

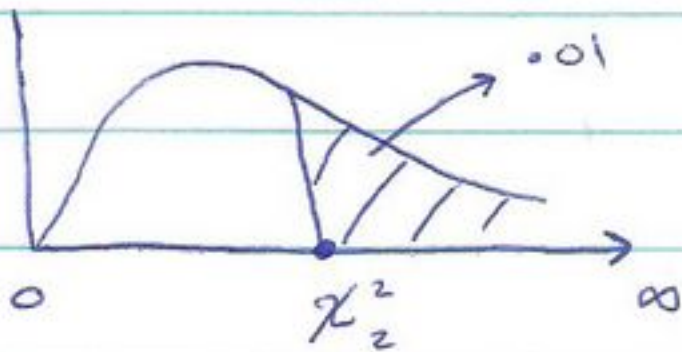
$$\Rightarrow 1 - P(\chi^2_v < \chi^2_2) = 0.025 \Rightarrow P(\chi^2_v < \chi^2_2) = 0.975 \Rightarrow \chi^2_{v,0.975} = \chi^2_2 = 12.83$$

(c)



$$P(\chi^2_v < \chi^2_1) = 0.1 \Rightarrow \chi^2_{v,0.1} = \chi^2_1 = 1.61$$

(d)



$$P(\chi^2_v > \chi^2_2) = 0.01$$

$$\Rightarrow 1 - P(\chi^2_v < \chi^2_2) = 0.01$$

$$\Rightarrow P(\chi^2_v < \chi^2_2) = 0.99$$

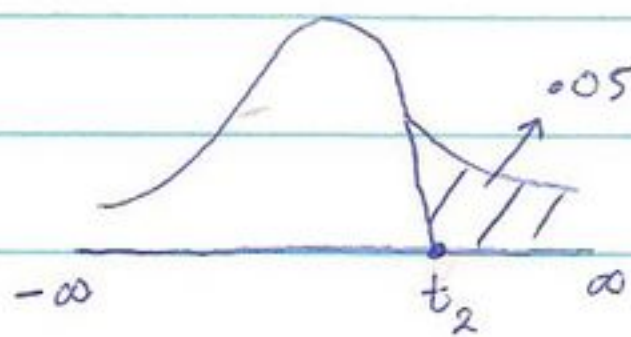
$$\Rightarrow \chi^2_{v,0.99} = \chi^2_2 = 15.09$$



Q6

$$X \sim T_{v=9}$$

a)

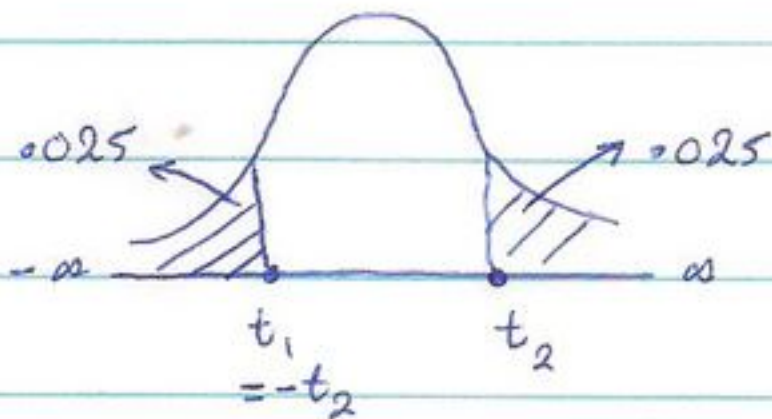


$$P(T_v > t_2) = 0.05 \Rightarrow 1 - P(T_v < t_2) = 0.05$$

$$\Rightarrow P(T_v < t_2) = 0.95 \Rightarrow t_{v, 0.95} = t_2 = 1.833$$

b)

$$\frac{0.05}{2} = 0.025$$



Since the distribution is symmetric

$$\text{i.e. } t_1 = -t_2$$

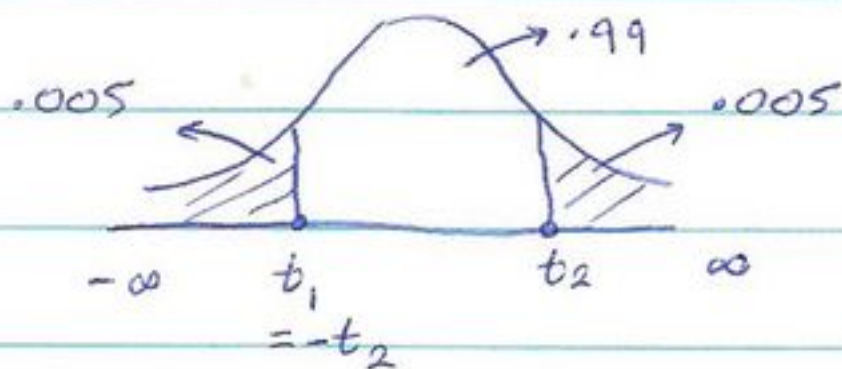
$$P(T_v > t_2) = 0.025 \Rightarrow 1 - P(T_v < t_2) = 0.025 \Rightarrow P(T_v < t_2) = 0.975$$

$$\Rightarrow t_{v, 0.975} = t_2 = 2.262$$

$$\therefore t_1 = -t_2 = -2.262$$

c)

$$1 - 0.99 = 0.01 \Rightarrow \frac{0.01}{2} = 0.005$$



Since the distribution is symmetric

$$\text{i.e. } t_1 = -t_2$$

$$P(T_v > t_2) = 0.005 \Rightarrow 1 - P(T_v < t_2) = 0.005 \Rightarrow P(T_v < t_2) = 0.995$$

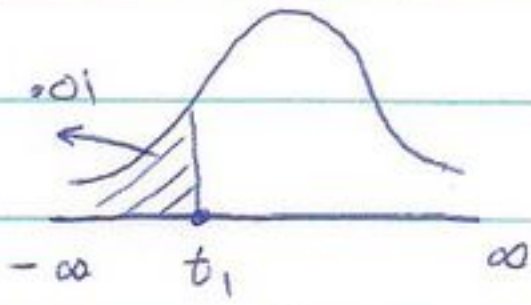
$$\Rightarrow t_{v, 0.995} = t_2 = 3.250$$

$$\therefore t_1 = -t_2 = -3.250$$

③



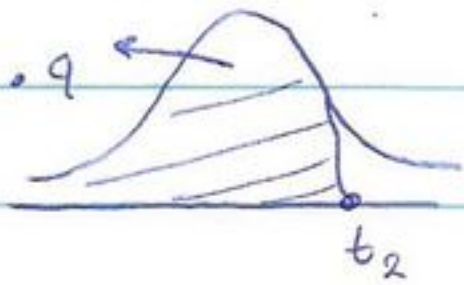
d)



$$P(T_v < t_1) = 0.01 \Rightarrow t_1 = t_{v, 0.01} = -t_{v, 1-0.01} = -t_{v, 0.99} = -2.821$$

(From the symmetry of T distribution)

e)



$$P(T_v < t_2) = 0.9 \Rightarrow t_2 = t_{v, 0.9} = 1.383$$

Q7

$X \sim \exp(\theta = \ln(3))$

$$f(x) = \theta e^{-\theta x} = \ln(3) e^{-\ln(3)x}, \quad x > 0$$

$$F(x) = 1 - e^{-\theta x} = 1 - e^{-\ln(3)x}, \quad x > 0$$

$$P(2 \leq X \leq 4) \xrightarrow{\text{pdf}} \int_2^4 f(x) dx = \ln(3) \int_2^4 e^{-\ln(3)x} dx = 0.0988 \quad (\text{Quellplatz})$$

$$\xrightarrow{\text{cdf}} = P(X \leq 4) - P(X \leq 2) = 1 - e^{-\ln(3) \cdot 4} - (1 - e^{-\ln(3) \cdot 2}) = 0.0988$$

Q8

$X \sim \exp(\theta = 1)$

$$f(x) = \theta e^{-\theta x} = e^{-x}, \quad x > 0$$

$$F(x) = 1 - e^{-\theta x} = 1 - e^{-x}, \quad x > 0$$

$$P(X > 2) \xrightarrow{\text{pdf}} \int_2^{\infty} f(x) dx = \int_2^{\infty} e^{-x} dx = \frac{e^{-x}}{-1} \Big|_2^{\infty} = -(e^{-\infty} - e^{-2}) = e^{-2}$$

$$\xrightarrow{\text{cdf}} = 1 - P(X < 2) = 1 - (1 - e^{-2}) = e^{-2}$$

Q9

$$X \sim \exp(\theta) \Rightarrow f(x) = \theta e^{-\theta x}, \quad x > 0; \quad F(x) = 1 - e^{-\theta x}, \quad x > 0; \quad E(X) = \frac{1}{\theta}$$

$$P(X < E(X)) = P(X < \frac{1}{\theta}) = F(\frac{1}{\theta}) = 1 - e^{-\theta \cdot \frac{1}{\theta}} = 1 - e^{-1}$$



$$Q10 \quad (a) \quad X \sim \chi^2_v \Rightarrow M_X(t) = \left( \frac{1}{1-2t} \right)^{\frac{v}{2}}, \quad t < \frac{1}{2}$$

$$\therefore \frac{v}{2} = 1 \Rightarrow v = 2$$

$$\text{or } X \sim \exp(\theta) \Rightarrow M_X(t) = \frac{\theta}{\theta - t}, \quad t < \theta$$

$$\therefore \frac{1}{1-2t} = \frac{\frac{1}{2}}{\frac{1}{2}-t} \Rightarrow \theta = \frac{1}{2}$$

$$(b) \quad X \sim N(\mu, \sigma^2) \Rightarrow M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

$$\therefore \mu = 3 \text{ and } \frac{1}{2}\sigma^2 = 2 \Rightarrow \sigma^2 = 4$$

(c)  $X$  and  $Y$  are indep.

$$M_{X+Y}(t) = \left( \frac{2}{2-t} \right)^3, \quad Y \sim \exp(\theta=2) \Rightarrow M_Y(t) = \frac{2}{2-t}$$

$$\therefore M_{X+Y}(t) = M_X(t) M_Y(t)$$

$$\Rightarrow \left( \frac{2}{2-t} \right)^3 = M_X(t) \left( \frac{2}{2-t} \right)$$

$$\Rightarrow M_X(t) = \frac{\left( \frac{2}{2-t} \right)^3}{\left( \frac{2}{2-t} \right)} = \left( \frac{2}{2-t} \right)^2$$

$\therefore X \sim \text{gamma}(\alpha=2, \beta=2)$

Q11  $X$  and  $Y$  are indep.

$$M_{X+Y}(t) = \frac{e^{2t} - 1}{2t - t^2}$$

$$X \sim \text{exp}(\lambda=2) \Rightarrow M_X(t) = \frac{2}{2-t}$$

$$M_{X+Y}(t) = M_X(t) M_Y(t)$$

$$\Rightarrow \frac{e^{2t} - 1}{2t - t^2} = \frac{2}{2-t} M_Y(t)$$

$$\begin{aligned} \Rightarrow M_Y(t) &= \frac{\frac{e^{2t} - 1}{t(2-t)}}{\frac{2}{2-t}} = \frac{e^{2t} - 1}{t(2-t)} \cdot \frac{2-t}{2} = \frac{e^{2t} - 1}{2t} \\ &= \frac{e^{2t} - e^0}{t(2-0)} \end{aligned}$$

$\therefore X \sim \text{uniform}(a=0, b=2)$