

1.

$$\begin{aligned}\langle ae^x + be^{-x}, x \rangle &= a \int_{-1}^1 xe^x dx + b \int_{-1}^1 xe^{-x} dx \\ &= a \left[xe^x \Big|_{-1}^1 - \int_{-1}^1 e^x dx \right] + b \left[xe^{-x} \Big|_{-1}^1 - \int_{-1}^1 e^{-x} dx \right] \\ &= 2e^{-1}(a-b) = 0 \Rightarrow a = b.\end{aligned}$$

$$\begin{aligned}\|ae^x + ae^{-x}\|^2 &= a^2 \int_{-1}^1 (e^{2x} + 2 + e^{-2x}) dx \\ &= a^2 (e^2 - e^{-2} + 4) \\ &= a^2 (2 \sinh 2 + 4) = 1 \\ \Rightarrow a &= \pm \frac{1}{\sqrt{2 \sinh 2 + 4}}.\end{aligned}$$

2. (i) $a_0 = \frac{1}{2} \int_{-1}^1 (x+1) dx = 1$.

$$a_n = \int_{-1}^1 (x+1) \cos n\pi x dx = \frac{1}{n^2\pi^2} \cos n\pi x \Big|_{-1}^1 = 1 \quad \forall n \geq 2.$$

$$b_n = \int_{-1}^1 (x+1) \sin n\pi x dx = -\frac{2}{n\pi} \cos n\pi + \frac{1}{n^2\pi^2} \sin n\pi x \Big|_{-1}^1 = \frac{2}{n\pi} (-1)^{n+1}.$$

$$\Rightarrow x+1 = 1 - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin n\pi x \quad \forall -1 < x < 1.$$

(ii) Since $f(-1) \neq f(1)$ the convergence is not uniform.

3.

$$\begin{aligned}\langle e^{-x/2}, L_n \rangle &= \int_0^\infty e^{-x/2} L_n(x) e^{-x} dx \\ &= \frac{1}{n!} \int_0^\infty e^{-x/2} \frac{d^n}{dx^n} (x^n e^{-x}) dx \\ &= \frac{1}{n!} \left[e^{-x/2} \frac{d^{n-1}}{dx^{n-1}} (x^n e^{-x}) \right]_0^\infty + \frac{1}{2} \int_0^\infty e^{-x/2} \frac{d^{n-1}}{dx^{n-1}} (x^n e^{-x}) dx \\ &= \dots \\ &= \frac{1}{n!2^n} \int_0^\infty e^{-x/2} x^n e^{-x} dx \\ &= \frac{1}{n!2^n} \int_0^\infty x^n e^{-3x/2} dx.\end{aligned}$$

$$\begin{aligned}
\int_0^\infty x^n e^{-3x/2} dx &= -\frac{2}{3} x^n e^{-3x/2} \Big|_0^\infty + \frac{2}{3} n \int_0^\infty x^{n-1} e^{-3x/2} dx \\
&= \dots \\
&= \left(\frac{2}{3}\right)^n n! \int_0^\infty e^{-3x/2} dx \\
&= \left(\frac{2}{3}\right)^{n+1} n!
\end{aligned}$$

$$\Rightarrow \langle e^{-x/2}, L_n \rangle = \frac{2}{3^{n+1}}, \quad e^{-x/2} = \sum_{n=0}^{\infty} \frac{\langle e^{-x/2}, L_n \rangle}{\|L_n\|^2} L_n(x) = 2 \sum_{n=0}^{\infty} 3^{-n-1} L_n(x).$$

$$4. \quad y' = -\frac{1}{2}x^{-3/2}u + x^{-1/2}u', \quad y'' = \frac{3}{4}x^{-5/2}u - x^{-3/2}u' + x^{-1/2}u''.$$

Direct substitution into Bessel's equation gives $u'' + u = 0$, whose general solution is

$$u(x) = c_1 \cos x + c_2 \sin x.$$

This implies that

$$y(x) = c_1 \frac{\cos x}{\sqrt{x}} + c_2 \frac{\sin x}{\sqrt{x}}.$$

5.

$$\begin{aligned}
J_{-1/2}(x) &= \left(\frac{x}{2}\right)^{-1/2} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \frac{1}{2})} \left(\frac{x}{2}\right)^{2m} \\
&= \sqrt{\frac{2}{x}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! (m - \frac{1}{2}) \cdots (1/2) \Gamma(1/2)} \left(\frac{x}{2}\right)^{2m} \\
&= \sqrt{\frac{2}{\pi x}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! (2m - 1) \cdots (1) \Gamma(1/2)} \frac{x^{2m}}{2^m} \\
&= \sqrt{\frac{2}{\pi x}} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m} = \sqrt{\frac{2}{\pi x}} \cos x.
\end{aligned}$$

The zeros of $J_{-1/2}(x)$ in $(0, \infty)$ are those of $\cos x$, namely $x_n = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$.

6. f is an odd function, hence $A(\xi) = 0$.

$$B(\xi) = 2 \int_0^\infty \sin x \sin x\xi \, dx = \int_0^\infty [\cos(1-\xi)x + \cos(1+\xi)x] \, dx = 2 \frac{\sin \pi \xi}{1 - \xi^2}. \text{ Therefore}$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin \pi \xi}{1 - \xi^2} \sin x\xi \, d\xi.$$

When $x = \pi$, we get

$$\frac{1}{2} = \frac{2}{\pi} \int_0^\infty \frac{\sin^2 \pi \xi}{1 - \xi^2} d\xi.$$