# *Logic Mathematic (Math 132)*

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# Chapter 3: Sets

## Definition 1.1

*A* **set** *is an unordered collection of objects, called elements or members of the set. A set is said to contain its elements.We write $a \in A$ to denote that $a$ is an element of the set $A$. The notation $a \notin A$ denotes that $a$ is not an element of the set $A$.*

It is common for sets to be denoted using uppercase letters. Lowercase letters are usually used to denote elements of sets.
There are several ways to describe a set. One way is to list all the members of a set, when this is possible. We use a notation where all members of the set are listed between braces. For example, the notation $\{a, b, c, d\}$ represents the set with the four elements $a, b, c$, and $d$. This way of describing a set is known as the **roster method**.

## Example 1.1

The set $V$ of all vowels in the English alphabet can be written as $V = \{a, e, i, o, u\}$.

## Example 1.2

The set $O$ of odd positive integers less than $10$ can be expressed by $O = \{1, 3, 5, 7, 9\}$.

Sometimes the roster method is used to describe a set without listing all its members. Some members of the set are listed, and then **ellipses** $(\ldots)$ are used when the general pattern of the elements is obvious.

## Example 1.3

The set of positive integers less than $100$ can be denoted by $\{1, 2, 3, \ldots, 99\}$.

## Example 1.4

Another way to describe a set is to use set builder notation. We characterize all those elements in the set by stating the property or properties they must have to be members. For instance, the set $O$ of all odd positive integers less than $10$ can be written as

$$O = \{x|x \text{ is an odd positive integer less than 10 }\},$$

or, specifying the universe as the set of positive integers, as

$$O = \{x \in \mathbb{Z}^+|x \text{ is odd and } x < 10\}.$$

We often use this type of notation to describe sets when it is impossible to list all the elements of the set. For instance, the set $\mathbb{Q}^+$ of all positive rational numbers can be written as

$$\mathbb{Q}^+ = \{x \in \mathbb{R}|x = \frac{p}{q}, \text{ for some positive integers } p \text{ and } q\}.$$

# Sets

These sets, each denoted using a boldface letter, play an important role in discrete mathematics:

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$, the set of **natural numbers**

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, the set of **integers**

$\mathbb{Z}^+ = \{1, 2, 3, \dots\}$, the set of **positive integers**

$\mathbb{Q} = \{\frac{p}{q} | p \in \mathbb{Z}, q \in \mathbb{Z}, \text{ and } q \neq 0\}$, the set of **rational numbers**

$\mathbb{R}$, the set of **real numbers**

$\mathbb{R}^+$, the set of **positive real numbers**

$\mathbb{C}$, the set of **complex numbers**.

Recall the notation for **intervals** of real numbers. When $a$ and $b$ are real numbers with $a < b$, we write

$[a, b] = \{x | a \leq x \leq b\}$ $\quad$ $(a, b] = \{x | a < x \leq b\}$

$[a, b) = \{x | a \leq x < b\}$ $\quad$ $(a, b) = \{x | a < x < b\}$

Note that $[a, b]$ is called the **closed interval** from $a$ to $b$ and $(a, b)$ is called the **open interval** from $a$ to $b$.

Sets can have other sets as members, as Example 4 illustrates.

## Example 1.5

The set $\{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$ is a set containing four elements, each of which is a set. The four elements of this set are $\mathbb{N}$, the set of natural numbers; $\mathbb{Q}$, the set of integers; $\mathbb{Q}$, the set of rational numbers; and $\mathbb{R}$, the set of real numbers.

## Remark 1.1

*Note that the concept of a datatype, or type, in computer science is built upon the concept of a set. In particular, a* **datatype** *or* **type** *is the name of a set, together with a set of operations that can be performed on objects from that set. For example, boolean is the name of the set $\{0, 1\}$ together with operators on one or more elements of this set, such as AND, OR, and NOT.*

# Sets

Because many mathematical statements assert that two differently specified collections of objects are really the same set, we need to understand what it means for two sets to be equal.

## Definition 1.2

*Two sets are equal if and only if they have the same elements. Therefore, if $A$ and $B$ are sets, then $A$ and $B$ are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if $A$ and $B$ are equal sets.*

## Example 1.6

The sets $\{1, 3, 5\}$ and $\{3, 5, 1\}$ are equal, because they have the same elements. Note that the order in which the elements of a set are listed does not matter. Note also that it does not matter if an element of a set is listed more than once, so $\{1, 3, 3, 3, 5, 5, 5, 5\}$ is the same as the set $\{1, 3, 5\}$ because they have the same elements.

## THE EMPTY SET

There is a special set that has no elements. This set is called the **empty set**, or null set, and is denoted by $\emptyset$. The empty set can also be denoted by $\{\}$. Often, a set of elements with certain properties turns out to be the null set. For instance, the set of all positive integers that are greater than their squares is the null set.

## SINGLETON SET

A set with one element is called a **singleton set**. A common error is to confuse the empty set $\emptyset$ with the set $\{\emptyset\}$, which is a singleton set

## Venn Diagrams

Sets can be represented graphically using Venn diagrams, named after the English mathematician JohnVenn, who introduced their use in 1881. InVenn diagrams the universal set U, which contains all the objects under consideration, is represented by a rectangle. (Note that the universal set varies depending on which objects are of interest.) Inside this rectangle, circles or other geometrical figures are used to represent sets. Sometimes points are used to represent the particular elements of the set.Venn diagrams are often used to indicate the relationships between sets.We show how a Venn diagram can be used in Example 6.

## Example 1.7

Draw a Venn diagram that represents $V$, the set of vowels in the English alphabet.
**Solution**: We draw a rectangle to indicate the universal set $U$, which is the set of the $26$ letters of the English alphabet. Inside this rectangle we draw a circle to represent $V$ . Inside this circle we indicate the elements of $V$ with points (see Figure 1).

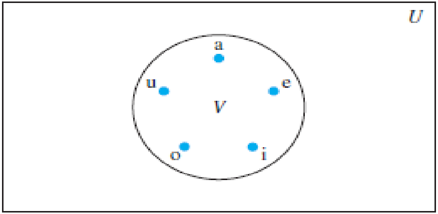


Figure: Venn Diagram for the Set of Vowels.

## Subsets

### Definition 1.3

*The set $A$ is a subset of $B$ if and only if every element of $A$ is also an element of $B$. We use the notation $A \subseteq B$ to indicate that $A$ is a subset of the set $B$.*

We see that $A \subseteq B$ if and only if the quantification

$$\forall x(x \in A \to x \in B)$$

is true. Note that to show that $A$ is not a subset of $B$ we need only find one element $x \in A$ with $x \notin B$. Such an $x$ is a counterexample to the claim that $x \in A$ implies $x \in B$. We have these useful rules for determining whether one set is a subset of another:

**Showing that $A$ is a Subset of $B$** To show that $A \subseteq B$, show that if $x$ belongs to $A$ then $x$ also belongs to $B$.
**Showing that $A$ is Not a Subset of $B$** To show that $A \nsubseteq B$, find a single $x \in A$ such that $x \notin B$.

### Example 1.8

The set of integers with squares less than $100$ is not a subset of the set of nonnegative integers because $-1$ is in the former set [as $(-1)^2 < 100$].

### Theorem 1.1

For every set $S$, (i ) $\emptyset \subseteq S$ and (ii ) $S \subseteq S$.

Showing Two Sets are Equal To show that two sets A and B are equal, show that $A \subseteq B$ and $B \subseteq A$.

Sets may have other sets as members. For instance, we have the sets $A = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $B = \{x | x$ is a subset of the set $\{a, b\}\}$. Note that these two sets are equal, that is, $A = B$. Also note that $\{a\} \in A$, but $a \notin A$.

## Definition 2.1

*Let $S$ be a set. If there are exactly $n$ distinct elements in $S$ where $n$ is a nonnegative integer, we say that $S$ is a finite set and that $n$ is the cardinality of $S$. The cardinality of $S$ is denoted by $|S|$.*

## Example 2.1

Let $A$ be the set of odd positive integers less than $10$. Then $|A| = 5$.

## Example 2.2

Let $S$ be the set of letters in the English alphabet. Then $|S| = 26$.

## Example 2.3

Because the null set has no elements, it follows that $|\emptyset| = 0$.

**Definition 2.2**

*A set is said to be* **infinite** *if it is not finite.*

**Example 2.4**

The set of positive integers is infinite.

## Power Sets

### Definition 2.3

*Given a set $S$, the power set of $S$ is the set of all subsets of the set $S$. The power set of $S$ is denoted by $\mathcal{P}(S)$.*

### Example 2.5

What is the power set of the set 0, 1, 2?

**Solution**: The power set $\mathcal{P}(\{0,1,2\})$ is the set of all subsets of $\{0,1,2\}$. Hence,

$$\mathcal{P}(\{0,1,2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}.$$

Note that the empty set and the set itself are members of this set of subsets.

### Example 2.6

What is the power set of the empty set? What is the power set of the set $\{\emptyset\}$?
**Solution**: The empty set has exactly one subset, namely, itself. Consequently,

$$\mathcal{P}(\emptyset) = \{\emptyset\}.$$

The set $\{\emptyset\}$ has exactly two subsets, namely, $\emptyset$ and the set $\{\emptyset\}$ itself. Therefore,

$$\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}.$$

If a set has $n$ elements, then its power set has $2^n$ elements.

The order of elements in a collection is often important. Because sets are unordered, a different structure is needed to represent ordered collections. This is provided by ordered $n$-**tuples**.

## Definition 3.1

*The* **ordered** $n$**-tuple** $(a_1, a_2, \ldots, a_n)$ *is the ordered collection that has* $a_1$ *as its first element,* $a_2$ *as its second element,* $\ldots$ *, and* $a_n$ *as its* $n$*th element.*

We say that two ordered $n$-tuples are equal if and only if each corresponding pair of their elements is equal. In other words, $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$ if and only if $a_i = b_i$ , for $i = 1, 2, \ldots, n$. In particular, ordered $2$-tuples are called ordered pairs. The ordered pairs $(a, b)$ and $(c, d)$ are equal if and only if $a = c$ and $b = d$. Note that $(a, b)$ and $(b, a)$ are not equal unless $a = b$.

## Definition 3.2

*Let $A$ and $B$ be sets. The Cartesian product of $A$ and $B$, denoted by $A \times B$, is the set of all ordered pairs $(a, b)$, where $a \in A$ and $b \in B$. Hence,*
$A \times B = \{(a, b) | a \in A \wedge b \in B\}$.

## Example 3.1

What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?
**Solution**: The Cartesian product $A \times B$ is
$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$.

## Example 3.2

Show that the Cartesian product $B \times A$ is not equal to the Cartesian product $A \times B$, where $A$ and $B$ are as in Example 3.1.
**Solution**: The Cartesian product $B \times A$ is
$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$.
This is not equal to $A \times B$, which was found in Example 14.

## Definition 3.3

*The Cartesian product of the sets* $A_1, A_2, \ldots, A_n$*, denoted by* $A_1 \times A_2 \times \cdots \times A_n$*, is the set of ordered* $n$*-tuples* $(a_1, a_2, \ldots, a_n)$*, where* $a_i$ *belongs to* $A_i$ *for* $i = 1, 2, \ldots, n$*. In other words,*

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}$$

## Example 3.3

What is the Cartesian product $A \times B \times C$, where $A = \{0, 1\}$, $B = \{1, 2\}$, and $C = \{0, 1, 2\}$ ?

**Solution**: The Cartesian product $A \times B \times C$ consists of all ordered triples $(a, b, c)$, where $a \in A$, $b \in B$, and $c \in C$. Hence,
$A \times B \times C = \{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2),$
$(1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}$.

# Sets

We use the notation $A^2$ to denote $A \times A$, the Cartesian product of the set $A$ with itself. Similarly, $A^3 = A \times A \times A$, $A^4 = A \times A \times A \times A$, and so on. More generally, $A^n = \{(a1, a2, ..., an) | a_i \in A$ for $i = 1, 2, \ldots, n\}$.

## Example 3.4

Suppose that $A = \{1, 2\}$. It follows that $A^2 = \{(1,1), (1,2), (2,1), (2,2)\}$ and $A^3 = \{(1,1,1), (1,1,2), (1,2,1), (1,2,2), (2,1,1), (2,1,2), (2,2,1), (2,2,2)\}$.

A subset $R$ of the Cartesian product $A \times B$ is called a relation from the set $A$ to the set $B$. The elements of $R$ are ordered pairs, where the first element belongs to $A$ and the second to $B$. For example, $R = \{(a,0), (a,1), (a,3), (b,1), (b,2), (c,0), (c,3)\}$ is a relation from the set $\{a, b, c\}$ to the set $\{0, 1, 2, 3\}$. A relation from a set $A$ to itself is called a relation on $A$.

## Example 3.5

What are the ordered pairs in the less than or equal to relation, which contains $(a, b)$ if $a \leq b$, on the set $\{0, 1, 2, 3\}$?

**Solution**: The ordered pair $(a, b)$ belongs to $R$ if and only if both $a$ and $b$ belong to $\{0, 1, 2, 3\}$ and $a - b$. Consequently, the ordered pairs in $R$ are $(0, 0), (0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 2), (2, 3)$, and $(3, 3)$.

Using Set Notation with Quantifiers

## Example 3.6

What do the statements $\forall x \in \mathbb{R}(x^2 \geq 0)$ and $\exists x \in \mathbb{Z}(x^2 = 1)$ mean?

**Solution**: The statement $\forall x \in \mathbb{R}(x^2 \geq 0)$ states that for every real number $x$, $x^2 \geq 0$. This statement can be expressed as "The square of every real number is nonnegative." This is a true statement.

The statement $\exists x \in \mathbb{Z}(x^2 = 1)$ states that there exists an integer $x$ such that $x^2 = 1$. This statement can be expressed as "There is an integer whose square is 1." This is also a true statement because $x = 1$ is such an integer (as is $-1$).