King Saud University,
College of Sciences
Mathematical Department.

Mid-Term 2/S1/2016
Full Mark: 25. Time 1H30mn
28/12/2016

Question 1[4,4] **a)** Find the largest interval for which the following initial value problem has a unique solution

$$\begin{cases} (x^2-4)y'' + xy' + 2y = \ln x \\ y(3) = 1,\ y'(3) = 2. \end{cases}$$

b) Solve the nonhomogeneous differential equation

$$y'' + y = \csc x,\ \ x \in (0, \frac{\pi}{2})$$

Question 2 [4,3]. **a)** Show that $y_1 = \sin x$ is a solution of the differential equation

$$y'' + (3\tan x)y' - 2y = 0, \quad x \in (0, \frac{\pi}{2}).$$

Find the second solution, then obtain the general solution.

b) Show whether the functions

$$f_1(x) = x, \quad f_2(x) = x\ln x,$$

are linearly independent or linearly dependent on $(0, \infty)$.

Question 3 [5] Find the general solution of the differential equation

$$x^2y'' - 2xy' + 2y = x^3 \ln x\ ; \quad x > 0.$$

Question 4 [5] Solve the following linear system of differential equations.

$$\begin{cases} x' = -x + 3y + e^t \\ y' = -2x + 4y. \end{cases}$$

Complete Solutions of M. 204.
Mid-2. Semester I 2016

Question 1

ⓐ $\begin{cases} (x^2-4)y'' + xy' + 2y = \ln x \; ; \; x > 0 \\ y(3) = 1 \, , \, y'(3) = 2 \end{cases}$

$a_2(x) = x^2 - 4 \neq 0$ if $x \in (0,2) \cup (2, \infty)$ (1)

$a_1(x) = x$, $a_0(x) = 2$, then a_0, a_1 and a_2 are Continuous on $(0, \infty)$ (1)

$R(x) = \ln x$ continuous on $(0, \infty)$

But $3 \in (2, \infty)$, so the IVP will have a unique solution on the largest interval $\boxed{I = (2, \infty)}$ (1)

(1)

ⓑ $y'' + y = \csc x \; ; \quad 0 < x < \frac{\pi}{2}$

1) We find the G. solution of $y'' + y = 0$, $y = e^{mx}$, then we have $m^2 + 1 = 0$. Hence $m = \mp i$, then

$$y_c = c_1 \sin x + c_2 \cos x.$$ (1)

2) The particular solution of $y'' + y = \csc x$ is the form

$$y_p = y_1 u_1 + y_2 u_2 \, , \text{ where } y_1 = \sin x \text{ and } y_2 = \cos x.$$

The functions u_1 and u_2 satisfy two D. Eqs.

$$\begin{cases} u_1' (\sin x) + u_2' (\cos x) = 0 \\ u_1' (\cos x) - u_2' \sin x = \csc x \; ; \quad 0 < x < \pi \end{cases}$$ (1)

$$W(x, y_1, y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1$$

$$u_1' = \frac{\begin{vmatrix} 0 & \cos x \\ \csc x & -\sin x \end{vmatrix}}{W} = \frac{-\cot x}{-1} = \cot x \Rightarrow u_1 = \int \frac{\cos x}{\sin x} dx = \boxed{\ln (\sin x)}$$ (1)

$$u_2' = \frac{\begin{vmatrix} \sin x & 0 \\ \cos x & \csc x \end{vmatrix}}{W} = \frac{1}{-1} = -1 \Rightarrow \boxed{u_2 = -x}$$

Thus the particular solution is

$$\boxed{y_p = (\sin x)(\ln \sin x) - x\cos x}$$

and the G. solution of the D. E is given by

$$\boxed{y = y_c + y_p = c_1 \sin x + c_2 \cos x + \sin x (\ln \sin x) - x \cos x}$$

Where c_1 and c_2 are two arbitrary constants. (1)

Question ②

ⓐ $y_1 = \sin x,\ y_1' = \cos x,\ y_1'' = -\sin x \qquad 0 < x < \pi/2$

$$y_1'' + (3\tan x)y_1' - 2y_1 = -\sin x + 3(\tan x)\cos x - 2\sin x =$$
$$= -3\sin x + 3\sin x = 0$$

Then $y_1 = \sin x$ is a solution of the D. E. (1)

$$y'' + (3\tan x)y' - 2y = 0 \qquad 0 < x < \pi/2.$$

Thus the second solution y_2 is given by

$$y_2 = y_1 \int \frac{e^{\int -P(x)dx}}{(y_1)^2} dx, \text{ where } P(x) = 3\tan x. \quad (1)$$

But $e^{\int -3\tan x\, dx} = e^{3\ln(\cos x)} = e^{\ln \cos^3 x} = \cos^3 x$, hence

$$y_2 = \sin x \int \frac{\cos^3 x}{\sin^2 x} dx = \sin x \int \frac{(1-\sin^2 x)\cos x}{\sin^2 x} dx$$

We put $\sin x = u$, $du = \cos x\, dx$, then

$$I = \int \frac{1-u^2}{u^2} du = \int \frac{du}{u^2} - \int du$$

$$I = -\frac{1}{\sin x} - \sin x \quad (1)$$

Hence $\boxed{y_2 = \sin x\left(\frac{-1}{\sin x} - \sin x\right) = -1 - \sin^2 x}$ and the G. solution of the D. E is

$$\boxed{y_c = c_1 \sin x - c_2(1 + \sin^2 x) = c_1 \sin x + c_3(1 + \sin^2 x)} \quad (1)$$

(b) $W(x, f_1(x), f_2(x)) = \begin{vmatrix} x & x\ln x \\ 1 & \ln x + 1 \end{vmatrix} = x \neq 0$ for all $x > 0$. 3

Then $f_1 = x$. and $f_2 = x\ln x$ are linearly independent on $(0, \infty)$

$\boxed{\text{Question (3)}}$

$$x^2 y'' - 2xy' + 2y = x^3 \ln x, \quad x > 0 \quad (1)$$

1) $x^2 y'' - 2xy' + 2y = 0, \quad y = x^m, \quad x > 0$

$m(m-1) - 2m + 2 = 0, \quad m^2 - 3m + 2 = (m-1)(m-2) = 0$

$m = 1, \; m = 2$, then

$y_c = c_1 x + c_2 x^2$, 1

2) let $y_1 = x$, $y_2 = x^2$, the particular sol. of the D.E (1) is

$y_p = u_1 y_1 + u_2 y_2$, where u_1 and u_2 satisfying

two D. Eqs. $\begin{cases} u_1'(x) + u_2'(x^2) = 0 \\ u_1'(1) + u_2'(2x) = x\ln x, \quad x > 0 \end{cases}$

$$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2$$

$$u_1' = \frac{\begin{vmatrix} 0 & x^2 \\ x\ln x & 2x \end{vmatrix}}{x^2} = -x\ln x \Rightarrow u_1 = -\int x\ln x\,dx = -\frac{x^2}{2}\ln x + \int \frac{x}{2}dx \quad 1$$

$$\boxed{u_1 = -\frac{x^2}{2}\ln x + \frac{x^2}{4}}$$

$$u_2' = \frac{\begin{vmatrix} x & 0 \\ 1 & x\ln x \end{vmatrix}}{x^2} = \ln x \Rightarrow \boxed{u_2 = x\ln x - x} \quad 1$$

$$y_p = x\left(-\frac{x^2}{2}\ln x + \frac{x^2}{4}\right) + x^2(x\ln x - x)$$

$$\boxed{y_p = \frac{1}{2}x^3\ln x - \frac{3}{4}x^3}$$

Thus the G. solution of (1) is $\boxed{y = y_c + y_p = c_1 x + c_2 x^2 + \frac{1}{2}x^3\ln x - \frac{3}{4}x^3}$ 2

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Question ④ $\begin{cases} x' = -x + 3y + e^t \quad \text{--- ①} \\ y' = -2x + 4y \quad \text{--- ②} \end{cases}$

$$(D+1)x - 3y = e^t$$
$$2x + (D-4)y = 0$$

We eliminate y: $(D-4)(D+1)x - 3(D-4)y = (D-4)(e^t)$

$$\underline{+ \quad 6x + 3(D-4)y = 0}$$

$$(D^2 - 3D - 4)x + 6x = e^t - 4e^t = -3e^t$$

$$\underline{x''(t) - 3x'(t) + 2x = -3e^t} \quad \text{--- (3)}$$

1) $x'' - 3x' + 2x = 0$, $x = e^{mt}$

$m^2 - 3m + 2 = (m-1)(m-2) = 0 \Rightarrow m = 1, 2$

$$x_c(t) = c_1 e^t + c_2 e^{2t} \quad \text{(4)}$$

2) $x_p = Ate^t$, $x_p' = Ae^t + Ate^t$

$$x_p'' = 2Ae^t + Ate^t$$

$$x_p'' - 3x_p' + 2x_p = 2Ae^t + Ate^t - 3Ae^t - 3Ate^t + 2Ate^t = -3e^t$$

$$-Ae^t = -3e^t \Rightarrow A = 3$$

Hence $\boxed{x_p = 3te^t}$ --- ⑤

Thus the G. solution of ③ is

$$\boxed{x(t) = x_c(t) + x_p(t) = c_1 e^t + c_2 e^{2t} + 3te^t} \quad \text{--- ⑥}$$

From ①, we have: $3y = x' + x - e^t$

$$3y = c_1 e^t + 2c_2 e^{2t} + 3e^t + 3te^t + c_1 e^t + c_2 e^{2t} + 3te^t - e^t$$

$$3y = 2c_1 e^t + 3c_2 e^{2t} + 6te^t + 2e^t$$

$$\boxed{y(t) = \tfrac{2}{3} c_1 e^t + c_2 e^{2t} + 2te^t + \tfrac{2}{3} e^t} \quad \text{--- ⑦}$$

Thus the G. solution of the system ① + ② is

$$\begin{cases} x(t) = c_1 e^t + c_2 e^{2t} + 3te^t \\ y(t) = \tfrac{2}{3} c_1 e^t + c_2 e^{2t} + 2te^t + \tfrac{2}{3} e^t \end{cases}$$

where c_1 and c_2 are two arbitrary constants