King Saud University,
College of Sciences
Mathematical Department.

Mid-Term 2/S1/2018
Full Mark: 25. Time 1H30mn
15/11/2018

Question 1[4,4] a) Find the largest interval for which the following initial value problem has a unique solution

$$\begin{cases} (x-2)y'' + 3y = x \\ y(0) = 0, \ y'(0) = 1. \end{cases}$$

b) Solve the nonhomogeneous differential equation

$$y'' - y = 2e^x - 2x^2 + 5$$

Question 2 [4,3]. a) If $y_1 = e^x$ is a solution of the differential equation

$$y'' + 3y' - 4y = x,$$

then use reduction of order method to obtain its general solution.

b) Determine a homogeneous linear differential equation with constant coefficients having the fundamental set of solutions:

$$y_1 = 7, \quad y_2 = 8x, \quad y_3 = e^{-x}\cos x, \quad y_4 = e^{-x}\sin x, \ y_5 = 5x^2.$$

Question 3 [5] Find the general solution of the differential equation

$$xy'' - 2y' + \frac{2}{x}y = 3x^3 + 2x; \quad x > 0.$$

Question 4 [5] Solve the following linear system of differential equations.

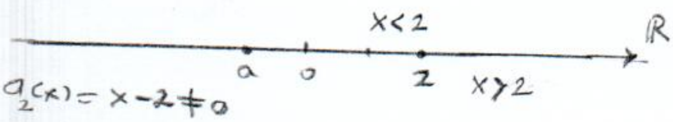
$$\begin{cases} 16x'' - y = 0 \\ y'' - 16x = 32t \end{cases}$$

Solutions Complete of Mid-Exam

M 204, First semester. 1439/1440H

Question 1

ⓐ $\begin{cases} (x-2)y'' + 3y = x \\ y(0) = 0,\ y'(0) = 1 \end{cases}$



$a_2(x) = x-2 \neq 0$

$a_2(x) = x-2$, $a_1(x) = 0$, $a_0(x) = 3$ and $g(x) = x$ are continuous on $\mathbb{R}$. (1)

But $a_2(x) \neq 0$ if $x \in (-\infty, 2)$ or $x \in (2, \infty)$. As $0 \in (-\infty, 2)$,

then the largest interval I for which the IVP has (1)

a unique solution is $I = (-\infty, 2)$. (2)

Note also that the IVP has a unique solution in $(-2, 2)$, centred at $x_0 = 0$.

ⓑ $y'' - y = -2x^2 + 2e^x + 5$

1) We find the solution of $y'' - y = 0$, $m^2 - 1 = 0$, $m = \mp 1$, then

$y_c = c_1 e^x + c_2 e^{-x}$ (1)

2) $y_p = (Ax^2 + Bx + C) + Dxe^x$ (1)

$y_p' = 2Ax + B + De^x + Dxe^x$, $y_p'' = 2A + 2De^x + Dxe^x$

$y_p'' - y_p = 2A + 2De^x + \cancel{Dxe^x} - Ax^2 - Bx - C - \cancel{Dxe^x} = -2x^2 + 2e^x + 5$

$2A - C = 5$, $-A = -2$, $2D = 2$, $B = 0$

$\boxed{A = 2,\ C = -1,\ D = 1,\ B = 0}$

Then $\boxed{y_p = 2x^2 - 1 + xe^x}$, and (1)

$\boxed{y = y_c + y_p = c_1 e^x + c_2 e^{-x} + 2x^2 - 1 + xe^x}$ (1)

Question ②

ⓐ $y'' + 3y' - 4y = x$, $y_1 = e^x$ is a given solution

We put $y = u y_1 = e^x u$

$y' = e^x u + e^x u'$, $y'' = e^x u + 2e^x u' + e^x u''$

$y'' + 3y' - 4y = \cancel{e^x u} + 2e^x u' + e^x u'' + \cancel{3e^x u} + 3e^x u' - \cancel{4e^x u} = x$

$e^x u'' + 5e^x u' = x$ or $u'' + 5u' = x e^{-x}$

Let $w = u'$, $w' = u''$

$w' + 5w = x e^{-x}$, $\mu(x) = e^{\int 5\,dx} = e^{5x}$ ①

$w e^{5x} = \int x e^{-x} \cdot e^{5x}\,dx = \int x e^{4x}\,dx$

$w e^{5x} = \frac{1}{4} x e^{4x} - \int \frac{1}{4} e^{4x}\,dx = \frac{1}{4} x e^{4x} - \frac{1}{16} e^{4x} + c_1$

$u' = w = \frac{1}{4} x e^{-x} - \frac{1}{16} e^{-x} + c_1 e^{-5x}$

$u = \int [\frac{1}{4} x e^{-x} - \frac{1}{16} e^{-x} + c_1 e^{-5x}]\,dx$ ⊘

$u = \frac{1}{4}[-x e^{-x} - e^{-x}] + \frac{1}{16} e^{-x} - \frac{c_1}{5} e^{-5x} + c_2$

$u = -\frac{x}{4} e^{-x} + (-\frac{1}{4} + \frac{1}{16}) e^{-x} - \frac{c_1}{5} e^{-5x} + c_2$

$\boxed{u = -\frac{x}{4} e^{-x} - \frac{3}{16} e^{-x} - \frac{c_1}{5} e^{-5x} + c_2}$ ⑨

$y = e^x u = -\frac{x}{4} - \frac{3}{16} - \frac{c_1}{5} e^{-5x} + c_2 e^x$

$\boxed{y = -\frac{x}{4} - \frac{3}{16} + c_3 e^{-4x} + c_2 e^x}$, $c_3 = \frac{-c_1}{5}$

(b) The D.Eq has a general solution :

$$y_G = c_1(7) + c_2(8x) + c_3(5x^2) + c_4 e^{-x}\cos x + c_5 e^{-x}\sin x$$

$$y_G = c_1^* + c_2^* x + c_3^* x^2 + c_4 e^{-x}\cos x + c_5 e^{-x}\sin x$$

1

So the roots of characteristic equation are

$m_1 = 0,\quad m_2 = 0,\quad m_3 = 0,\quad m_4 = -1+i,\quad m_5 = -1-i$

Then the charateristic equation is

1

$$m^3(m-(-1+i))(m-(-1-i)) = 0$$

$$m^3((m+1)+i)((m+1)-i) = 0$$

$$m^3((m+1)^2+1) = \boxed{m^3(m^2+2m+2)=0}$$

2

$m^5 + 2m^4 + 2m^3 = 0$, hence the D.E is

$$\boxed{y^{(5)} + 2y^{(4)} + 2y''' = 0}$$

$\boxed{\text{Question 3}}$

$$xy'' - 2y' + \frac{2}{x}y = 3x^3 + 2x\ ; \qquad x>0$$

$$x^2y'' - 2xy' + 2y = 3x^4 + 2x^2\ ; \qquad x>0$$

1) $x^2y'' - 2xy' + 2y = 0$, $y = x^m$, $m(m-1) - 2m + 2 = 0$

$$m^2 - 3m + 2 = (m-1)(m-2) = 0\ , \quad m=1,\ m=2$$

$y_c = c_1 x + c_2 x^2$, $y_1 = x$, $y_2 = x^2$

2) $y_p = y_1u_1 + y_2u_2$ s.t. $xu_1' + x^2u_2' = 0$

1

$$u_1' + 2xu_2' = 3x^2+2$$

$$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2, \qquad W_1 = \begin{vmatrix} 0 & x^2 \\ 3x^2+2, & 2x \end{vmatrix} = -3x^4 - 2x^2$$

$$u_1' = \frac{W_1}{W} = \frac{-3x^4-2x^2}{x^2} = -3x^2-2 \ , \ \boxed{u_1 = -x^3-2x}$$

(1)

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & 3x^2+2 \end{vmatrix} = 3x^3+2x \ , \quad u_2' = \frac{W_2}{W} = \frac{3x^3+2x}{x^2}$$

$$u_2' = 3x + \frac{2}{x} \ , \ \boxed{u_2 = \frac{3}{2}x^2 + 2\ln x}$$

(1)

$$y_p = xu_1 + x^2u_2 = -x^4 - 2x^2 + \frac{3}{2}x^4 + 2x^2\ln x$$

$$\boxed{y_p = \frac{1}{2}x^4 - 2x^2 + 2x^2\ln x}$$

(2)

$$\boxed{y = y_c + y_p = c_1x + c_2x^2 + \frac{1}{2}x^4 - 2x^2 + 2x^2\ln x}$$ is the general solution of the D.E.

Question (4)

$$\begin{cases} 16\ddot{x} - y = 0 \\ \ddot{y} - 16x = 32t \end{cases} \Rightarrow \begin{matrix} 16D^2x - y = 0 \\ -16x + D^2y = 32t \end{matrix} \ , \text{ then}$$

$$\begin{array}{l} 16D^4x - D^2y = 0 \\ -16x + D^2y = 32t \\ \hline \end{array} \qquad \begin{array}{l} 16x^{(4)}(t) - 16x = 32t \\ x^{(4)} - x = 2t \end{array}$$

(1)

1) $x^{(4)} - x = 0$, $x = e^{mt}$, $(m-1)(m+1)(m+i)(m-i) = 0$

$m = 1$, $m = -1$, $m = i$, $m = -i$

$$\boxed{x_c(t) = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t}$$

(1)

$x_p = At + B$, $x_p = -2t$

$$\boxed{x(t) = x_c + x_p = c_1e^t + c_2e^{-t} + c_3\cos t + c_4\sin t - 2t}$$

(1)

But $y = 16\ddot{x}$, then $\dot{x} = c_1e^t - c_2e^{-t} - c_3\sin t + c_4\cos t - 2$

$$\ddot{x} = c_1e^t + c_2e^{-t} - c_3\cos t - c_4\sin t$$

(2)

$$\boxed{y(t) = 16c_1e^t + 16c_2e^{-t} - 16c_3\cos t - 16c_4\sin t}$$