King Saud University,
College of Sciences
Mathematical Department.

Mid-Term1 /S1/2016
Full Mark:25. Time 1H30mn
02/02/1438

**Question 1**[4,4]. **a)** Determine the local region in the $xy-$plane for which the following differential equation

$$\sqrt{9-y^2}\frac{dy}{dx} = \ln(4-x^2),$$

would have a unique solution through the origin $(0,0)$.

**b)** Find the solution of the differential equation:

$$(x^2-x-2)\frac{dy}{dx} = (x-2)^2 + 3y, \quad x > 2.$$

**Question 2**[4,4]. **a)** Verify that the differential equation

$$(x^2+y^2-2)dx + (x^2-2xy)dy = 0, \quad x(x-2y) \neq 0.$$

is not exact. Find a suitable integrating factor to convert it to an exact equation, and then solve it.

**b)** Solve the initial value problem

$$\begin{cases} 5xy^2y' + y^3 = 32(1+\ln x)y^{-2}, \quad x>0,\ y\neq 0 \\ y(1) = 1 \end{cases}$$

**Question 3**[4]. Solve the differential equation

$$\frac{dy}{dx} = \frac{1-x-y}{x+y}, \quad x+y \neq 0.$$

**Question 4**[5]. Initially 100 mg of a radioactive substence was present. After 8 hours the mass has decreased by 4%. If the rate of decay is proportional to the amount of the substence present at time $t$. Find the amount of the remaining after 50 hours.

Completes solution of M. term 1

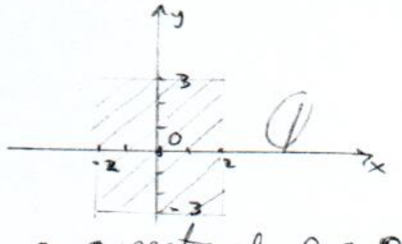
M. 204. First Semester 37/38H

$\boxed{\text{Question 1}}$ (a) $y'(x) = \frac{dy}{dx} = f(x,y) = \frac{\ln(4-x^2)}{\sqrt{9-y^2}}$, $\frac{\partial f}{\partial y} = -\frac{y}{(9-y^2)^{3/2}} \cdot \ln(4-x^2)$. 2

Thus, $f$ and $\frac{\partial f}{\partial y}$ are continuous on $-2 < x < 2$ and $-3 < y < 3$.

As $(0,0)$ is also interior to this rectangle

$R = \{(x,y) ; -2 < x < 2, -3 < y < 3\}$. Then 1




the IVP has a unique solution defined locally, on a rectangle $R_1 \subset R$ s. t. $(0,0) \in R_1$

(b) $(x^2 - x - 2) y' = (x-2)^2 + 3y$ ; $x > 2$

$$\frac{dy}{dx} - \frac{3}{(x+1)(x-2)} y = \frac{(x-2)}{x+1} \text{ which is a linear D.E}$$

$$\mu(x) = e^{\int \frac{-3}{(x+1)(x-2)} dx}, \quad \frac{-3}{(x+1)(x-2)} = \frac{+1}{x+1} - \frac{1}{x-2} \quad 2$$

$$\mu(x) = e^{-\ln(x-2) + \ln(x+1)} = e^{\ln\left(\frac{x+1}{x-2}\right)} = \frac{x+1}{x-2}.$$

Then

$$y\mu(x) = y\left(\frac{x+1}{x-2}\right) = \int \mu(x) \frac{x-2}{(x+1)} = \int dx$$

$$y\left(\frac{x+1}{x-2}\right) = x + c$$

is the general solution of the D.E or $\boxed{y = x\left(\frac{x-2}{x+1}\right) + \left(\frac{x-2}{x+1}\right)c}$ 2

$\boxed{\text{Question 2}}$ (a) $\underbrace{(x^2+y^2-2)}_{M} dx + \underbrace{(x^2 - 2xy)}_{N} dy = 0$ ; $x(x-2y) \neq 0$

$\frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = 2x - 2y$, then the D.E is not exact.

$$f(x) = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{4y-2x}{x(x-2y} = \frac{-2(x-2y)}{x(x-2y)} = \frac{-2}{x}$$

$$\mu(x) = e^{\int -\frac{2dx}{x}} = e^{\ln(1/x^2)} = \frac{1}{x^2} \text{ is an integrating factor.} \quad 2$$

Thus, the D.E. $\left(\overbrace{1+\frac{y^2}{x^2}-\frac{2}{x^2}}^{M}\right)dx+\left(\overbrace{1-\frac{2y}{x}}^{N}\right)dy=0$ is exact

$\frac{\partial M}{\partial y}=\frac{2y}{x^2}=\frac{\partial N}{\partial x}$, hence there exists a function $F$ of $x$ and $y$ s.t

$$\frac{\partial F}{\partial x}=1+\frac{y^2}{x^2}-\frac{2}{x^2}, \quad \frac{\partial F}{\partial y}=1-\frac{2y}{x}$$

$$F(x,y)=x-\frac{y^2}{x}+\frac{2}{x}+\phi(y)$$

$$\frac{\partial F}{\partial y}=\frac{-2y}{x}+\phi'(y)=1-\frac{2y}{x} \Rightarrow \phi(y)=y+c,$$

Then the solution of the D.E is

$$F(x,y)=\boxed{\frac{-y^2}{x}+x+\frac{2}{x}+y+C=0}$$

Question 2 (b) $\begin{cases} 5xy^2y' + y^3 = 32(1+\ln x)y^{-2}; & x>0, y\neq 0 \\ y(1)=1 \end{cases}$

$y' + \frac{1}{5x}y = \frac{32}{5}\left(\frac{1+\ln x}{x}\right)y^{-4}$ is Bernoulli's equation $n=-4$

Let $u = y^{1-(n)} = y^5$, $u' = 5y^4y'$, $\frac{u'}{5} = y^4y'$

(1) $y'y^4 + \frac{1}{5x}y^5 = \frac{32}{5}\left(\frac{1+\ln x}{x}\right)$

$\frac{u'}{5} + \frac{u}{5x} = \frac{32}{5}\left(\frac{1+\ln x}{x}\right)$

$u' + \frac{1}{x}u = 32\left(\frac{1+\ln x}{x}\right)$ is linear D.E. of first order

$\mu(x) = e^{\int \frac{dx}{x}} = e^{\ln x} = x$

(2) $\mu(x)u = xu = 32\int x\left(\frac{1+\ln x}{x}\right)dx$

$= 32\int(1+\ln x)dx = 32\left(x + (x\ln x - x)\right) + c$

$xu = 32x\ln x + C \Rightarrow u = 32\ln x + \frac{C}{x}$

Then $y^5 = 32\ln x + \frac{C}{x}$ is the solution of the D.E.

(3) Now $y(1)=1 \Rightarrow 1 = 32\ln(1) + C \Rightarrow C=1$

Thus the solution of the IVP is $\boxed{y^5 - 32\ln x - \frac{1}{x} = 0}$

Question 3

$$\frac{dy}{dx} = \frac{1-x-y}{x+y} = \frac{1-(x+y)}{x+y}, \quad x+y \neq 0$$

Let $u = x+y$, $u' - 1 = y'$, hence

$$y' = u' - 1 = \frac{1-u}{u}$$

$$u' = 1 + \frac{1-u}{u} = \frac{1}{u}$$

2

$$\frac{du}{\frac{1}{u}} = dx \Rightarrow u\,du = dx$$

2

$$\frac{1}{2}u^2 = x + c \quad \text{or} \quad u^2 = 2x + c_1 \; (2c)$$

Thus the solution of the D.E is $\boxed{(x+y)^2 - 2x = c_1}$ or

$$x^2 + y^2 + 2xy - 2x = c_1$$

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Question 4

Let $m = m(t)$ be the mass of radio active at time $t$.

Then

$$\frac{\frac{dm}{dt}}{m} = k \Rightarrow \frac{dm}{m} = k\,dt$$

1

$$\ln(m(t)) = kt + c$$

$$m(t) = e^{kt} \cdot e^{c} = e^{kt} c_1 \; ; \; (e^{c} = c_1).$$

1 We have $m(0) = 100$ m.g. $\Rightarrow m(t) = 100\, e^{kt}$. But $m(8) = 100 - 4 = 96$,

then $m(8) = 96 = 100\, e^{8k} \Rightarrow 0.96 = e^{8k}$, or $k = \frac{1}{8}\ln(0.96)$

2

$$k \simeq -5.1027$$

$$\boxed{m(t) \simeq 100\, e^{-(0.0051)t}}$$

So 1 $m(50) \simeq 100\, e^{(-0.0051)(50)} = 100\, e^{-0.255}$

$$\simeq \boxed{77.4916 \text{ m.g.}}$$