King Saud University,
College of Sciences
Mathematical Department.

Mid-Term1 /S2/2017
Full Mark:25. Time 1H30mn
13/07/1438

Question 1[4,4]. **a)** Determine the largest local region in the $xy-$plane for which the following differential equation

$$(x^2 - x - 6)\frac{dy}{dx} = \ln(4 - y^2),$$

would have a unique solution through the point $(1,1)$.

b) Solve the differential equation:

$$x\ln(x+1)\frac{dy}{dx} - 2x + \frac{xy}{x+1} + y\ln(x+1) = 0, \quad x > -1.$$

Question 2[4,4]. **a)** Solve the following differential equation by using a suitable integrating factor

$$(xy+1)dx + (x^2y + x^2/2 + 2x)dy = 0.$$

b) Write the differential equation in the form of Bernoulli's equation, hence solve it

$$(xy^3 - y^3 - x^2e^x)dx + 3xy^2dy = 0, \quad x > 0, \ y \neq 0.$$

Question 3[4]. Solve the differential equation

$$\left(x - y\tan^{-1}(\frac{y}{x})\right)dx + x\tan^{-1}(\frac{y}{x})dy = 0, \quad x > 0.$$

Question 4[5]. A cake is removed from a 350^0F oven and placed to cool in a room with temperature 75^0F. In 15 minutes the pie has a temperature of 150^0F. Determine the time required to cool the cake to a temperature of 80^0F so that it may be eaten.

Q1 a): $f(x,y) = \dfrac{\ln(4-y^2)}{x^2-x-6} = \dfrac{\ln(4-y^2)}{(x-3)(x+2)}$, $\dfrac{\partial f}{\partial y} = \dfrac{-2y}{(4-y^2)(x-3)(x+2)}$

$\therefore f$ is continuous on $R_1 = \{(x,y) \in \mathbb{R}^2, |y| < 2, x \neq 3, x \neq -2\}$

and $\dfrac{\partial f}{\partial y}$ is continuous on $R_2 = \{(x,y) \in \mathbb{R}^2, |y| \neq 2, x \neq 3, x \neq -2\}$

$\Rightarrow f$ and $\dfrac{\partial f}{\partial y}$ are continuous on $R_1 \cap R_2 = R_1$

Since $(1,1) \in R_3 = \{(x,y) \in \mathbb{R}^2 : -2 < y < 2, -2 < x < 3\}$

the given IVP has a unique solution in R_3 which is the largest region.

Q1 b): $\dfrac{dy}{dx} - \dfrac{2\cancel{x}}{\cancel{x}\ln(x+1)} + \dfrac{xy}{(x+1)x\ln x} + \dfrac{y}{x} = 0$, $x > -1$

$$\frac{dy}{dx} + \left(\frac{1}{x} + \frac{x}{(x+1)\ln x}\right)y = \frac{2}{\ln(x+1)}$$

$$y' + \left(\frac{1}{x} + \frac{x}{(x+1)\ln x}\right)y = \frac{2}{\ln(x+1)}$$

which is a linear DE of first order.

Where $P(x) = \dfrac{1}{x} + \dfrac{x}{(x+1)\ln x}$, $Q(x) = \dfrac{2}{\ln(x+1)}$

$$\mu(x) = e^{\int \frac{dx}{x} + \int \frac{dx}{(x+1)\ln(x+1)}} = e^{\ln x + \ln(\ln(x+1))}$$
$$= e^{\ln x}\, e^{\ln(\ln(x+1))}$$
$$= x \ln(x+1)$$

$\int \dfrac{dx}{(x+1)\ln(x+1)}$

Let $u = \ln(x+1)$

$\dfrac{du}{dx} = \dfrac{1}{x+1}$

$\Rightarrow A = \int \dfrac{(x+1)}{(x+1)u}\,du$

$= \ln u$

$= \ln \ln(x+1)$

$$\Rightarrow y\,\mu(x) = \int \mu(x) Q(x)\,dx + C$$

$$x\ln(x+1)\,y = \int x\cancel{\ln(x+1)}\left[\frac{2}{\cancel{\ln(x+1)}}\right]dx + C$$

$$x\ln(x+1)\,y = 2\int x\,dx + C$$

$$\boxed{x\ln(x+1)\,y = x^2 + C}$$

Q_2 a): $\frac{\partial M}{\partial y} = x$, $\frac{\partial N}{\partial x} = 2xy + x + 2 \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\Rightarrow$ the DE is not Exact

$$\frac{N_x - M_y}{M} = \frac{(2xy + x + 2) - x}{xy+1}$$
$$= \frac{2xy+2}{xy+1} = \frac{2(xy+1)}{xy+1} = 2y^{\circ} = 2$$

$$\Rightarrow \mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$$
$$= e^{\int 2\,dy} = e^{2y}$$

$$\therefore \mu(y) = e^{2y}$$

$$\Rightarrow (xye^{2y} + e^{2y})dx + (x^2y + \frac{x^2}{2} + 2x)e^{2y} dy = 0 \quad (*)$$

$$\left.\begin{array}{l} \frac{\partial M^*}{\partial y} = xe^{2y} + 2xye^{2y} + 2e^{2y} \\ \frac{\partial N^*}{\partial x} = 2xye^{2y} + xe^{2y} + 2e^{2y} \end{array}\right\} \therefore \frac{\partial M^*}{\partial y} = \frac{\partial N^*}{\partial x}$$

$\Rightarrow$ the (*) DE is exact

$$\int (xye^{2y} + e^{2y})\,dx = \frac{x^2}{2}ye^{2y} + xe^{2y}$$

$$\int (x^2ye^{2y} + \frac{x^2}{2}e^{2y} + 2xe^{2y})\,dy$$
$$= \frac{x^2}{2}ye^{2y} - \cancel{\frac{x^2}{4}e^{2y}} + \cancel{\frac{x^2}{4}e^{2y}} + xe^{2y}$$
$$= \frac{x^2}{2}ye^{2y} + xe^{2y}$$

$$\Rightarrow \boxed{\frac{x^2}{2}ye^{2y} + xe^{2y} = C}$$

$*\int x^2ye^{2y}\,dy$

$u = x^2y \quad dv = e^{2y}dy$

$du = x^2dy \Rightarrow v = \frac{1}{2}e^{2y}$

$* = \frac{x^2}{2}ye^{2y} - \frac{x^2}{2}\int e^{2y}dy$

$= \frac{x^2}{2}ye^{2y} - \frac{x^2}{4}e^{2y}$

(2)

Q_2 b):

$$y' = \frac{x^2 e^x + y^3 - x y^3}{3 x y^2} = \frac{y}{3}\left(\frac{1}{x} - 1\right) + \frac{x}{3} e^x y^{-2}$$

$$y' + \frac{1}{3}\left(1 - \frac{1}{x}\right) y = \frac{x}{3} e^x y^{-2} \quad (BE)$$

$$\Rightarrow y' y^2 + \frac{1}{3}\left(1 - \frac{1}{x}\right) y^3 = \frac{x}{3} e^x$$

New we let $v = y^3 \Rightarrow 3y^2 y' = v'$

~~H~~hence $v' + \left(1 - \frac{1}{x}\right) v = x e^x \quad (CE)$ ⊛

$$\mu(x) = e^{\int (1 - \frac{1}{x}) dx} = \frac{e^x}{x}$$

Times ⊛ by $\mu(x)$ we have

$$\frac{d}{dx}\left(\frac{e^x}{x} v\right) = e^{2x}$$

$$\Rightarrow \frac{e^x}{x} y^3 = \frac{1}{2} e^x + C$$

Q3 : $\underbrace{\left(x - y\tan^{-1}\left(\frac{y}{x}\right)\right)}_{M} dx + \underbrace{x\tan^{-1}\left(\frac{y}{x}\right)}_{N} dy = 0$

M and N have the same degree

$\Rightarrow$ the DE is homogeneous.

$$\frac{dy}{dx} = \frac{y\tan^{-1}(y/x) - x}{x\tan^{-1}(y/x)} = \frac{(y/x)\tan^{-1}(y/x) - 1}{\tan^{-1}(y/x)}$$

let $u = \frac{y}{x} \Rightarrow y' = xu' + u$

then we have

$$xu' + u = \frac{u\tan^{-1}u - 1}{\tan^{-1}u} \Rightarrow x\frac{du}{dx} = -\frac{1}{\tan^{-1}u}$$

$$\Rightarrow \tan^{-1}u\, du = -\frac{dx}{x}$$

$$\Rightarrow \int \tan^{-1}u\, du = -\ln x + c$$

Integration by parts we get

$$u\tan^{-1}u - \frac{1}{2}\ln(1+u^2) + \ln x = c$$

Then $(y/x)\tan^{-1}(y/x) - \frac{1}{2}\ln\left(1+(y/x)^2\right) + \ln x = c$

Q4) $\frac{dT}{dt} = K(T - T_s)$

$\Rightarrow T(t) = T_s + ce^{kt}$

$T_0 = 350 F^\circ$

$T_s = 75 F^\circ$

$T(0) = 75 + c \Rightarrow c = 350 - 75 = 275$

$T(t) = 75 + 275e^{kt}$

$T(15) = 150 \Rightarrow 150 = 75 + 275e^{15k}$

$\Rightarrow e^{15k} = \frac{3}{11} \Rightarrow k = \frac{1}{15}\ln\left(\frac{3}{11}\right)$

Hence $T(t) = 75 + 275e^{\frac{t}{15}\ln(3/11)}$

$80 = 75 + 275e^{\frac{t}{15}\ln(3/11)}$

$\Rightarrow t = \frac{15\ln\left(\frac{5}{275}\right)}{\ln\left(\frac{3}{11}\right)} \simeq 46.264$ mins