# SAMC 2013

SAUDI ARABIA MATHEMATICAL COMPETITIONS

مسابقات الرّياضيات للمملكة العربيّة السّعوديّة

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# Introduction

This booklet contains problems used in the training and selection of the Saudi team for the International Mathematics Olympiad. The training was supported by The Ministry of Education, which commissioned **Mawhiba**, the main establishment in Saudi Arabia that cares for the gifted students, to do the task. **Mawhiba** is an independent establishment, presided by the King of Saudi Arabia, with the Minister of Education as Vice President. We thank King Saud University for giving the opportunity to trainers to contribute in the training of the Saudi Team.

The Saudi team had four main training camps during the academic year 2012-2013 beside the full-time training period that started on April 1, 2013. The team participated in the Asian Pacific Olympiad in March 12, 2013, and in the Gulf Mathematical Olympiad, which was held in Qatar in the period April 1-5, 2013.

It is our pleasure to share these training and selection problems with other IMO teams, hoping it will contribute to a future cooperation.

Dr. Fawzi A. Al-Thukair
Leader of the Saudi Math Olympiad Team

# مقدّمة

يحوي هذا الكتيّب على مسائل التّدريب والاختيار للفريق السّعودي للاولمبياد الدّولي للرّياضيات والّذي يعقد في شهر يوليو ٢٠١٣ في جمهوريّة كولومبيا. لقد كان التّدريب مدعوما من وزارة التّربية والتّعليم في المملكة العربيّة السّعوديّة والّتي كلّفت مؤسّسة الملك عبدالعزيز ورجاله للموهبة والابداع " موهبة " للقيام بالمهمّة. تعتبر مؤسّسة " موهبة " مؤسّسة مستقلّة يرأسها الملك عبدالّله بن عبدالعزيز رعاه الّله وينوب عنه سمو وزير التّربية والتّعليم. وتجدر الاشارة إلى دعم جامعة الملك سعود الّتي أتاحت الفرصة لبعض الاساتذة للقيام بتدريب الفريق.

تمّ تنظيم أربع ملتقيات تدريبيّة للفريق السّعودي خلال العام الدّراسي ٢٠١٢ - ٢٠١٣، بالإضافة إلى فترة التّفريغ الكلّي للتّدريب الّتي بدأج في ١ - ٤ - ٢٠١٣. كما شارك الفريق في اولمبياد الرّياضيات لدول آسيا والباسيفيك *APMO* في ١٢ مارس ٢٠١٣، واولمبياد دول الخليج *GMO* الّذي عقد في دولة قطر في الفترة ١ - ٥ أبريل ٢٠١٣.

يسعدني أن أهدي لكم مسائل هذا الكتيّب، راجيا أن تكون بداية تعاون بيننا في مجال الاولمبياد وغيره من حقول المنافسة العلميّة.

د. فوزي بن أحمد الذكير
رئيس الفريق السّعودي للاولمبياد الدّولي.

# Acknowledgement

Problems for the 2013 KSA MO contests were chosen by Abdullah Alghamdi, Abdulaziz Bin Obaid, Safwat Eltanany, Zuming Feng (co-chair), Yunhao Fu, Abdallah Laradji, Ian Le, Carlos Shine, Malik Talbi (co-chair), and Pin Yu. Many thanks to all of these persons, especially those who contributed additional solutions that were helpful in grading these tests and construction of this booklet.

# Contents

# Part I

# Problems

# Chapter 1

# Preselection tests for the full-time training

The students who attended the full-time training that started on April 1, 2013, were chosen according to their combined performance in this set of tests during the October camp at Riyadh. These tests are prepared by Abdullah Alghamdi, Abdulaziz Bin Obaid, Safwat Eltanany, Abdallah Laradji, and Malik Talbi.

## 1.1 Day I - October 16, 2013

*Allowed time: 3 hours*

1. Let $-1 \leq x, y \leq 1$. Prove the inequality

$$2\sqrt{(1-x^2)(1-y^2)} \leq 2(1-x)(1-y)+1.$$

2. Let $x, y$ be two non-negative integers. Prove that 47 divides $3^x - 2^y$ if and only if 23 divides $4x + y$.

3. Ten students take a test consisting of 4 different papers in Algebra, Geometry, Number Theory and Combinatorics. First, the proctor distributes randomly the Algebra paper to each student. Then the remaining papers are distributed one at a time in the following order: Geometry, Number Theory, Combinatorics in such a way that no student receives a paper before he finishes the previous one. In how many ways can the proctor distribute the test papers given that a student may for example finish the Number Theory paper before

another student receives the Geometry paper, and that he receives the Combinatorics paper after that the same other student receives the Combinatorics papers.

4. $ABC$ is a triangle, $G$ its centroid and $A', B', C'$ the midpoints of its sides $BC, CA, AB$, respectively. Prove that if the quadrilateral $AC'GB'$ is cyclic then
$$AB \cdot CC' = AC \cdot BB'.$$

## 1.2 Day II - October 19, 2013

*Allowed time: 3 hours*

1. Prove that if $a$ is an integer relatively prime with 35 then
$$(a^4 - 1)(a^4 + 15a^2 + 1) \equiv 0 \mod 35.$$

2. The quadratic equation $ax^2 + bx + c = 0$ has its roots in the interval $[0, 1]$. Find the maximum of
$$\frac{(a-b)(2a-b)}{a(a-b+c)}.$$

3. The positive integer $a$ is relatively prime with 10. Prove that for any positive integer $n$, there exists a power of $a$ whose last $n$ digits are $\underbrace{0\cdots0}_{n-1}1$.

4. $\Delta ABC$ is a triangle and $I_b, I_c$ its excenters opposite to $B, C$. Prove that $\Delta ABC$ is right at $A$ if and only if its area is equal to $\frac{1}{2}AI_b \cdot AI_c$.

## 1.3 Day III - October 21, 2013

*Allowed time: 3 hours*

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(f(x)) = 4x + 1$ for all real number $x$. Prove that the equation $f(x) = x$ has a unique solution.

2. Let $a_1, a_2, \ldots, a_9$ be integers. Prove that if 19 divides $a_1^9 + a_2^9 + \cdots + a_9^9$ then 19 divides the product $a_1 a_2 \cdots a_9$.

3. The points of the plan have been colored by 2013 different colors. We say that a triangle $\Delta ABC$ has the color $X$ if its three vertices $A, B, C$ has the color $X$. Prove that there are infinitely many triangles with the same color and the same area.

4. $\Delta ABC$ is a triangle with $AB < BC$, $\mathcal{C}$ its circumcircle, $K$ the midpoint of the minor arc $\widehat{CA}$ of the circle $\mathcal{C}$ and $T$ a point on $\mathcal{C}$ such that $KT$ is perpendicular to $BC$. If $A', B'$ are the intouch points of the incircle of $\Delta ABC$ with the sides $BC, AC$, prove that the lines $AT, BK, A'B'$ are concurrent.

## 1.4 Day IV - October 23, 2013

*Allowed time: 3 hours*

1. Let $a_1, a_2, a_3, ...$ be a sequence of real numbers which satisfy the relation
$$a_{n+1} = \sqrt{a_n^2 + 1}.$$
Suppose that there exists a positive integer $n_0$ such that $a_{2n_0} = 3a_{n_0}$. Find the value of $a_{46}$.

2. Let $x, y$ be two integers. Prove that if 2013 divides $x^{1433} + y^{1433}$ then 2013 divides $x^7 + y^7$.

3. How many permutations $(s_1, s_2, \cdots, s_n)$ of $(1, 2, \ldots, n)$ are there satisfying the condition $s_i > s_j$ for all $i \geq j + 3$ when $n = 5$ and when $n = 7$?

4. $\Delta ABC$ is a triangle, $M$ the midpoint of $BC$, $D$ the projection of $M$ on $AC$ and $E$ the midppoint of $MD$. Prove that the lines $AE, BD$ are orthogonal if and only if $AB = AC$.

# Chapter 2

# Selection tests for the Gulf Mathematical Olympiad 2013

The KSA 2013 Gulf MO team members were chosen according to their combined performance in this set of tests during the January camp at Riyadh. These tests are prepared by Zuming Feng, Ian Le, Carlos Shine, Malik Talbi, and Pin Yu.

Members of the KSA 2013 Gulf MO team were Sameh Zawawi, Mahdi Alshaikh, Alzubair Habibullah, Ibraheem Khan, Salman Tawfik, Abdallah Alnashwan. The total team scores was 162 out of 240, and was ranked 1st place among the 7 participating teams. The team's individual performances were as follows:

| | |
|---|---|
| Sameh Zawawi | GOLD Medallist |
| Mahdi Alshaikh | GOLD Medallist |
| Alzubair Habibullah | SILVER Medallist |
| Ibraheem Khan | SILVER Medallist |
| Salman Tawfik | SILVER Medallist |
| Abdallah Alnashwan | SILVER Medallist |

## 2.1 Day I - January 22, 2013

*Allowed time: 4 hours and half*

1. Tarik wants to choose some distinct numbers from the set $S = \{2, \ldots, 111\}$ in such a way that each of the chosen numbers cannot be written as the product of two other distinct chosen numbers. What is the maximum number of numbers Tarik can choose?

2. For positive real numbers $a, b$ and $c$, prove that

$$\frac{a^3}{a^2+ab+b^2} + \frac{b^3}{b^2+bc+c^2} + \frac{c^3}{c^2+ca+a^2} \geq \frac{a+b+c}{3}.$$

3. Define a *regular $n$-pointed star* to be a union of $n$ lines segments $P_1P_2$, $P_2P_3$, $\ldots$, $P_nP_1$ such that

   - the points $P_1, P_2, \ldots, P_n$ are coplanar and no three of them are collinear;
   - each of the $n$ line segments intersects at least one of the other line segments at a point other than an endpoint;
   - all of the angles at $P_1, P_2, \ldots, P_n$ are congruent;
   - all of the $n$ line segments $P_1P_2$, $P_2P_3$, $\ldots$, $P_nP_1$ are congruent; and
   - the path $P_1P_2 \ldots P_nP_1$ turns counterclockwise at an angle less than $180°$ at each vertex.

   There are no regular 3-pointed, 4-pointed, or 6-pointed stars. All regular 5-pointed star are similar, but there are two non-similar regular 7-pointed stars. Find all possible values of $n$ such that there are exactly 29 non-similar regular $n$-pointed stars.

4. In acute triangle $ABC$, points $D$ and $E$ are the feet of the perpendiculars from $A$ to $BC$ and $B$ to $CA$, respectively. Segment $AD$ is a diameter of circle $\omega$. Circle $\omega$ intersects sides $AC$ and $AB$ at $F$ and $G$ (other than $A$), respectively. Segment $BE$ intersects segments $GD$ and $GF$ at $X$ and $Y$ respectively. Ray $DY$ intersects side $AB$ at $Z$. Prove that lines $XZ$ and $BC$ are perpendicular.

## 2.2 Day II - January 26, 2013

*Allowed time: 4 hours and half*

1. An acute triangle $ABC$ is inscribed in circle $\omega$ centered at $O$. Line $BO$ and side $AC$ meet at $B_1$. Line $CO$ and side $AB$ meet at $C_1$. Line $B_1C_1$ meets circle $\omega$ at $P$ and $Q$. If $AP = AQ$, prove that $AB = AC$.

2. Let $f(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + p$ be a polynomial of integer coefficients where $p$ is a prime number. Assume that

$$p > \sum_{i=1}^{n} |a_i|.$$

   Prove that $f(X)$ is irreducible.

3. Find the largest integer $k$ such that $k$ divides $n^{55} - n$ for all integer $n$.

4. Let $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$, for all positive integer $n$, be the Fibonacci sequence. Prove that for any positive integer $m$ there exist infinitely many positive integers $n$ such that

$$F_n + 2 \equiv F_{n+1} + 1 \equiv F_{n+2} \mod m.$$

## 2.3 Day III - January 29, 2013

*Allowed time: 3 hours and half*

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy

$$\begin{aligned} f\left(\tfrac{\sqrt{3}}{3}x\right) &= \sqrt{3}f(x) - \tfrac{2\sqrt{3}}{3}x, \\ f(x)f(y) &= f(xy) + f\left(\tfrac{x}{y}\right), \end{aligned}$$

   for all $x, y \in \mathbb{R}$, with $y \neq 0$.

2. Find all values of $n$ for which there exists a convex cyclic non-regular polygon with $n$ vertices such that the measures of all its internal angles are equal.

3. $ABC$ is a triangle, $H$ its orthocenter, $I$ its incenter, $O$ its circumcenter and $\omega$ its circumcircle. Line $CI$ intersects circle $\omega$ at point $D$ different from $C$. Assume that $AB = ID$ and $AH = OH$. Find the angles of triangle $ABC$.

4. Find all pairs of positive integers $(a, b)$ such that $a^2 + b^2$ divides both $a^3 + 1$ and $b^3 + 1$.

# Chapter 3

# Selection tests for the Balkan Mathematical Olympiad 2013

The KSA 2013 Balkan MO team members were chosen according to their combined performance in this set of tests during the March-April camp at Riyadh. These tests are prepared by Zuming Feng, Yunhao Fu, and Malik Talbi.

Members of the KSA 2013 Balkan MO team were Abdallah Alnashwan, Ali Alnasser, Mahdi Alshaikh, Alzubair Habibullah, Ibraheem Khan, and Sameh Zawawi. However, the team could not take part to the competition this year because of its schedule.

Members of the KSA 2012 Balkan MO team were Saleh Algamdi, Abdulrahman Alharbi, Doha Aljeddawi, Hasan Eid, Husain Eid, Alyazeed Basuni. Dr. Fawzi Al-Thukair (King Saud University, Riyadh) and Dr. Abdul Aziz bin Obaid (Mawhiba, Riyadh) served as team leader and deputy leader, respectively. The team was also accompanied by Dr. Malik Talbi (King Saud University, Riyadh), Dr. Abdulaziz Al-Harthi (MAWHIBA, Riyadh), Adel Mohammad Alghadir, and Mrs. Abeer Kawther as observers of the KSA delegation. The total team scores was 131 out of 240, and was ranked 14th place among the 22 participating teams. The team's individual performances were as follows:

| | |
|---|---|
| Alyazeed Basuni | SILVER Medallist |
| Saleh Algamdi | BRONZE Medallist |
| Abdulrahman Alharbi | BRONZE Medallist |
| Doha Aljeddawi | BRONZE Medallist |
| Hasan Eid | BRONZE Medallist |
| Husain Eid | |

## 3.1 Day I - April 7, 2013

*Time allowed: 5 hours*

1. The set $G$ is defined by the points $(x, y)$ with integer coordinates, $1 \leq x \leq 5$ and $1 \leq y \leq 5$. Determine the number of five-point sequences $(P_1, P_2, P_3, P_4, P_5)$ such that for $1 \leq i \leq 5$, $P_i = (x_i, i)$ is in $G$ and $|x_1 - x_2| = |x_2 - x_3| = |x_3 - x_4| = |x_4 - x_5| = 1$.

2. For positive integers $a$ and $b$, $\gcd(a, b)$ denote their greatest common divisor and $\operatorname{lcm}(a, b)$ their least common multiple. Determine the number of ordered pairs $(a, b)$ of positive integers satisfying the equation
$$ab + 63 = 20 \operatorname{lcm}(a, b) + 12 \gcd(a, b).$$

3. Solve the following equation where $x$ is a real number:
$$\lfloor x^2 \rfloor - 10\lfloor x \rfloor + 24 = 0.$$

4. $ABCDEF$ is an equiangular hexagon of perimeter 21. Given that $AB = 3$, $CD = 4$, and $EF = 5$, compute the area of hexagon $ABCDEF$.

5. Let $k$ be a real number such that the product of real roots of the equation
$$X^4 + 2X^3 + (2 + 2k)X^2 + (1 + 2k)X + 2k = 0$$
is $-2013$. Find the sum of the squares of these real roots.

6. Let $ABC$ be a triangle with incenter $I$, and let $D, E, F$ be the midpoints of sides $BC, CA, AB$, respectively. Lines $BI$ and $DE$ meet at $P$, and lines $CI$ and $DF$ meet at $Q$. Line $PQ$ meets sides $AB$ and $AC$ at $T$ and $S$, respectively. Prove that $AS = AT$.

7. Ayman wants to color the cells of a $50 \times 50$ chessboard into black and white so that each $2 \times 3$ or $3 \times 2$ rectangle contains an even number of white cells. Determine the number of ways Ayman can color the chessboard.

8. Prove that the ratio
$$\frac{1^1 + 3^3 + 5^5 + \cdots + (2^{2013} - 1)^{(2^{2013}-1)}}{2^{2013}}$$
is an odd integer.

## 3.2 Day II - April 9, 2013

*4 Time allowed: 5 hours*

1. In triangle $ABC$, $AB = AC = 3$ and $\angle A = 90°$. Let $M$ be the midpoint of side $BC$. Points $D$ and $E$ lie on sides $AC$ and $AB$ respectively such that $AD > AE$ and $ADME$ is a cyclic quadrilateral. Given that triangle $EMD$ has area 2, find the length of segment $CD$.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy for all $x, y \in \mathbb{R}$ the relation
$$f(f(f(x) + y) + y) = x + y + f(y).$$

3. Find all positive integers $x, y, z$ such that
$$2^x + 21^y = z^2.$$

4. Ten students are standing in a line. A teacher wants to place a hat on each student. He has two colors of hats, red and white, and he has 10 hats of each color. Determine the number of ways in which the teacher can place hats such that among any set of consecutive students, the number of students with red hats and the number of students with blue hats differ by at most 2.

5. We call a positive integer *good* if it doesn't have a zero digit and the sum of the squares of its digits is a perfect square. For example, 122 and 34 are good and 304 and 12 are not not good. Prove that there exists a $n$-digit good number for every positive integer $n$.

6. Let $a, b, c$ be positive real numbers such that $ab + bc + ca = 1$. Prove that
$$a\sqrt{b^2 + c^2 + bc} + b\sqrt{c^2 + a^2 + ca} + c\sqrt{a^2 + b^2 + ab} \geq \sqrt{3}.$$

7. The excircle $\omega_B$ of triangle $ABC$ opposite $B$ touches side $AC$, rays $BA$ and $BC$ at $B_1$, $C_1$ and $A_1$, respectively. Point $D$ lies on major arc $\overset{\frown}{A_1C_1}$ of $\omega_B$. Rays $DA_1$ and $C_1B_1$ meet at $E$. Lines $AB_1$ and $BE$ meet at $F$. Prove that line $FD$ is tangent to $\omega_B$ (at $D$).

8. A social club has 101 members, each of whom is fluent in the same 50 languages. Any pair of members always talk to each other in only one language. Suppose that there were no three members such that they use only one language among them. Let $A$ be the number of three-member subsets such that the three distinct pairs among them use different languages. Find the maximum possible value of $A$.

## 3.3 Day III - April 14, 2013

*Time allowed: 4 hours*

1. $ABCD$ is a cyclic quadrilateral and $\omega$ its circumcircle. The perpendicular line to $AC$ at $D$ intersects $AC$ at $E$ and $\omega$ at $F$. Denote by $\ell$ the perpendicular line to $BC$ at $F$. The perpendicular line to $\ell$ at $A$ intersects $\ell$ at $G$ and $\omega$ at $H$. Line $GE$ intersects $FH$ at $I$ and $CD$ at $J$. Prove that points $C, F, I$, and $J$ are concyclic.

2. Define Fibonacci sequence $\{F\}_{n=0}^{\infty}$ as $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for every integer $n > 1$. Determine all quadruples $(a, b, c, n)$ of positive integers with $a < b < c$ such that each of $a, b, c, a+n, b+n, c+2n$ is a term of the Fibonacci sequence.

3. Let $T$ be a real number satisfying the property: For any nonnegative real numbers $a, b, c, d, e$ with their sum equal to 1, it is possible to arrange them around a circle such that the products of any two neighboring numbers are no greater than $T$. Determine the minimum value of $T$.

4. Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be a function which satisfies for all integer $n \geq 0$:

$$(a) \quad f(2n+1)^2 - f(2n)^2 = 6f(n) + 1, \qquad (b) \quad f(2n) \geq f(n);$$

where $\mathbb{Z}_{\geq 0}$ is the set of nonnegative integers. Solve the equation $f(n) = 1000$.

## 3.4 Day IV - April 16, 2013

*Time allowed: 1 hour 30 minutes*

1. $ABCD$ is a cyclic quadrilateral such that $AB = BC = CA$. Diagonals $AC$ and $BD$ intersect at $E$. Given that $BE = 19$ and $ED = 6$, find the possible values of $AD$.

2. The base-7 representation of number $n$ is $\overline{abc}_{(7)}$, and the base-9 representation of number $n$ is $\overline{cba}_{(9)}$. What is the decimal (base-10) representation of $n$?

3. Find the area of the set of points of the plane whose coordinates $(x, y)$ satisfy

$$x^2 + y^2 \leq 4|x| + 4|y|.$$

4. Find all positive integers $n < 589$ for which 589 divides $n^2 + n + 1$.

# Chapter 4

# Selection tests for the International Mathematical Olympiad 2013

The KSA 2013 IMO team members were chosen according to their combined performance in this set of tests during the May camp at Riyadh. These tests are prepared by Zuming Feng, Carlos Shine, and Malik Talbi.

Members of the KSA 2013 IMO team were Alyazeed Basyoni, Sameh Zawawi, Ibraheem Khan, Abdulrahman Alharbi, Ali Alnasser, Alzubair Habibullah. Dr. Fawzi Al-Thukair (King Saud University, Riyadh) and Dr. Najla Altwaijry (King Saud University, Riyadh) served as team leader and deputy leader, respectively. The team was also accompanied by Dr. Malik Talbi (King Saud University, Riyadh), Dr. Abdulaziz Al-Harthi (MAWHIBA, Riyadh), Mansour Almoaigel, and Dr. Abdulrahman Albarak (Ministry of Education) as observers of the KSA delegation.

Members of the KSA 2012 IMO team were Alyazeed Basuni, Husain Eid, Saleh Algamdi, Abdulrahman Alharbi, Wael Alsaeed, and Hasan Eid. Dr. Fawzi Al-Thukair (King Saud University, Riyadh) and Dr. Najla Altwaijry (King Saud University, Riyadh) served as team leader and deputy leader, respectively. The team was also accompanied by Dr. Malik Talbi (King Saud University, Riyadh), Dr. Abdulaziz Al-Harthi (MAWHIBA, Riyadh), Palmer Mebane, Mansour Almoaigel, and Dr. Abdulrahman Albarak (Ministry of Education) as observers of the KSA delegation. The total team scores was 105 out of 252, and was ranked 29th place among the 100 participating teams. The team's individual performances were as follows:

| | |
|---|---|
| Alyazeed Basuni | SILVER Medallist |
| Husain Eid | SILVER Medallist |
| Saleh Algamdi | BRONZE Medallist |
| Abdulrahman Alharbi | BRONZE Medallist |
| Wael Alsaeed | BRONZE Medallist |
| Hasan Eid | |

## 4.1 Day I - May 28, 2013

*4 hours 30 minutes*

1. Triangle $ABC$ is inscribed in circle $\omega$. Point $P$ lies inside triangle $ABC$. Lines $AP, BP$ and $CP$ intersect $\omega$ again at points $A_1, B_1$ and $C_1$ (other than $A, B, C$), respectively. The tangent lines to $\omega$ at $A_1$ and $B_1$ intersect at $C_2$. The tangent lines to $\omega$ at $B_1$ and $C_1$ intersect at $A_2$. The tangent lines to $\omega$ at $C_1$ and $A_1$ intersect at $B_2$. Prove that the lines $AA_2, BB_2$ and $CC_2$ are concurrent.

2. Let $S = \{0, 1, 2, 3, \ldots\}$ be the set of the non-negative integers. Find all strictly increasing functions $f\colon S \to S$ such that $n + f(f(n)) \leq 2f(n)$ for every $n$ in $S$.

3. A Saudi company has two offices. One office is located in Riyadh and the other in Jeddah. To insure the connection between the two offices, the company has designated from each office a number of correspondents so that

   (a) each pair of correspondents from the same office share exactly one common correspondent from the other office.

   (b) there are at least 10 correspondents from Riyadh.

   (c) Zayd, one of the correspondents from Jeddah, is in contact with exactly 8 correspondents from Riyadh.

   What is the minimum number of correspondents from Jeddah who are in contact with the correspondent Amr from Riyadh?

4. Determine whether it is possible to place the integers $1, 2, \ldots, 2012$ in a circle in such a way that the 2012 products of adjacent pairs of numbers leave pairwise distinct remainders when divided by 2013.

## 4.2 Day II - May 29, 2013

*4 hours 30 minutes - 4 problems*

1. Find the maximum and the minimum values of
$$S = (1 - x_1)(1 - y_1) + (1 - x_2)(1 - y_2)$$
for real numbers $x_1, x_2, y_1, y_2$ with $x_1^2 + x_2^2 = y_1^2 + y_2^2 = 2013$.

2. Let $ABC$ be an acute triangle, and let $AA_1, BB_1$, and $CC_1$ be its altitudes. Segments $AA_1$ and $B_1C_1$ meet at point $K$. The perpendicular bisector of segment $A_1K$ intersects sides $AB$ and $AC$ at $L$ and $M$, respectively. Prove that points $A, A_1, L$, and $M$ lie on a circle.

3. For a positive integer $n$, we consider all its divisors (including 1 and itself). Suppose that $p\%$ of these divisors have their unit digit equal to 3 (For example $n = 117$, has six divisors, namely 1,3,9,13,39,117. Two of these divisors, namely 3 and 13, have unit digits equal to 3. Hence for $n = 117$, $p = 33.33\cdots$). Find, when $n$ is any positive integer, the maximum possible value of $p$.

4. Determine if there exists an infinite sequence of positive integers
$$a_1, a_2, a_3, \ldots$$
such that
   (i) each positive integer occurs exactly once in the sequence, and
   (ii) each positive integer occurs exactly once in the sequence $|a_1 - a_2|$, $|a_2 - a_3|$, ..., $|a_k - a_{k+1}|$, ...

## 4.3 Day III - May 30, 2013

*4 hours 30 minutes*

1. Adel draws an $m \times n$ grid of dots on the coordinate plane, at the points of integer coordinates $(a, b)$ where $1 \le a \le m$ and $1 \le b \le n$. He proceeds to draw a closed path along $k$ of these dots, $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$, such that $(a_i, b_i)$ and $(a_{i+1}, b_{i+1})$ (where $(a_{k+1}, b_{k+1}) = (a_1, b_1)$) are 1 unit apart for each $1 \le i \le k$. Adel makes sure his path does not cross itself, that is, the $k$ dots are distinct. Find, with proof, the maximum possible value of $k$ in terms of $m$ and $n$.

2. Given an integer $n \geq 2$, determine the number of ordered $n$-tuples of integers $(a_1, a_2, \ldots, a_n)$ such that

   (a) $a_1 + a_2 + \cdots + a_n \geq n^2$; and

   (b) $a_1^2 + a_2^2 + \cdots + a_n^2 \leq n^3 + 1$.

3. Let $ABC$ be an acute triangle, $M$ be the midpoint of $BC$ and $P$ be a point on line segment $AM$. Lines $BP$ and $CP$ meet the circumcircle of $ABC$ again at $X$ and $Y$, respectively, and sides $AC$ at $D$ and $AB$ at $E$, respectively. Prove that the circumcircles of $AXD$ and $AYE$ have a common point $T \neq A$ on line $AM$.

4. Find all polynomials $p(x)$ with integer coefficients such that for each positive integer $n$, the number $2^n - 1$ is divisible by $p(n)$.

# Part II

# Problems in arabic

# اختبارات ترشيح الطّلاب للتّدريب المكثّف 2013

## مسائل اليوم الأوّل ـ 30 ذوالقعدة 1433 هـ

1 ـ لتكن $-1 \le x, y \le 1$ . أثبت المتباينة

$$2\sqrt{(1-x^2)(1-y^2)} \le 2(1-x)(1-y)+1.$$

2 ـ ليكن $x, y$ عددين صحيحين غير سالبين. أثبت أنّ المقدار $3^x - 2^y$ يقبل القسمة على 47 إذا وفقط إذا كان المقدار $4x+y$ يقبل القسمة على 23 .

3 ـ يشارك عشر طلّاب في اختبار في أربع مواد مختلفة: جبر، وهندسة، ونظريّة أعداد، وتركيبات. طبعت مسائل كلّ مادة في ورقة مستقلّة. يبدأ المراقب بتوزيع مسائل الجبر على الطلّاب بترتيب عشوائي. كلّما يتنهي طالب من حلّ مسائل الجبر يعطيه المراقب مسائل الهندسة، وإذا انتهى من حلّ مسائل الهندسة يعطيه مسائل نظريّة الأعداد ثمّ في الأخير يعطيه مسائل التركيبات بهذا الترتيب. بكم طريقة يمكن للمراقب توزيع المسائل على الطلّاب، علما بأنّه يمكن لطالب أن يحصل مثلا على مسائل نظرية الأعداد قبل أن يحصل زميله على مسائل الهندسة، وأن يحصل على مسائل التركيبات بعد أن يحصل نفس الزميل على مسائل التركيبات.

4 ـ $\Delta ABC$ مثلّث، و $G$ مركزه المتوسّط و $A', B', C'$ منتصفات أضلاعه $BC, CA, AB$ على الترتيب. أثبت أنّه إذا كان الرباعي $AC'GB'$ دائريّا فإنّ لدينا العلاقة: $AB \cdot CC' = AC \cdot BB'$ .

## مسائل اليوم الثّاني ـ 3 ذوالحجّة 1433 هـ

1 ـ إذا كان $a$ عددا صحيحا أوّليّا نسبيّا مع 35 ، فأثبت أنّ
$$(a^4-1)(a^4+15a^2+1) \equiv 0 \mod 35.$$

2 ـ لتكن $ax^2+bx+c=0$ معادلة تربيعيّة جذراها على الفترة $[0,1]$ . جد القيمة العظمى للمقدار
$$\frac{(a-b)(2a-b)}{a(a-b+c)}.$$

3 ـ ليكن $a$ عددا صحيحا موجبا أوّليّا نسبيّا مع 10 . أثبت أنّه لكلّ عدد صحيح موجب $n$ ، توجد قوّة للعدد $a$ تنتهي خاناتها الـ $n$ بـ $\cdots 00\cdots 01$ .

4 ـ $\Delta ABC$ مثلّث، و $I_b, I_c$ مركزا الدائرتيه الخارجيّتين المقابلتين للرأسين $B, C$ . أثبت أنّ المثلّث $\Delta ABC$ قائم عند الرأس $A$ إذا وفقط إذا كانت قيمة مساحته تساوي $\frac{1}{2}AI_b \cdot AI_c$ .

## مسائل اليوم الثّالث ـ 5 ذوالحجّة 1433 هـ

1 ـ لتكن $f:\mathbb{R}\to\mathbb{R}$ دالة تحقّق $f(f(x))=4x+1$ لكلّ $x$ عدد حقيقي. أثبت أنّ المعادلة $f(x)=x$ لها حلّ وحيد.

2 ـ لتكن $a_1, a_2, \ldots, a_9$ أعدادا صحيحة. أثبت أنّه إذا كان المقدار $a_1^9+a_2^9+\cdots+a_9^9$ يقبل القسمة على 19 فإنّ حاصل الضرب $a_1a_2\cdots a_9$ يقبل القسمة على 19 .

3 ـ تمّ تلوين نقط المستوي بـ 2013 لونا مختلفا. نقول إنّ المثلّث $\Delta ABC$ له لون كذا إذا كانت رؤوسه الثلاثة $A, B, C$ بهذا اللّون. أثبت أنّه يوجد عدد غير منته من المثلّثات لها نفس اللّون ونفس المساحة.

4 ـ $\Delta ABC$ مثلّث يحقّق $AB<BC$ ، و $\mathcal{C}$ دائرته المحيطة، و $K$ نقطة منتصف القوس $\widehat{CA}$ من الدائرة $\mathcal{C}$ ، و $T$ نقطة على $\mathcal{C}$ بحيث $KT$ عمودي على $BC$ . إذا كانت

$A', B'$ نقطتي تماس الدائرة الداخليّة للمثلّث $\Delta ABC$ مع الضلعين $BC, AC$ على الترتيب، فأثبت أنّ المستقيمات $AT, BK, A'B'$ تلتقي في نقطة واحدة.

## مسائل اليوم الرّابع ـ 7 ذوالحجّة 1433 هـ

1 ـ لتكن $a_1, a_2, a_3, \ldots$ متتالية أعداد حقيقيّة تحقّق العلاقة
$$a_{n+1} = \sqrt{a_n^2 + 1}.$$
لنفرض وجود عدد صحيح موجب $n_0$ بحيث $a_{2n_0} = 3a_{n_0}$ . أوجد قيمة $a_{46}$ .

2 ـ ليكن $x, y$ عددين صحيحين. أثبت أنّ المقدار $x^7 + y^7$ يقبل القسمة على 2013 إذا كان المقدار $x^{1433} + y^{1433}$ يقبل القسمة على 2013 .

3 ـ كم توجد من تبديلات $(s_1, s_2, \cdots, s_n)$ لـ $(1, 2, \ldots, n)$ تحقّق الشرط $s_i > s_j$ لكلّ $i \geq j + 3$ عندما $n = 5$ وعندما $n = 7$ ؟

4 ـ $\Delta ABC$ مثلّث، و $M$ نقطة المنتصف للضلع $BC$ ، و $D$ مسقط النقطة $M$ على الضلع $AC$ ، و $E$ نقطة المنتصف للضلع $MD$ . أثبت أنّ المستقيمين $AE, BD$ عموديّان إذا وفقط إذا كان $AB = AC$ .

# اختبارات ترشيح الفريق السّعودي لأولمبياد الرياضيات لدول الخليج 2013

## مسائل اليوم الأوّل - 10 ربيع الأوّل 1434 هـ

1- يريد طارق أن يختار بعض الأعداد المختلفة من المجموعة $S = \{1, 2, \ldots, 111\}$ بحيث لا يمكن كتابة أيّ عدد مختار كحاصل ضرب عددين مختارين آخرين مختلفين. ما هو أكبر كمّ من الأعداد يمكن لطارق أن يختارها؟

2- لكلّ الأعداد الحقيقيّة الموجبة $a$ و $b$ و $c$ ، أثبت المتباينة

$$\frac{a^3}{a^2+ab+b^2}+\frac{b^3}{b^2+bc+c^2}+\frac{c^3}{c^2+ca+a^2}\geq\frac{a+b+c}{3}.$$

3- نعرّف **النجمة المنتظمة ذات** $n$ **من الرؤوس** على أنّها اتّحاد $n$ من القطع المستقيمة $P_1P_2, P_2P_3, \ldots, P_{n-1}P_n, P_nP_1$ تحقّق

- النقط $P_1, P_2, \ldots, P_n$ تقع في نفس المستوى ولا توجد ثلاثة نقط منها على استقامة واحدة؛
- كلّ قطعة مستقيمة تقطع على الأقل قطعة مستقيمة أخرى في غير الرؤوس؛
- كلّ الزوايا عند الرؤوس $P_1, P_2, \ldots, P_n$ متطابقة؛
- كلّ القطع المستقيمة $P_1P_2, P_2P_3, \ldots, P_nP_1$ متطابقة؛ وأخيرا
- الخطّ المضلّع $P_1P_2 \ldots P_nP_1$ يستدير باتّجاه معاكس لعقارب السّاعة بزاوية أقل من $180°$ عند كلّ رأس.

لاتوجد نجوم منتظمة ذات 3 ، أو 4 ، أو 6 رؤوس. كلّ النجوم المنتظمة ذات 5 رؤوس متشابهة فيما بينها، وتوجد نجمتان منتظمتان غير متشابهتين ذات 7 رؤوس. جد كلّ قيم $n$ الممكنة بحيث توجد بالضبط 29 نجمة منتظمة غير متشابهة ذات $n$ رأس.

4- في مثلّث حاد $ABC$ ، النقطتان $D$ و $E$ قدما الارتفاعين من $A$ على $BC$ ، ومن $B$ على $CA$ ، على الترتيب. القطعة المستقيمة $AD$ قطر لدائرة $\omega$ . الدائرة $\omega$ تقطع الضلعين

$AC$ و $AB$ عند نقطتين $F$ و $G$ ، مختلفتين عن $A$ ، على الترتيب. القطعة المستقيمة $BE$ تقطع القطعتين المستقيمتين $GD$ و $GF$ عند $X$ و $Y$ على الترتيب. الشعاع $DY$ يقطع الضلع $AB$ عند $Z$ . أثبت أنّ المستقيم $XZ$ عموديّ على $BC$ .

## مسائل اليوم الثّاني - 14 ربيع الأوّل 1434 هـ

1- $ABC$ مثلّث حاد، و $\omega$ الدّائرة المارة برؤوسه، و $O$ مركز الدّائرة $\omega$ . يتقاطع المستقيم $BO$ مع الضلع $AC$ عند $B_1$ . يتقاطع المستقيم $CO$ مع الضلع $AB$ عند $C_1$ . المستقيم $B_1C_1$ يقطع الدّائرة $\omega$ عند $P$ و $Q$ . إذا كان $AP = AQ$ ، فأثبت أنّ $AB = AC$ .

2- لتكن $f(X) = a_nX^n + a_{n-1}X^{n-1} + \cdots + a_1X + p$ كثيرة حدود معاملاتها أعداد صحيحة و $p$ عدد أوّليّ. لنفرض أنّ

$$p > \sum_{i=1}^{n} |a_i|.$$

أثبت أنّ $f(X)$ غير قابلة للتّحليل (للاختزال).

3- جد أكبر عدد صحيح $k$ بحيث $k$ يقسم $n^{55} - n$ لكلّ عدد صحيح $n$ .

4- لتكن $F_0 = 0$ ، و $F_1 = 1$ ، و $F_{n+1} = F_n + F_{n-1}$ ، لكلّ عدد صحيح موجب $n$ ، متتابعة فيبوناتشي. أثبت أنّه لكلّ عدد صحيح $m$ يوجد عدد غير منته من الأعداد الصّحيحة الموجبة $n$ الّتي تحقّق

$$F_n + 2 \equiv F_{n+1} + 1 \equiv F_{n+2} \mod m.$$

## مسائل اليوم الثّالث ـ 17 ربيع الأوّل 1434 هـ

1- جد كلّ الدّوال $f:\mathbb{R}\to\mathbb{R}$ التي تحقّق

$$f\left(\frac{\sqrt{3}}{3}x\right) = \sqrt{3}f(x)-\frac{2\sqrt{3}}{3}x,$$

$$f(x)f(y) = f(xy)+f\left(\frac{x}{y}\right),$$

لكلّ $x,y\in\mathbb{R}$ ، حيث $y\neq 0$ .

2- جد كلّ قيم $n$ التي لأجلها يوجد مضلّع دائريّ محدّب غير منتظم له $n$ رأس وقيم كلّ زواياه الدّاخليّة متساوية.

3- $ABC$ مثلّث، و $H$ ملتقى ارتفاعاته، و $I$ مركزه الدّاخليّ، و $O$ مركزه المحيط، و $\omega$ دائرته المحيطة (الّتي تمرّ برؤوسه). المستقيم $CI$ يقطع الدائرة $\omega$ عند نقطة $D$ مختلفة عن $C$. لنفرض أنّ $AB=ID$ و $AH=OH$ . جد زوايا المثلّث $ABC$ .

4- جد كلّ أزواج الأعداد الصحيحة الموجبة $(a,b)$ الّتي لأجلها يقسم المقدار $a^2+b^2$ المقدارين $a^3+1$ و $b^3+1$ .

# اختبارات ترشيح الفريق السّعودي لأولمبياد الرياضيات لدول البلقان 2013

## مسائل اليوم الأوّل ـ 26 جمادى الأولى 1434 هـ

1– لتكن $G$ مجموعة النقط $(x, y)$ ذات الاحداثيّات الصّحيحة $1 \leq x \leq 5$ و $1 \leq y \leq 5$ . حدّد عدد متتابعات النقط الخمس $(P_1, P_2, P_3, P_4, P_5)$ التّي تحقّق لكلّ $1 \leq i \leq 5$ ، $P_i = (x_i, i)$ في $G$ و $|x_1 - x_2| = |x_2 - x_3| = |x_3 - x_4| = |x_4 - x_5| = 1$ .

2– لكلّ الأعداد الصّحيحة $a$ و $b$ ، يرمز $\gcd(a, b)$ لأكبر قاسم مشترك بينهما ويرمز $lcm(a, b)$ لأصغر مضاعف مشترك بينهما. حدّد عدد الأزواج المرتّبة $(a, b)$ من الأعداد الصحيحة الموجبة التّي تحقّق المعادلة

$$ab + 63 = 20\ lcm(a, b) + 12\gcd(a, b).$$

3– حلّ المعادلة التّالية حيث $x$ عدد حقيقي

$$\lfloor x^2 \rfloor - 10\lfloor x \rfloor + 24 = 0.$$

4– السّداسي $ABCDEF$ متطابق الزّوايا ومحيطه يساوي 21 . إذا علمت أنّ $AB = 3$ ، و $CD = 4$ ، و $EF = 5$ ، فاحسب مساحة السّداسي $ABCDEF$ .

5– ليكن $k$ عددا حقيقيّا بحيث يساوي حاصل ضرب الجذور الحقيقيّة للمعادلة

$$X^4 + 2X^3 + (2 + 2k)X^2 + (1 + 2k)X + 2k = 0$$

2013– . احسب حاصل جمع مربّعات هذه الحلول.

6– $ABC$ مثلّث و $I$ مركز دائرته المحاطة بأضلاعه (الدّاخليّة)، و $D$ ، $E$ ، $F$ نقط منتصف أضلاعه $BC$ ، $CA$ ، $AB$ ، على التّرتيب. يتقاطع المستقيمان $BI$ و $DE$ عند $P$ ، ويتقاطع المستقيمان $CI$ و $DF$ عند $Q$ . المستقيم $PQ$ يقطع الضلعين $AB$ و $AC$ عند $T$ و $S$ ، على التّرتيب. أثبت أنّ $AS = AT$ .

7– يريد أيمن أن يلوّن خانات طاولة $50 \times 50$ شطرنج باللّونين الأبيض والأسود بحيث يشتمل كلّ مستطيل من القياس $3 \times 2$ أو $2 \times 3$ على عدد زوجي من الخانات البيضاء. حدّد عدد الطرق التّي يمكن لأيمن أن يلوّن بها طاولة الشطرنج.

8– أثبت أنّ حاصل القسمة

$$\frac{1^1 + 3^3 + 5^5 + \cdots + (2^{2013} - 1)^{(2^{2013}-1)}}{2^{2013}}$$

عدد فردي.

## مسائل اليوم الثّاني ـ 28 جمادى الأولى 1434 هـ

1– في المثلّث $ABC$ ، $AB = AC = 3$ و $\angle A = 90°$ . لتكن $M$ نقطة منتصف الضّلع $BC$ . النقطتان $D$ و $E$ تقعان على الضّلعين $AC$ و $AB$ على التّرتيب بحيث يكون $AD > AE$ ويكون $ADME$ رباعيّا دائريّا. إذا علمت أنّ مساحة المثلّث $EMD$ تساوي 2 ، جد طول القطعة المستقيمة $CD$ ؟

2– جد كلّ الدّوال $f : \mathbb{R} \to \mathbb{R}$ التّي تحقّق لكلّ $x, y \in \mathbb{R}$ العلاقة

$$f(f(f(x) + y) + y) = x + y + f(y).$$

3– جد كلّ الأعداد الصّحيحة الموجبة $x, y, z$ التّي تحقّق

$$2^x + 21^y = z^2.$$

4– هناك عشر طلّاب متصافّون في خط مستقيم. أراد مدرّس أن يضع غطرة على رأس كلّ طالب. لديه لونان من الغطرات، لون أحمر ولون أبيض، ولديه عشر غطرات من كلّ لون. جد عدد الطّرق التّي تمكّن المدرّس من وضع الغطرات بحيث إذا اعتبرنا أي مجموعة طلّاب متتاليّين، الفارق بين عدد الطّلاب الّذين على رأسهم غطرة حمراء وعدد الطّلاب الّذين على رأسهم غطرة بيضاء لا يتجاوز 2 .

5– نقول لعدد صحيح موجب إنّه طيّب إذا كانت خاناته مختلفة عن صفر ومجموع

مربّعات خاناته مربّع كامل. فمثلا، 122 و 34 عددان طيّبان، في حين أنّ 304 و 12 عددان غير طيّبين. أثبت أنّه يوجد عدد طيّب مكوّن من $n$ خانة لكلّ عدد صحيح موجب $n$ .

6– لتكن $a, b, c$ أعدادا حقيقيّة موجبة تحقّق $ab+bc+ca=1$ . أثبت أنّ
$$a\sqrt{b^2+c^2+bc}+b\sqrt{c^2+a^2+ca}+c\sqrt{a^2+b^2+ab}\geq\sqrt{3}.$$

7– الدّائرة $\omega_B$ المحاطة خارج المثلّث $ABC$ مقابلة الرّأس $B$ ، تمسّ الضلع $AC$ ، والشعاعين $BA$ و $BC$ عند $B_1$ ، و $C_1$ ، و $A_1$ ، على التّرتيب. النقطة $D$ تقع على القوس الكبير $\widehat{A_1C_1}$ لـ $\omega_B$ . الشّعاعان $DA_1$ و $C_1B_1$ يلتقيان عند $E$ . المستقيمان $AB_1$ و $BE$ يلتقيان عند $F$ . أثبت أنّ المستقيم $FD$ مماس لـ $\omega_B$ (عند $D$ ).

8– يضمّ نادي اجتماعي 101 عضوا كلّهم يتقنون نفس الـ 50 لغة. اختار كلّ زوج من الأعضاء لغة واحدة يتحدّثان بها فيما بينهما. لنفرض عدم وجود ثلاث أعضاء يتحدّثون فيما بينهم بلغة واحدة. ليكن $A$ عدد المجموعات المكوّنة من ثلاث أعضاء يتكلّمون فيما بينهم بثلاث لغات مختلفة. جد أكبر قيمة ممكنة لـ $A$ .

## مسائل اليوم الثّالث - 4 جمادى الآخرة 1434 هـ

1– $ABCD$ رباعي دائري و $\omega$ الدائرة المحيطة برؤوسه. المستقيم العمودي على $AC$ والمار بـ $D$ يقطع $AC$ عند $E$ ويقطع $\omega$ عند $F$ . ليكن $\ell$ المستقيم العمودي على $BC$ والمار بـ $F$ . المستقيم العمودي على $\ell$ والمار بـ $A$ يقطع $\ell$ عند $G$ ويقطع $\omega$ عند $H$ . المستقيم $GE$ يقطع $FH$ عند $I$ ويقطع $CD$ عند $J$ . أثبت أنّ النّقط $C$ ، و $F$ ، و $I$ ، و $J$ على دائرة واحدة.

2– نعرّف متتابعة فيبوناتشي $\{F\}_{n=0}^{\infty}$ بـ $F_0=0$ ، و $F_1=1$ ، و $F_{n+1}=F_n+F_{n-1}$ لكلّ عدد صحيح $n>1$ . حدد كلّ رباعيّات $(a,b,c,n)$ الأعداد الصّحيحة الموجبة الّتي تحقّق $a<b<c$ ويكون فيها كلّ من $a$ ، $b$ ، $c$ ، $a+n$ ، $b+n$ ، و $c+2n$ عنصرا من عناصر متتابعة فيبوناتشي؟

3- ليكن $T$ عددا حقيقيّا يحقّق الخاصيّة التّالية: لكلّ أعداد حقيقيّة غير سالبة $a$ ، $b$ ، $c$ ، $d$ ، $e$ حاصل جمعها 1 ، يمكن ترتيبها على دائرة بحيث لا يكبر حاصل ضرب كلّ عددين متتاليين العدد $T$ . حدّد أصغر قيمة ممكنة لـ $T$ .

4- لتكن $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ دالة تحقّق لكلّ عدد صحيح $n \geq 0$ :

$$(a) \quad f(2n+1)^2 - f(2n)^2 = 6f(n) + 1, \qquad (b) \quad f(2n) \geq f(n);$$

حيث يرمز $\mathbb{Z}_{\geq 0}$ لمجموعة الأعداد الصحيحة غير السالبة. حلّ المعادلة $f(n) = 1000$ .

## مسائل اليوم الرّابع - 6 جمادى الآخرة 1434 هـ

1- $ABCD$ رباعي دائري يحقّق $AB = BC = CA$ . القطران $AC$ و $BD$ يلتقيان عند $E$ . إذا علمت أنّ $BE = 19$ و $ED = 6$ ، فجد كلّ القيم الممكنة لـ $AD$ .

2- يكتب العدد $n$ في القاعدة 7 على شكل $\overline{abc}_{(7)}$ ، ويكتب في القاعدة 9 على شكل $\overline{cba}_{(9)}$ . ما هي كتابة العدد $n$ في القاعدة 10 ؟

3- جد مساحة مجموعة نقط المستوي الّتي تحقّق إحداثيّاتها $(x, y)$ المتباينة

$$x^2 + y^2 \leq 4|x| + 4|y|.$$

4- جد كلّ الأعداد الصحيحة الموجبة $n < 589$ الّتي تجعل 589 يقسم المقدار $n^2 + n + 1$ .

# اختبارات ترشيح الفريق السّعودي لأولمبياد الرياضيات الدّولي 2013

## مسائل اليوم الأوّل ـ 18 رجب 1434 هـ

1– ليكن $ABC$ مثلّثا محاطا بالدّائرة $\omega$ . النقطة $P$ تقع داخل المثلّث $ABC$ . المستقيمات $AP$ و $BP$ و $CP$ تقطع الدّائرة $\omega$ مرّة ثانية عند النقط $A_1$ و $B_1$ و $C_1$ ، على التّرتيب، مختلفة عن $A$ و $B$ و $C$ . المستقيمان الماسّان للدّائرة $\omega$ عند النقطتين $A_1$ و $B_1$ يتقاطعان عند النقطة $C_2$ . المستقيمان الماسّان للدّائرة $\omega$ عند النقطتين $B_1$ و $C_1$ يتقاطعان عند النقطة $A_2$ . المستقيمان الماسّان للدّائرة $\omega$ عند النقطتين $C_1$ و $A_1$ يتقاطعان عند النقطة $B_2$ . أثبت أنّ المستقيمات $AA_2$ و $BB_2$ و $CC_2$ تلتقي عند نقطة واحدة.

2– لتكن $S = \{0, 1, 2, 3, \ldots\}$ مجموعة الأعداد الصّحيحة غير السّالبة. جد كلّ الدّوال $f \colon S \to S$ المتزايدة قطعا والّتي تحقّق $n + f(f(n)) \leq 2f(n)$ لكلّ عدد $n$ في $S$ .

3– لدى شركة سعوديّة مكتبان، أحدهما في الرّياض والآخر في جدّه. حتّى يتمّ التّنسيق بين المكتبين، قامت الشّركة بتعيين في كلّ مكتب عددا من المراسلين بحيث يتحقّق ما يلي

ا ـ كلّ مراسلين اثنين من نفس المكتب يشتركان في مراسل واحد فقط من المكتب الثّاني.

ب ـ هناك على الأقل 10 مراسلون في الرّياض.

ج ـ أحد المراسلين في جدّة، واسمه زيد، على اتّصال مع 8 مراسلين فقط من الرّياض.

ما هو أصغر عدد ممكن من المراسلين في جدّه الّذين هم على اتّصال مع المراسل عمرو من الرّياض؟

4– حدّد ما إذا كان ممكنا إعادة ترتيب الأعداد الصّحيحة $1, 2, \ldots, 2012$ على دائرة بحيث تشكّل بواقي القسمة على 2013 لكلّ عددين متجاورين، وعدد هذه البواقي 2012 ، أعدادا مختلفة فيما بينها.

# مسائل اليوم الثّاني ـ 19 رجب 1434 هـ

1- جد أكبر وأصغر قيمة للمقدار
$$S=(1-x_1)(1-y_1)+(1-x_2)(1-y_2)$$
لمّا تحقّق الأعداد الحقيقيّة $x_1$ ، $x_2$ ، $y_1$ ، $y_2$ العلاقة $x_1^2+x_2^2=y_1^2+y_2^2=2013$ .

2- ليكن $ABC$ مثلّثا حادّا و $AA_1$ و $BB_1$ و $CC_1$ ارتفاعاته. تلتقي القطعتان المستقيمتان $AA_1$ و $B_1C_1$ عند النقطة $K$ . يقطع المنصّف العموديّ على القطعة المستقيمة $A_1K$ الضلعين $AB$ و $AC$ عند النقطتين $L$ و $M$ ، على التّرتيب. أثبت أنّ النقط $A$ و $A_1$ و $L$ و $M$ تقع على دائرة واحدة.

3- نعتبر لكلّ عدد صحيح $n$ قائمة قواسمه (بما فيهم 1 وذات العدد). لنفرض أنّ خانات الآحاد لـ $p\%$ من هذه القواسم تساوي 3 . فمثلا، إذا كان $n=117$ ، فإنّ لديه 6 قواسم، وهي 1 ، 3 ، 9 ، 13 ، 39 ، 117 . اثنان من هذه القواسم، وهي 3 و 13 ، خانتا آحاديها تساوي 3 . إذن لدينا $p=33.33\cdots$ لمّا $n=117$ . جد أكبر قيمة ممكنة لـ $p$ إذا كان $n$ أي عدد صحيح موجب.

4- حدّد هل توجد متسلسلة غير متهية من الأعداد الصّحيحة الموجبة
$$a_1,a_2,a_3,\ldots$$
الّتي تحقّق

ا ـ كلّ عدد صحيح موجب يظهر مرّة واحدة فقط في المتسلسلة،

ب ـ كلّ عدد صحيح موجب يظهر مرّة واحدة فقط في المتسلسلة $|a_1-a_2|$ ، $|a_2-a_3|$ ، $\ldots$ ، $|a_k-a_{k+1}|$ ، $\ldots$

# مسائل اليوم الثّالث ـ 20 رجب 1434 هـ

1– رسم عادل شبكة $m \times n$ رؤوس على المستوي. إحداثيّات هذه الرؤوس أعداد صحيحة $(a,b)$ تحقّق $1 \le a \le m$ و $1 \le b \le n$ . قام عادل برسم مسار مغلق يمرّ بـ $k$ من هذه الرؤوس، $(a_1,b_1)$ ، $(a_2,b_2)$ ، ... ، $(a_k,b_k)$ ، بحيث يساوي طول المسافة بين $(a_i,b_i)$ و $(a_{i+1},b_{i+1})$ واحدا لكلّ $1 \le i \le k$ (حيث $(a_{k+1},b_{k+1}) = (a_1,b_1)$ ). تأكّد عادل أنّ مساره غير متداخل، أي أنّ كلّ رؤوسه الـ $k$ مختلفة فيما بينها. احسب بالإثبات أكبر قيمة ممكنة لـ $k$ بدلالة $m$ و $n$ .

2– ليكن $n$ عددا صحيحا يحقّق $n \ge 2$ . حدّد عدد المجموعات المرتّبة $(a_1, a_2, \ldots, a_n)$ من الأعداد الصحيحة التّي تحقّق

ا ـ $a_1 + a_2 + \cdots + a_n \ge n^2$ ؛

ب ـ و $a_1^2 + a_2^2 + \cdots + a_n^2 \le n^3 + 1$ .

3– ليكن $ABC$ مثلّثا حادّا، و $M$ نقطة المنتصف للضلع $BC$ ، و $P$ نقطة على القطعة المستقيمة $AM$ . المستقيمان $BP$ و $CP$ يقطعان الدائرة المحيطة بالمثلّث $ABC$ مرّة ثانية عند $X$ و $Y$ ، على التّرتيب، ويقطعان، الأوّل منهما الضلع $AC$ عند $D$ والآخر الضّلع $AB$ عند $E$. أثبت انّ الدّائرتين المحيطتين بالمثلّثين $AXD$ و $AYE$ تتقاطعان مرّة ثانية عند نقطة $T \neq A$ على المستقيم $AM$ .

4– جد كلّ كثيرات الحدود $p(x)$ ذوات المعاملات الصحيحة التّي تجعل لكلّ عدد صحيح موجب $n$ ، المقدار $2^n - 1$ يقبل القسمة على $p(n)$ .

# Part III
# Solutions

# Chapter 5

# Solutions to the preselection tests for the full-time training

## 5.1 Solutions to the selection test of day I

1. *First solution.* Because $-1 \le x, y \le 1$, we have $(1-x^2), (1-y^2) \ge 0$. Therefore, by applying AM-GM, we obtain

$$2\sqrt{(1-x^2)(1-y^2)} \le (1-x^2) + (1-y^2).$$

So, it remains to prove that

$$(1-x^2) + (1-y^2) \le 2(1-x)(1-y) + 1$$

which is equivalent to

$$x^2 + y^2 - 2x - 2y + 2xy + 1 \ge 0.$$

But

$$x^2 + y^2 - 2x - 2y + 2xy + 1 = (x+y-1)^2.$$

This ends the proof.
The equality holds when $1-x^2 = 1-y^2$ and $x+y-1=0$. This is equivalent to $x = y = \frac{1}{2}$.
*Second solution.* Because $-1 \le x, y \le 1$, there exist $-\frac{\pi}{2} \le \theta, \phi \le \frac{\pi}{2}$ such that $x = \sin\theta$ and $y = \sin\phi$. The inequality to prove becomes

$$2\cos\theta\cos\phi \le 2(1-\sin\theta)(1-\sin\phi) + 1,$$

or, equivalently
$$2[\cos(\theta+\phi)+\sin\theta+\sin\phi] \le 3.$$

Using the relations
$$\cos(\theta+\phi) = 1-2\sin^2\frac{\theta+\phi}{2} \qquad \text{and} \qquad \sin\theta+\sin\phi = 2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},$$

the inequality simplifies to
$$\cos^2\frac{\theta-\phi}{2} \le 1+\left(2\sin\frac{\theta+\phi}{2}-\cos\frac{\theta-\phi}{2}\right)^2,$$

which is satisfied.
The equality holds when $\theta=\phi=\frac{\pi}{6}$, which means $x=y=\frac{1}{2}$.

2. Because 2 and 47 are relatively prime numbers, 47 divides $3^x-2^y$ if and only if 47 divides $2^{4x}(3^x-2^y)$. But
$$2^{4x}(3^x-2^y) = 48^x-2^{4x+y} \equiv 1-2^{4x+y} \mod 47.$$
Therefore 47 divides $3^x-2^y$ if and only if $2^{4x+y} \equiv 1 \mod 47$.
On the other hand, we have $2^{23} \equiv 49^{23} \equiv 7^{46} \equiv 1 \mod 47$. We deduce that the prime number 23 is the order of 2 modulo 47. This implies that $2^{4x+y} \equiv 1 \mod 47$ if and only if 23 divides $4x+y$.

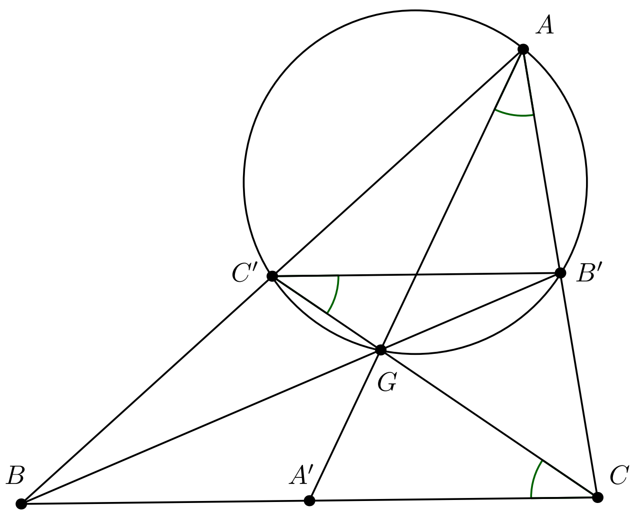
3. First, since the proctor distribute randomly the Algebra paper to each student, he has 10! ways to do it depending on how he orders the students.
For the other three papers, we order the students from 1 to 30, each student receiving three positions corresponding to the three papers. Since the first position a student receives corresponds to the Geometry paper, the second to the Number Theory paper and the third to the Combinatorics paper, the problem is equivalent to count the number of partitions of the set $\{1,2,\ldots,30\}$ into 10 subsets, each of 3 elements. This is equal to
$$\binom{30}{3,\ldots,3} = \frac{30!}{3!^{10}}.$$
Therefore, the number of ways the proctor can distribute the test papers is
$$\frac{10!30!}{3!^{10}}.$$

4. *Remark.* This problem has a converse:

$$\text{If } AB \cdot CC' = AC \cdot BB', \text{ then either } AB = AC \text{ or } AC'GB' \text{ is cyclic.}$$

The third solution contains a proof for the coverse.
*First solution.* Because $AC'GB'$ is cyclic, we have $\measuredangle GAB' = \measuredangle GC'B'$. Because $B'C'$ is parallel to $BC$, we have $\measuredangle CC'B' = \measuredangle C'CB$. We deduce that

$$\sin \measuredangle A'AC = \sin \measuredangle C'CB.$$




But triangles $AA'C, BCC'$ have the same area which is half of the area of triangle $ABC$. We deduce that

$$\frac{1}{2}AA' \cdot AC \sin \measuredangle A'AC = \frac{1}{2}CC' \cdot BC \sin \measuredangle C'CB,$$

and therefore

$$\frac{AC}{CC'} = \frac{BC}{AA'}.$$

We proceed in a similar way, by considering triangles $ABA'$ and $BCB'$, obtaining

$$\frac{AB}{BB'} = \frac{BC}{AA'},$$

and deduce the relation

$$AB \cdot CC' = AC \cdot BB'.$$

*Second solution.* Lets us consider the power of the point $B$ with respect to the circumcircle of the cyclic quadrilateral $AC'GB'$. We have

$$\mathcal{P}(B) = \overline{BC'} \cdot \overline{BA} = \overline{BG} \cdot \overline{BB'}.$$

But $\overline{BC'} = \frac{1}{2}\overline{BA}$ and $\overline{BG} = \frac{2}{3}\overline{BB'}$. We deduce that

$$BB' = \frac{\sqrt{3}}{2}AB.$$

Similarly, by considering the power of the point $C$ with respect to the circumcircle of the cyclic quadrilateral $AC'GB'$ we obtain

$$CC' = \frac{\sqrt{3}}{2}AC.$$

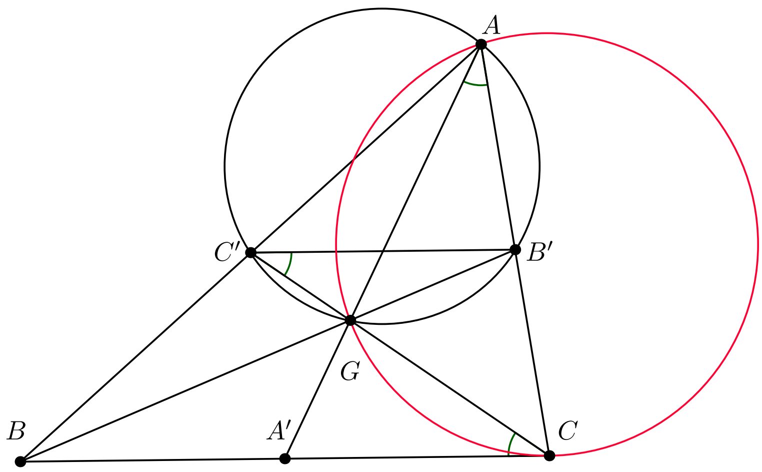
From these two relations we deduce easily that

$$AB \cdot CC' = AC \cdot BB'.$$

*Third solution (Contains also a proof for the converse).* We can see from the first solution that the quadrilateral $AC'GB'$ is cyclic if and only

$$\measuredangle GAC = \measuredangle GCB.$$

This is equivalent to saying that the line $BC$ is tangent to the circumcircle of triangle $AGC$.



This property is equivalent to saying that $A'C^2 = \overline{A'G} \cdot \overline{A'A}$ by using the power of the point $A'$ with respect to this circumcircle. But

$$A'C = \frac{1}{2}BC, \quad \overline{A'G} = \frac{1}{3}\overline{A'A} \quad \text{and} \quad AA' = \sqrt{\frac{2AB^2 + 2AC^2 - BC^2}{4}}.$$

We deduce that $AC'GB'$ is cyclic if and only if

$$2BC^2 = AB^2 + AC^2.$$

Now, using the formulas for the medians

$$BB' = \sqrt{\frac{2BC^2 + 2AB^2 - CA^2}{4}} \quad \text{and} \quad CC' = \sqrt{\frac{2CA^2 + 2BC^2 - AB^2}{4}},$$

we deduce that the relation

$$AB \cdot CC' = AC \cdot BB'$$

is equivalent to

$$AB^2 \cdot (2CA^2 + 2BC^2 - AB^2) = AC^2 \cdot (2BC^2 + 2AB^2 - CA^2),$$

which is equivalent to

$$(AB^2 - AC^2) \cdot (2BC^2 - AB^2 - AC^2) = 0,$$

which is equivalent to either $AB = AC$ or $AC'GB'$ is cyclic.

## 5.2 Solutions to the selection test of day II

1. If $a$ is relatively prime with 35 then it is relatively prime with both $5, 7$. Since $a$ is relatively prime with 5 then, by Fermat, $a^4 \equiv 1 \mod 5$ which implies
$$(a^4 - 1)(a^4 + 15a^2 + 1) \equiv 0 \mod 5.$$
Since $a$ is relatively prime with 7 then $a^6 \equiv 1 \mod 7$. Hence
$$\begin{aligned}(a^4 - 1)(a^4 + 15a^2 + 1) &\equiv (a^2 + 1)(a^2 - 1)(a^4 + a^2 + 1) \\ &\equiv (a^2 + 1)(a^6 - 1) \equiv 0 \qquad \mod 7.\end{aligned}$$
But $5, 7$ are relatively prime and therefore
$$(a^4 - 1)(a^4 + 15a^2 + 1) \equiv 0 \mod 35.$$

2. Let $u, v$ be the roots of the quadratic equation $ax^2 + bx + c = 0$ such that $0 \leq u \leq v \leq 1$. We have the relations $b = -a(u + v)$ and $c = uv$. Therefore

$$\begin{aligned}\frac{(a-b)(2a-b)}{a(a-b+c)} &= \frac{(1+u+v)(2+u+v)}{1+u+v+uv}\\ &= 2 + \frac{u}{1+v} + \frac{v}{1+u}\\ &\leq 2 + \frac{u}{1+u} + \frac{1}{1+u} = 3.\end{aligned}$$

Clearly, when $u = v = 1$, the equality holds. Thus, 3 is the maximum.

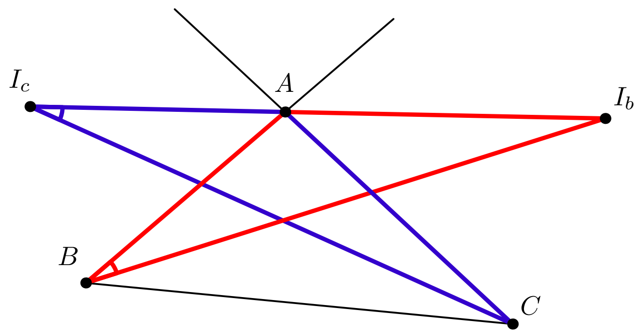
3. This is equivalent to prove that there exists a positive integer $k$ such that $10^n$ divides $a^k - 1$.
*First solution.* Consider the remainders of the division of the $10^n + 1$ powers $a^1, a^2, \ldots, a^{10^n+1}$ of $a$ by $10^n$. Since there are at most $10^n$ possible reminders, by the pigeonhole principle there exist at least two powers $a^i, a^j$, $i < j$, having the same remainder. Therefore $10^n$ divides $a^j - a^i = a^i(a^{j-i} - 1)$. But $10^n$ is relatively prime with $a^i$. This proves that $10^n$ divides $a^{j-i} - 1$.
*Second solution.* Since $a$ and $10^n$ are relatively prime, by Euler's theorem

$$a^{\phi(10^n)} = a^{4\cdot 10^{n-1}} \equiv 1 \mod 10^n.$$

Therefore, $10^n$ divides $a^{4\cdot 10^{n-1}} - 1$.

4. *First solution.* Let $\Delta ABC$ be any triangle. Considering triangle $AI_cC$, we have

$$\begin{aligned}\measuredangle CI_cA &= 180^\circ - (\measuredangle ACI_c + \measuredangle I_cAC)\\ &= 180^\circ - (\tfrac{1}{2}\measuredangle ACB + \measuredangle BAC + \tfrac{1}{2}(180^\circ - \measuredangle BAC))\\ &= 90^\circ - \tfrac{1}{2}(\measuredangle ACB + \measuredangle BAC)\\ &= \measuredangle I_bBA.\end{aligned}$$



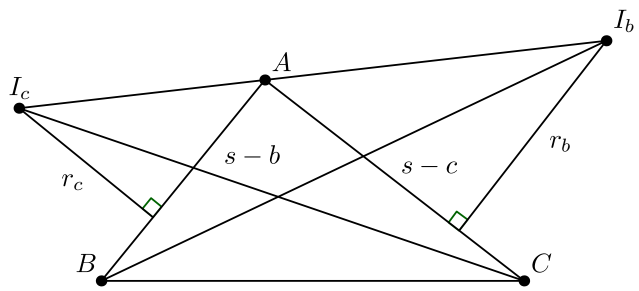
On the other hand, $\measuredangle I_cAC = \measuredangle BAI_b$. We deduce that triangles $ABI_b, AI_cC$ are similar and therefore

$$\frac{AI_c}{AB} = \frac{AC}{AI_b} \quad \Leftrightarrow \quad AI_b \cdot AI_c = AB \cdot AC.$$

Thus, the area of triangle $ABC$ is equal to $\frac{1}{2}AI_b \cdot AI_c \sin \measuredangle BAC$. Hence, triangle $\Delta ABC$ is right at $A$ if and only if its area is equal to $\frac{1}{2}AI_b \cdot AI_c$.

*Second solution.* Let $\Delta ABC$ be any triangle. Denote $a, b, c$ the lengths of the opposite sides to the vertices $A, B, C$ respectively, $s$ the semiperimeter, $r$ the inradius, $r_b, r_c$ the exradius opposite to the vertices $B, C$ respectively. We have

$$AI_b^2 = r_b^2 + (s-c)^2 \quad \text{and} \quad AI_c^2 = r_c^2 + (s-b)^2.$$




But the area of the triangle $\Delta ABC$ is equal to

$$K = rs = r_b(s-b) = r_c(s-c) = \sqrt{s(s-a)(s-b)(s-c)},$$

and that $r_b r_c = s(s-a)$ (which follows from the above formulas for the area). Hence

$$AI_b^2 \cdot AI_c^2 = (r_b^2 + (s-c)^2)(r_c^2 + (s-b)^2) = s^2(s-a)^2 + 2K^2 + (s-b)^2(s-c)^2.$$

Therefore, $K = \frac{1}{2}AI_b \cdot AI_c$ is equivalent to

$$2K^2 = s^2(s-a)^2 + (s-b)^2(s-c)^2.$$

But $2K^2 = 2s(s-a)(s-b)(s-c)$. Hence, this is equivalent to

$$((s-b)(s-c) - s(s-a))^2 = 0,$$

which simplifies to

$$a^2 = b^2 + c^2.$$

## 5.3 Solutions to the selection test of day III

1. Let $x_0$ be a a solution of the equation $f(x) = x$. We have

$$x_0 = f(x_0) = f(f(x_0)) = 4x_0 + 1.$$

Therefore, $x_0 = -\frac{1}{3}$. This proves the uniqueness of the solution.
On the other hand,

$$f\left(f\left(-\frac{1}{3}\right)\right) = 4\left(-\frac{1}{3}\right) + 1 = -\frac{1}{3}.$$

We deduce that

$$f\left(-\frac{1}{3}\right) = f\left(f\left(f\left(-\frac{1}{3}\right)\right)\right) = 4f\left(-\frac{1}{3}\right) + 1,$$

and therefore,

$$f\left(-\frac{1}{3}\right) = -\frac{1}{3}.$$

This prove the existence of the solution.

*Remark.* There are infinitely many functions $f$ satisfying the relation $f(f(x)) = 4x + 1$, for all $x \in \mathbb{R}$. For example, the function given by

$$f(x) = 2\left(x + \frac{1}{3}\right) - \frac{1}{3}, \quad \text{for all } x \in \mathbb{R},$$

or the function given by

$$f(x) = -2\left(x + \frac{1}{3}\right) - \frac{1}{3}, \quad \text{for all } x \in \mathbb{R},$$

or the function given by

$$f(x) = \begin{cases} 2\left(x + \frac{1}{3}\right) - \frac{1}{3} & \text{if} \quad x \in \mathbb{Q} \\ \\ -2\left(x + \frac{1}{3}\right) - \frac{1}{3} & \text{if} \quad x \notin \mathbb{Q} \end{cases},$$

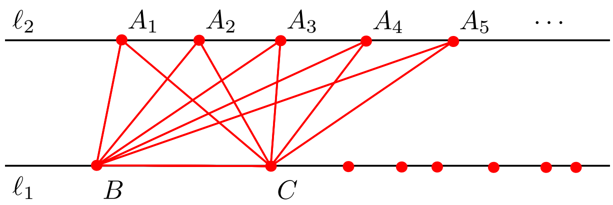
all satisfy the relation. More generaly, if $A \subseteq \mathbb{R}$ has the property: $ra \in A$ whenever $r \in \mathbb{Q},\ a \in A$, the function given by

$$f(x) = \begin{cases} 2\left(x + \frac{1}{3}\right) - \frac{1}{3} & \text{if} \quad x \in A \\ \\ -2\left(x + \frac{1}{3}\right) - \frac{1}{3} & \text{if} \quad x \notin A \end{cases},$$

satisfies the relation.

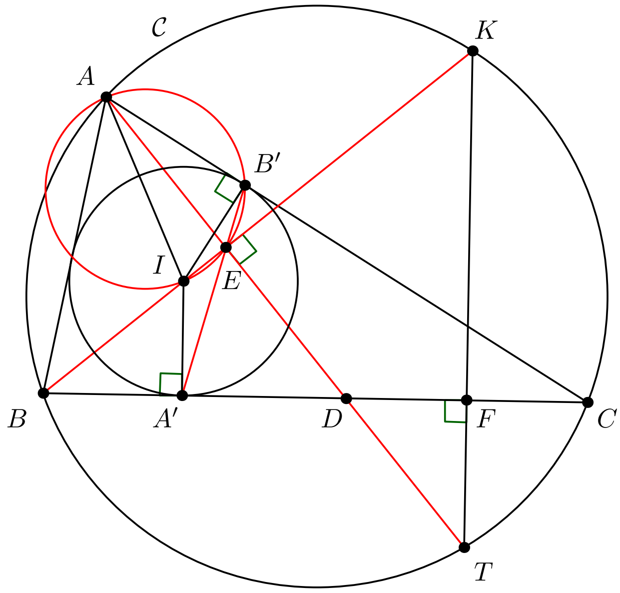
2. Assume that 19 does not divide the product $a_1a_2\cdots a_9$. This means that $a_1, a_2, \ldots, a_9$ are relatively prime with 19. Using Fermat,
$$a_1^{18} \equiv a_2^{18} \equiv \cdots a_9^{18} \equiv 1 \mod 19.$$
But $a_i^{18} \equiv 1 \mod 19$, is equivalent to $(a_i^9 - 1)(a_i^9 + 1) \equiv 0 \mod 19$. Since 19 is a prime number, this implies that $a_i^9 \equiv \pm 1 \mod 19$, for $i = 1, \ldots, 9$, and therefore $a_1^9 + a_2^9 + \cdots + a_9^9 \equiv \pm k \mod 19$ for some odd number $k$ between 1 and 9. This is a contradiction. Thus 19 divides the product $a_1a_2\cdots a_9$.

3. Consider 2014 parallel lines. Each line contains infinitely many points. Since the number of the colors is finite, by the pigeonhole principle, there exist on each line infinitely many points of the same color. Choose for each line one color for which there exist infinitely many points. Since there are 2013 colors and 2014 lines, by the pigeonhole principle there exist at least two lines $\ell_1, \ell_2$ for which the same color $\mathcal{C}$ have been choosen. Choose two points $B, C$ from the first line $\ell_1$ of this color $\mathcal{C}$. Choose infinitely many points $A_1, A_2, \ldots$ from the second line $\ell_2$ of this color $\mathcal{C}$. Triangles $A_1BC, A_2BC, \ldots$ are all of the same color $\mathcal{C}$ and have the same area since $\ell_1, \ell_2$ are parallel.




4. Let $E$ be the intersection point of $BK$ with $AT$. The problem is equivalent to prove that points $A', E, B'$ are collinear.

*First solution.* Let $F$ be the intersection point of $BC$ with $KT$. Since $K$ is the midpoint of $\overset{\frown}{CA}$, by expressing the angles in terms of arc lengths we obtain
$$\measuredangle KEA = \frac{1}{2}\left(\overset{\frown}{KA} + \overset{\frown}{BT}\right) = \frac{1}{2}\left(\overset{\frown}{CK} + \overset{\frown}{BT}\right) = \measuredangle CFK = 90^\circ.$$
But $B'$ is an intouch point. This implies that
$$\measuredangle AB'I = 90^\circ = \measuredangle AEI,$$
and the points $A, I, E, B'$ are concyclic.




We deduce from this that

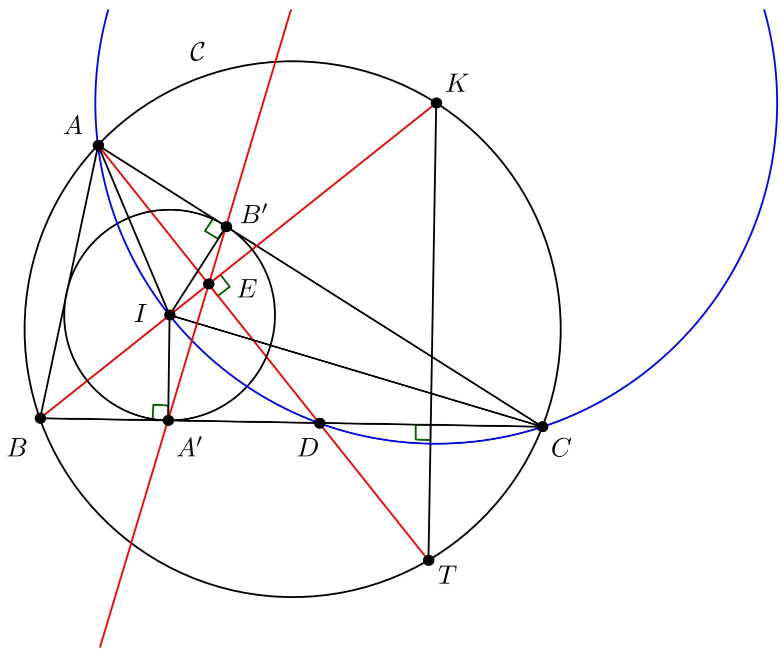
$$\measuredangle IB'E = \measuredangle IAE = 90^\circ - \measuredangle EIA = 90^\circ - \frac{1}{2}\measuredangle BAC - \frac{1}{2}\measuredangle CBA = \frac{1}{2}\measuredangle ACB.$$

On the other hand, triangle $IA'B'$ is isosceles ($IA' = IB'$). Therefore,

$$\measuredangle IB'A' = \frac{1}{2}(180^\circ - \measuredangle A'IB') = \frac{1}{2}\measuredangle ACB = \measuredangle IB'E,$$

since $\measuredangle CA'I = 90^\circ$. This proves that the points $A', E, B'$ are collinear.

*Second solution.* We prove as in the first solution that $\measuredangle KEA = 90^\circ$. Let $D$ be the intersection point of $AT$ with $BC$. The pedal triangle of $I$ with respect to triangle $ADC$ is $A'B'E$. Proving that points $A', B', E$ are collinear is equivalent to proving that $I$ is on the circumcircle of triangle $ADC$ and therefore, the line $A'B'E$ will be the Simson line of point $I$ with respect to triangle $ADC$.




We have

$$\measuredangle CDA = \measuredangle DBA + \measuredangle BAD = \measuredangle CBA + 90^\circ - \frac{1}{2}\measuredangle CBA = 90^\circ + \frac{1}{2}\measuredangle CBA.$$

On the other hand

$$\measuredangle CIA = 180^\circ - \frac{1}{2}\measuredangle BAC - \frac{1}{2}\measuredangle ACB = 90^\circ + \frac{1}{2}\measuredangle CBA = \measuredangle CDA.$$

This proves that $I$ is on the circumcircle of triangle $ADC$ and thus $A', E, B'$ are collinear.

## 5.4 Solutions to the selection test of day IV

1. We have $a_{n+1}^2 - a_n^2 = 1$ for all natural number $n$. Using a telescopic sum we obtain

$$n_0 = \sum_{k=n_0}^{2n_0-1} (a_{k+1}^2 - a_k^2) = a_{2n_0}^2 - a_{n_0}^2 = 9a_{n_0}^2 - a_{n_0}^2 = 8a_{n_0}^2.$$

Hence

$$a_{n_0}^2 = \frac{n_0}{8}.$$

On the other hand,

$$n_0 - 1 = \sum_{k=1}^{n_0-1}(a_{k+1}^2 - a_k^2) = a_{n_0}^2 - a_1^2 = \frac{n_0}{8} - a_1^2,$$

or equivalently

$$a_1^2 = \frac{8 - 7n_0}{8}.$$

Since $a_1^2 \geq 0$, we deduce that $n_0 = 1$ and $a_1^2 = \frac{1}{8}$.

Therefore

$$45 = \sum_{k=1}^{45}(a_{k+1}^2 - a_k^2) = a_{46}^2 - a_1^2 = a_{46}^2 - \frac{1}{8},$$

which leads to

$$a_{46} = \frac{19\sqrt{2}}{4}.$$

2. Since $2013 = 3 \times 11 \times 61$, we will prove that for $p = 3, 11, 61$, if $p$ divides $x^{1433} + y^{1433}$ then $p$ divides $x^7 + y^7$.
   Let $p = 3, 11, 61$, and assume that $p$ divides $x^{1433} + y^{1433}$.
   If $p$ divides $x$, then it divides $x^7$ and $x^{1433}$. But $p$ divides $x^{1433} + y^{1433}$. Then it divides $y^{1433}$. Since $p$ is a prime number, we deduce that it divides $y$ and $y^7$. Therefore, it divides $x^7 + y^7$. In a similar way we prove that if $p$ divides $y$, then it divides $x^7 + y^7$.
   Assume now that $p$ is relatively prime with $x, y$. Then, using Fermat,

   $$x^{p-1} \equiv y^{p-1} \equiv 1 \mod p.$$

   But $p - 1$ divides 1440 for $p = 3, 11, 61$. We deduce that

   $$x^{1440} \equiv y^{1440} \equiv 1 \mod p.$$

   Hence

   $$x^7 + y^7 \equiv x^7 y^{1440} + y^7 x^{1440} \equiv x^7 y^7 (y^{1433} + x^{1433}) \equiv 0 \mod p.$$

   This proves that $p$ divides $x^7 + y^7$.

3. Let $N_n$ be the number of permutations satisfying these conditions.

*For $n = 5$,* the conditions are $s_4 > s_1$ and $s_5 > s_2, s_1$. Among the 5! permutations of $(1, \cdots, 5)$, half of them satisfy the condition $s_4 > s_1$. Among these permutations, half of them satisfy also the condition $s_5 > s_2$. Therefore, there are 30 permutations satisfying both conditions $s_4 > s_1$ and $s_5 > s_2$.
To compute $N_5$, it is easier to substract from 30 the number of permutations which do not satisfy the condition $s_5 > s_1$. These permutations satisfy the condition $s_4 > s_1 > s_5 > s_2$ with no condition on $s_3$. Since there are 5 possibilities for $s_3$, there are 5 such permutations and

$$N_5 = 30 - 5 = 25.$$

*For $n = 7$.* There are more conditions and the method used for $n = 5$ becomes complicated. That is why, we will use a different method based on an inductive relation for $N_n$.
For $n = 1, 2, 3$ there are no conditions on the permutations. So

$$N_1 = 1! = 1, \quad N_2 = 2! = 2, \quad N_3 = 3! = 6.$$

For $n = 4$, there is only one condition: $s_4 > s_1$. This gives,

$$N_4 = \frac{1}{2} 4! = 12.$$

For $n \geq 5$, we have the conditions $s_n \geq s_1, s_2, \ldots, s_{n-3}$. This means that at most there are only $s_{n-1}, s_{n-2}$ at most which can be greater than $s_n$. Therefore $s_n \in \{n, n-1, n-2\}$.

(a) When $s_n = n$, there are as many permutations as for $n-1$, that is $N_{n-1}$.

(b) When $s_n = n - 1$, because of the conditions $s_n \geq s_1, s_2, \ldots, s_{n-3}$, we have $n \in \{s_{n-1}, s_{n-2}\}$.
   - If $s_{n-1} = n$, there are as many permutations as for $n - 2$, that is $N_{n-2}$.
   - If $s_{n-2} = n$, because of the conditions $s_{n-1} \geq s_1, s_2, \ldots, s_{n-4}$ we have $s_{n-1} \in \{n-2, n-3\}$.
     i. If $s_{n-1} = n - 2$, there are as many permutations as for $n - 3$, that is $N_{n-3}$.
     ii. If $s_{n-1} = n - 3$ then $s_{n-3} = n - 2$ and there are as many permutations as for $n - 4$, that is $N_{n-4}$.

(c) When $s_n = n - 2$, because of the conditions $s_n \geq s_1, s_2, \ldots, s_{n-3}$, we have either $s_{n-1} = n,\ s_{n-2} = n - 1$ or $s_{n-1} = n - 1,\ s_{n-2} = n$. In each case there are as many permutations as for $n - 3$, that is $N_{n-3}$.

Hence, we obtain the inductive relation

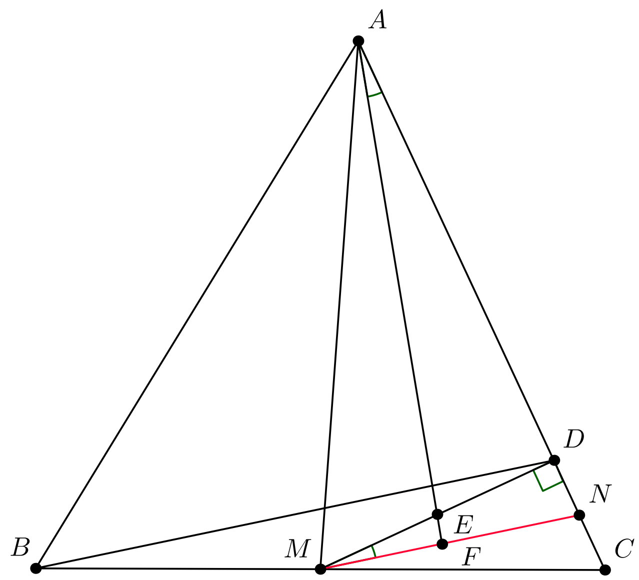
$$N_n = N_{n-1} + N_{n-2} + 3N_{n-3} + N_{n-4}.$$

Therefore

$$N_5 = 12 + 6 + 3 \times 2 + 1 = 25,$$
$$N_6 = 25 + 12 + 3 \times 6 + 2 = 57,$$
$$N_7 = 57 + 25 + 3 \times 12 + 6 = 124.$$

*Remark.* If the condition in this problem is replaced by $s_i > s_j$ for all $i \geq j+2$, $N_n$ will be given by the Fibonacci sequence.

4. *First solution.* Let $N$ be the midpoint of $CD$ and $F$ the intersection point of lines $AE$ and $MN$. Since $M$ is the midpoint of $BC$, the segment $MN$ is parallel to $BD$.




Therefore, $AE, BD$ are perpendicular if and only if $AF, MN$ are perpendicular.
Equivalently, triangles $AFN$ and $MDN$ are similar since they already share the same angle at $N$.
Since $AFN$ and $ADE$ are similar, then triangles $ADE$ and $MDN$ are similar, which is equivalent to the fact that triangles $ADM$ and $MDC$ are similar

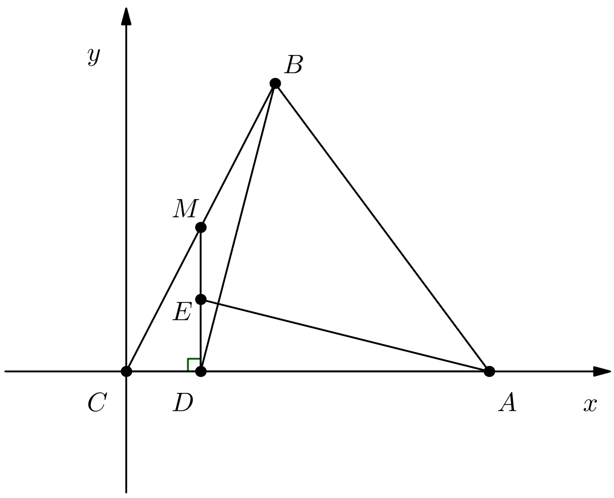
since $E$ is the midpoint of $DM$ and $N$ is the midpoint of $DC$.
In conclusion, $AE$ and $BD$ are perpendicular if and only if the median $AM$ of triangle $ABC$ is also an altitude. That is $AB = AC$.

*Second solution.* By coordinates or equivalently by complex numbers. Fix the origin at $C$, and the $x-$ axis to be $CA$. The affixes of the points are

$$C(0), \qquad A(a) \quad \text{and} \quad B(b+ic),$$

where $a, b, c$ are positive real numbers. Hence, the affixes of the other points are

$$M\left(\frac{b}{2}+i\frac{c}{2}\right), \qquad D\left(\frac{b}{2}\right) \quad \text{and} \quad E\left(\frac{b}{2}+i\frac{c}{4}\right).$$




Therefore, the affixes of the vectors are

$$\overrightarrow{DB}\left(\frac{b}{2}+ic\right) \quad \text{and} \quad \overrightarrow{EA}\left(a-\frac{b}{2}-i\frac{c}{4}\right).$$

On the other hand, the lengths of the sides are equal to

$$AB = \sqrt{(a-b)^2+c^2} \quad \text{and} \quad AC = a.$$

The segments $AE, BD$ are perpendicular if and only if

$$Re\left(\left(\frac{b}{2}+ic\right)\left(a-\frac{b}{2}+i\frac{c}{4}\right)\right) = 0,$$

which is equivalent to $2ab - b^2 - c^2 = 0$, or, after a straight computation, to $AB = AC$.

*Third solution.* By using scalar products of vectors. Because $M$ is the midpoint of $BC$ and $E$ the midpoint of $MD$, we have

$$\overrightarrow{DM} = \frac{1}{2}(\overrightarrow{DB} + \overrightarrow{DC}), \text{ or, equivalently } \overrightarrow{BD} = 2\overrightarrow{MD} + \overrightarrow{DC},$$

and

$$\overrightarrow{AE} = \overrightarrow{AM} + \overrightarrow{ME} = \frac{1}{2}(\overrightarrow{AM} + \overrightarrow{AD}).$$

Because $MD$ and $AC$ are perpendicular, we have

$$\overrightarrow{AD} \cdot \overrightarrow{MD} = \overrightarrow{ME} \cdot \overrightarrow{DC} = 0.$$

Therefore

$$\begin{aligned}\overrightarrow{AE} \cdot \overrightarrow{BD} &= 2\overrightarrow{AE} \cdot \overrightarrow{MD} + \overrightarrow{AE} \cdot \overrightarrow{DC} \\ &= (\overrightarrow{AM} + \overrightarrow{AD}) \cdot \overrightarrow{MD} + (\overrightarrow{AM} + \overrightarrow{ME}) \cdot \overrightarrow{DC} \\ &= \overrightarrow{AM} \cdot \overrightarrow{MD} + \overrightarrow{AM} \cdot \overrightarrow{DC} = \overrightarrow{AM} \cdot \overrightarrow{MC} \\ &= \tfrac{1}{2}\overrightarrow{AM} \cdot \overrightarrow{BC}.\end{aligned}$$

Hence, $AE$ and $BD$ are perpendicular if and only if the median $AM$ of the triangle $ABC$ is an altitude. This is equivalent to $AB = AC$.

# Chapter 6

# Solutions to the selection tests for the Gulf Mathematical Olympiad 2013

## 6.1 Solutions to the selection test of day I

- **Problem 1.** First, we see that it is possible for Tarik to choose the 101 numbers 11,12,...,111, since the product $11 \times 12 > 111$.

  Assume that Tarik has choosen $k$ numbers and let $d$ be the smallest among these numbers. If $d \geq 11$, then clearly, $k \leq 101$.

  If $2 \leq d \leq 6$, from each of the 9 sets $\{9, 9d\}; \{10, 10d\}; \ldots; \{17, 17d\}$, Tarik can choose at most one number. Because $9d > 17$, these sets are pairwise disjoint. Because $17d \leq 102$, there are at least 9 numbers between 9 and 102 that Tarik could not choose. Therefore $k \leq 101$.

  If $3 \leq d \leq 10$, from each of the sets $\{d+1, d(d+1)\}$; $\{d+2, d(d+2)\}$; $\ldots$; $\{11, 11d\}$, Tarik can choose at most one element. Because $d(d+1) \geq 12$, these sets are pairwise disjoint. Because $11d < 111$, There are at least $11-d$ numbers from these sets that Tarik could not choose. But Tarik didn't choose the numbers $2, \ldots, d-1$. Therefore, Tarik didn't choose at least $11-d+d-2 = 9$ numbers. Hence $k \leq 101$.

Therefore, the maximum number of numbers Tarik can choose is 101.

- **Problem 2.** We have

$$\begin{aligned}
&\frac{a^3}{a^2+ab+b^2}+\frac{b^3}{b^2+bc+c^2}+\frac{c^3}{c^2+ca+a^2}\\
&\qquad=\frac{a^4}{a^3+a^2b+ab^2}+\frac{b^4}{b^3+b^2c+bc^2}+\frac{c^4}{c^3+c^2a+ca^2}\\
&\qquad\geq\frac{(a^2+b^2+c^2)^2}{a^3+ab^2+ac^2+ba^2+b^3+bc^2+ca^2+cb^2+c^3},
\end{aligned}$$

by applying Cauchy-Schwarz inequality.

On the other hand

$$\begin{aligned}
\frac{(a^2+b^2+c^2)^2}{a^3+ab^2+ac^2+ba^2+b^3+bc^2+ca^2+cb^2+c^3} &= \frac{a^2+b^2+c^2}{a+b+c}\\
&\geq \frac{a+b+c}{3},
\end{aligned}$$

by applying again Cauchy-Schwarz inequality.

The equality holds when $a=b=c$.

- **Problem 3.** Because all angles are congruent and all line segments are congruent, the regular n-pointed star is cyclic. Indeed, consider any four consecutive vertices $P_i, P_{i+1}, P_{i+2}, P_{i+3}$, for some $i \in \{1,2,\ldots,n\}$. They form an isosceles trapezoid. So they are cocyclic. By induction, all vertices $P_1, P_2, \ldots, P_n$ are on the same circle. Let $O$ be the center of this circle.

Because all the line segments are congruent and the path turns counter-clockwise, there exists a positive real number $0° < \theta < 180°$ such that $\angle P_iOP_{i+1} = \theta$ for all $i = 1,2,\ldots,n$. Because $P_{n+1} = P_1$, there exists a positive integer $k$ such that $n\theta = 360k°$. But $\theta < 180°$. Therefore $k < \frac{n}{2}$. Because each of the $n$ line segments intersects at least one of the other line segments at a point other than an endpoint, $k > 1$. Because all the vertices $P_1, P_2, \ldots, P_n$, are different, the integers $k$ and $n$ have no common divisors.

Conversely, for a given circumradius, if $1 < k < \frac{n}{2}$ is an integer relatively prime to $n$, there exists a unique regular $n$-pointed star of angle $\theta = \frac{360k}{n}^{\circ}$. Hence, the number of non-similar regular $n$-pointed stars is the number of such integers $k$, that is $\frac{\phi(n)}{2} - 1$, where $\phi$ is the Euler totient function. Therefore, it remains to solve the equation $\phi(n) = 60$.

There are three cases. The first case is when there exists a prime $p$ divisor of $n$ such that $3 \times 5$ divides $p-1$. Here, the only possibilities are $p = 61$ or 31.

If $p = 61$, there are 2 solutions for $n$. Either $n = 61$ or $n = 2 \times 61 = 122$.

If $p = 31$, there are 3 solutions for $n$. Either $n = 3 \times 31 = 93$, or $n = 2 \times 3 \times 31 = 186$, or $n = 2^2 \times 31 = 124$.

The second case is when there exist two prime numbers $p$ and $q$ such that 3 divides $p - 1$ and 5 divides $q - 1$. In this case, the only possibility is $p = 7$ and $q = 11$. This gives only two solutions for $n$. Either $n = 7 \times 11 = 77$ or $n = 2 \times 7 \times 11 = 154$.

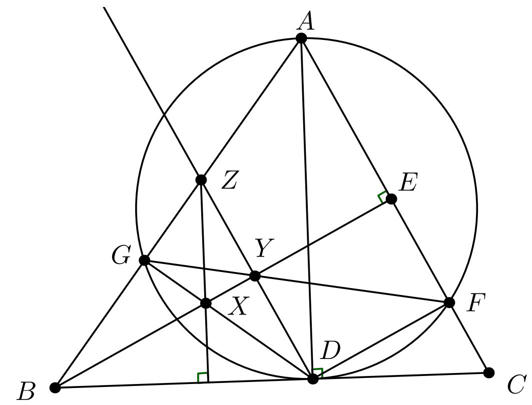
The third case is when there exists an odd prime number $p$ such that $p^2$ divides $n$. Then $p = 3$ or 5. If $n = 5^2 m$ and $m$ is relatively prime to 5, we obtain $\phi(m) = 3$ which is impossible since 3 is an odd number. If $n = 3^2 m$ and $m$ is relatively prime to 3, we obtain $\phi(m) = 10$. Therefore, $m = 11$ or 22. Hence $n = 99$ or 198.

In conclusion, all possible values for $n$ are 61,77,93,99,122,124,154,186 and 198.

- **Problem 4.** We present for this problem two solutions.

**First solution.** We have $\angle AFG = \angle ADG$ since $AGDF$ is cyclic. On the other hand $\angle DGA = \angle AFD = 90^\circ$, since $AD$ is a diameter. We deduce that triangles $AGD$ and $ADB$ are similar, and therefore $\angle AFG = \angle CBA$.

Because $\angle AEB = \angle ADB = 90^\circ$, quadrilateral $ABDE$ is cyclic. Therefore $\angle DEB = \angle CBA = \angle EFY$. But $\angle EFD = \angle YEF = 90^\circ$. We deduce that $DFEY$ is a rectangle and therefore $BY$ is an altitude in triangle $BDZ$. Line segment $DG$ is also an altitude in triangle $BDZ$ which intersects $BY$ at $X$. We deduce that $ZX$ and $BC$ are perpendicular.

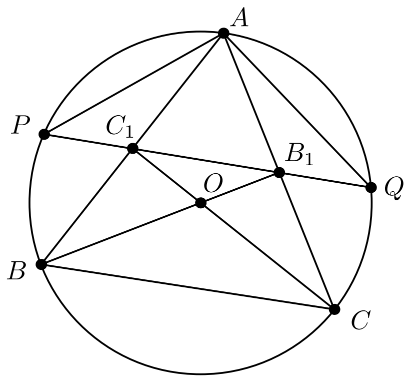



**Second solution.** Because $\angle ADB = \angle AEB = 90°$, the quadrilateral $ABDE$ is cyclic. Therefore, the projections of the point $D$ on the lines $AB, BE$, and $EA$ are collinear (Simson line). But $G$ and $F$ are the projections of $D$ on lines $AB$ and $EA$, respectively, since $AD$ is a diameter of the circle $\omega$. Then the projection of $D$ on $BE$ is $Y$, the intersection point of $BE$ with $GF$. Therefore, $BY$ is perpendicular to $DZ$. Hence, in triangle $BDZ$, line segments $BY$ and $DG$ are altitudes and intersect at $X$. Thus $ZX$ is also an altitude and therefore $ZX$ and $BC$ are perpendicular.

## 6.2 Solutions to the selection test of day II

- **Problem 1.** Assume $AP = AQ$. Because $OP = OQ$, the line $AO$ is perpendicular to $PQ$. But since $\angle C_1AO = 90° - \angle ACB$ then $\angle B_1C_1A = \angle ACB$, and therefore, quadrilateral $BCB_1C_1$ is cyclic.

  Using the power of the point $O$ with respect to the circumcircle of $BCB_1C_1$ we get
$$OB_1 = \frac{OB_1 \cdot OB}{R} = \frac{OC_1 \cdot OC}{R} = OC_1,$$
  where $R$ is the circumradius of triangle $ABC$. Therefore, $BB_1 = CC_1$, that is the minors $\widehat{BB_1}$ and $\widehat{CC_1}$ of the circumcircle of $BCB_1C_1$ have the same length. This is equivalent to saying that $\angle CBC_1 = \angle B_1CB$, and therefore $AB = AC$.




- **Problem 2.** Assume that there exist two non constant polynomials $g(X)$ and $h(X)$ with integer coefficients such that $f(X) = g(X)h(X)$. Because, $p = g(0)h(0)$ is prime, we can assume that $|g(0)| = 1$. Because the modulus of the product of the complex roots of $g(X)$ is equal to 1, at least one of these roots, say $\omega_0$, has modulus less than or equal to 1. But $f(\omega_0) = 0$. We deduce that

$$\begin{aligned} p &= |a_n\omega_0^n + a_{n-1}\omega_0^{n-1} + \cdots + a_1\omega_0| \\ &\leq |a_n| \cdot |\omega_0|^n + |a_{n-1}| \cdot |\omega_0|^{n-1} + \cdots + |a_1| \cdot |\omega_0| \\ &\leq |a_n| + |a_{n-1}| + \cdots + |a_1|, \end{aligned}$$

which is a contradiction. Therefore, $f(X)$ is irreducible.

- **Problem 3.** Let $p$ be a prime divisor of $n^{55} - n$ for all integer $n$. Whenever $n$ is not divisible with $p$, we have

$$n^{54} \equiv 1 \mod p.$$

In this case, the order of $n$ modulo $p$ divides 54. But there exists an integer $n$ of order $p-1$ modulo $p$. We deduce that $p-1$ divides 54. But the only primes $p$ such that $p-1$ divides 54 are $p = 2, 3, 7$ and 19. Conversely, all these four primes $2, 3, 7$ and 19 divide $n^{55}-n$ for all integer $n$ by Fermat's little theorem.

Notice that for $p(p^{55} - 1)$ is not divisible by $p^2$ for all prime numbers $p$ since $p^{54} - 1$ is relatively prime with $p$. Therefore, the greatest integer $k$ which divides $n^{55} - n$ for all integer $n$ is $2 \times 3 \times 7 \times 19 = 798$.

- **Problem 4.** Let $m$ be a positive integer and consider the infinite set of pairs $(F_k, F_{k+1})$, for $k \in \mathbb{N}$. By the pigeonhole principle, there exists a pair $(a, b)$ of integers $0 \le a, b \le m-1$ and an infinite sequence of integers $0 < k_1 < k_2 < \cdots$ such that

$$(F_{k_i}, F_{k_i+1}) \equiv (a, b) \mod m, \quad \text{for all} \quad i \ge 1.$$

Therefore

$$(F_{k_i-1}, F_{k_i}) = (F_{k_i+1} - F_{k_i}, F_{k_i}) \equiv (b-a, a) \mod m, \quad \text{for all} \quad i \ge 1.$$

We keep descending in this way until we get

$$(F_{k_i-k_1+2}, F_{k_i-k_1+3}) \equiv (F_2, F_3) \equiv (1, 2) \mod m, \quad \text{for all} \quad i \ge 1.$$

Let $n_i = k_i - k_1 + 2$, for all $i \ge 1$. Clearly, the infinite sequence $2 = n_1 < n_2 < \cdots$ is increasing and we have

$$F_{n_i} + 2 \equiv F_2 + 2 \equiv 3 \mod m, \qquad F_{n_i+1} + 1 \equiv F_3 + 1 \equiv 3 \mod m,$$

$$\text{and} \quad F_{n_i+2} \equiv F_{n_i+1} + F_{n_i} \equiv F_3 + F_2 \equiv 3 \mod m.$$

## 6.3 Solutions to the selection test of day III

- **Problem 1.** Let $f : \mathbb{R} \to \mathbb{R}$ be a function which satisfies the two functional equations.

  Because $f\left(\frac{\sqrt{3}}{3}x\right) - \sqrt{3}f(1) = -\frac{2\sqrt{3}}{3} \neq 0$ , either $f(1) \neq 0$ or $f\left(\frac{\sqrt{3}}{3}x\right) \neq 0$.

  Fix $x_0 \in \mathbb{R}$ with $f(x_0) \neq 0$ and let $y \in \mathbb{R}$ with $y \neq 0$. We have

  $$f(x_0)f\left(\frac{1}{y}\right) = f\left(\frac{x_0}{y}\right) + f(x_0y) = f(x_0)f(y).$$

  Therefore, $f\left(\frac{1}{y}\right) = f(y)$ for all $y \neq 0$.

Let $x \neq 0$. We have

$$\begin{aligned}
f\left(\frac{\sqrt{3}}{3}x\right) &= \sqrt{3}f(x) - \frac{2\sqrt{3}}{3}x \\
&= \sqrt{3}f\left(\frac{\frac{\sqrt{3}}{3}}{\frac{\sqrt{3}}{3}x}\right) - \frac{2\sqrt{3}}{3}x \\
&= \sqrt{3}\left(\sqrt{3}f\left(\frac{1}{\frac{\sqrt{3}}{3}x}\right) - \frac{2}{x}\right) - \frac{2\sqrt{3}}{3}x \\
&= 3f\left(\frac{\sqrt{3}}{3}x\right) - \frac{2\sqrt{3}}{x} - \frac{2\sqrt{3}}{3}x.
\end{aligned}$$

We deduce that

$$f\left(\frac{\sqrt{3}}{3}x\right) = \frac{\sqrt{3}}{x} + \frac{\sqrt{3}}{3}x,$$

and therefore

$$f(x) = x + \frac{1}{x},$$

for all $x \neq 0$.

For $x = 0$ and $y = 2$, we have from the second functional equation

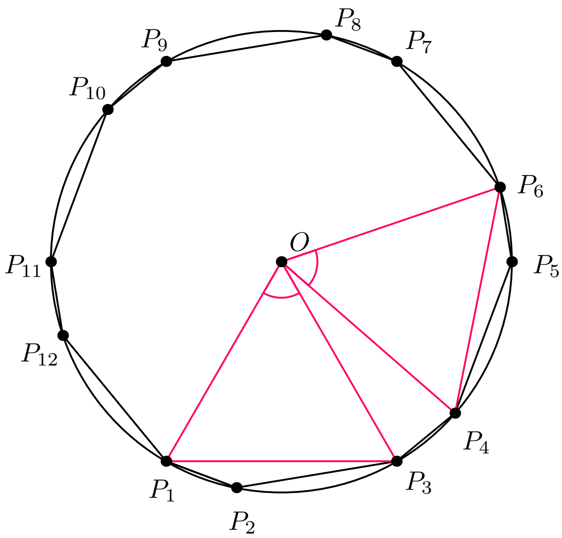
$$\frac{5}{2}f(0) = f(0) + f(0),$$

which implies that $f(0) = 0$. Hence

$$f(x) = \begin{cases} x + \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Conversely, it is easy to check that this function satisfies the two functional equations.

- **Problem 2.** Let $P_1P_2\cdots P_n$ be a convex cyclic non-regular polygon with the measures of all its internal angles are equal and let $O$ be its circumcenter. Because, all the angles are equal, all the arcs $\widehat{P_iP_{i+2}}$, for $i = 1, \ldots, n$, have the same length. Hence, $\angle P_iOP_{i+2} = \frac{4\pi}{n}$, for all $i = 1, \ldots, n$, since the polygon is convex.




Let $\theta = \angle P_1OP_2$. We have

$$\angle P_{2i-1}OP_{2i} = \angle P_1OP_2 + \angle P_2OP_{2i} - \angle P_1OP_{2i-1} = \angle P_1OP_2 = \theta.$$

If $n$ is an odd integer,

$$\theta = \angle P_nOP_1 = \frac{n+1}{2}\angle P_nOP2 = \frac{n+1}{2}\cdot\frac{4\pi}{n} = \frac{2\pi}{n} = \angle P_1OP_2 \mod 2\pi.$$

This means that the polygon is regular, which contradicts the hypothesis.

If $n$ is even, any value of $\theta$ with $0 < \theta < \frac{4\pi}{n}$ and $\theta \neq \frac{2\pi}{n}$ defines a unique non-regular such a polygon.

Therefore, the possible values of $n$ are all even positive integer $n \geq 4$.

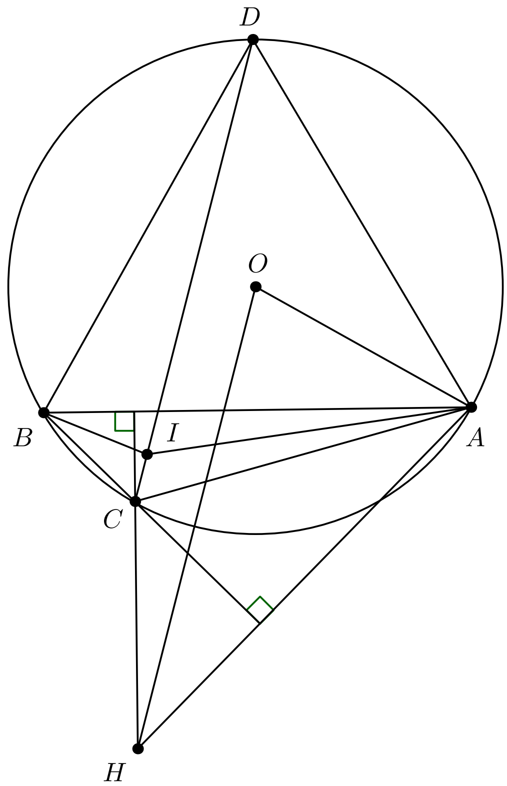
- **Problem 3.** We already know that $DA = DB = DI$. Because $DI = AB$, triangle $ADB$ is equilateral. But $\angle ACB + \angle BDA = 180°$, we deduce that $\angle ACB = 120°$.

  Because $\angle BOA = 2\angle BDA = 120°$, we deduce that $\angle OAB = 30°$. On the other hand, we have $\angle CAH = 90° - \angle CBA - \angle BAC = 30°$. By applying cosine law in the triangle $AOH$ we obtain

$$OH^2 = AH^2 + R^2 - 2R\cdot AH\cos(\angle BAC + 60°),$$

where $R$ is the circumradius of triangle $ABC$. This is equivalent to

$$2AH\cos(\angle BAC + 60^\circ) = R.$$




Because $HC$ is perpendicular to $AB$ and $BC$ is perpendicular to $AH$, we have $\angle AHC = \angle CBA$. On the other hand, we have $\angle HCA = 180^\circ - \angle AHC - 30^\circ = 90^\circ + \angle BAC$. We deduce, by applying sine law to the triangle $ACH$ that

$$\frac{AH}{\sin(90^\circ + \angle BAC)} = \frac{AC}{\sin \angle CBA} = 2R.$$

Therefore, $AH = 2R\cos \angle BAC$. By plugging this into our previous relation we obtain

$$2\cos \angle BAC \cdot \cos(\angle BAC + 60^\circ) = \frac{1}{2}.$$

This is equivalent to

$$\cos(2\angle BAC + 60^\circ) + \cos 60^\circ = \frac{1}{2},$$

and therefore, $\angle BAC = 15°$. Thus

$$\angle BAC = 15°, \quad \angle CBA = 45°, \quad \text{and } \angle ACB = 120°.$$

- **Problem 4.** We have

$$0 \equiv (a^3+1) - (b^3+1) \equiv (a-b)(a^2+ab+b^2) \equiv (a-b)ab \mod (a^2+b^2).$$

Let $d$ be a common divisor of $a$ and $a^2+b^2$. Then $d$ divides $a^3+1$ and $a^3$, so it divides 1. Hence $a$ and $a^2+b^2$ are coprime. In a similar way $b$ and $a^2+b^2$ are coprime. Thus $a-b \equiv 0 \mod (a^2+b^2)$.

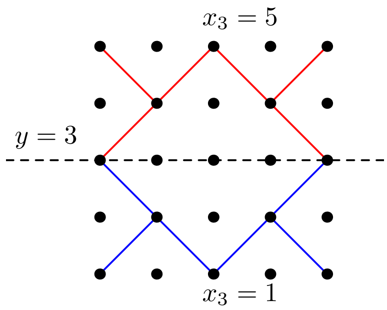
If $a \neq b$ then $a^2+b^2 \leq |a-b| \leq (a-b)^2 < a^2+b^2$, since $ab \geq 1$, which is a contradiction. Hence $a = b = 1$, since $a$ and $a^2+b^2$ are coprime.

# Chapter 7

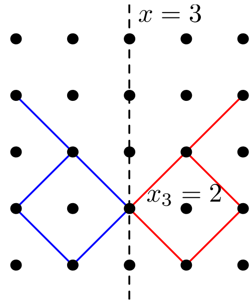
# Solutions to the selection tests for the Balkan Mathematical Olympiad 2013

## 7.1 Solutions to the selection test of day I

- **Problem 1.** Let us count the number of five-point sequences according to the position of point $P_3$.



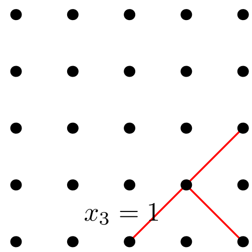

Because the horizontal line of equation $y = 3$ is a symmetry axis for this figure, the number of sequences with $x_3 = i$ is equal to the number of sequences with $x_3 = 6 - i$, for $i = 1, 2$.



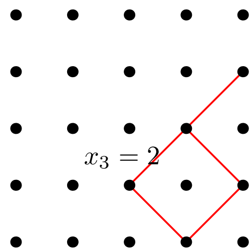

Similarly, the vertical line of equation $x = 3$ is a symmetry axis for this figure, so that the number $N_i$ of three-point sequences $(P_1, P_2, P_3)$ ending at $P_3$ with $x_3 = i$ is equal to the number of three-point sequences $(P_3, P_4, P_5)$ starting at $P_3$, with $x_3 = i$, for $i = 1, 2, 3$. Therefore, the number of five-point sequences $(P_1, P_2, P_3, P_4, P_5)$ with $x_3 = i$ is $N_i^2$. Hence, the total number of five-point sequences $(P_1, P_2, P_3, P_4, P_5)$ is

$$2(N_1^2 + N_2^2) + N_3^2.$$

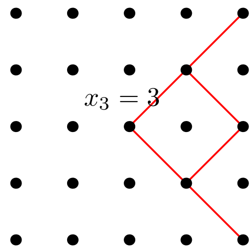
It is easy to see that $N_1 = 2$, since the only possibilites for $x_4, x_5$ are $x_4 = 2$ and $x_5 = 1$ or 3.




It is also easy to see that $N_2 = 3$, since $x_4 = 1$ or 3 and $x_5 = 2$ in the first case and 2 or 4 in the second case.



Finaly, $N_3 = 4$, since $x_4 = 2$ or 4, and $x_5 = 1$ or 3 in the first case, and 3 or 5 in the second case.




Hence, the number of sequences is $2\left(2^2 + 3^2\right) + 4^2 = 42$.

- **Problem 2.** Let $d = \gcd(a, b)$ and $a = da'$ and $b = db'$ with $a', b'$ two relatively prime positive integers. The equation becomes

$$a'b'd^2 + 63 = 20a'b'd + 12d.$$

Therefore, $d$ divides 63 and we have

$$a'b'd + \frac{63}{d} = 20a'b' + 12.$$

If $5 < d < 20$, then $a'b'd + \frac{63}{d} < 20a'b' + 12$, and this is impossible. Hence, $d = 1, 3, 21$, or 63.

1. If $d = 1$, the equation is equivalent to $51 = 19a'b'$, which is impossible since 19 does not divide 51.
2. If $d = 3$, the equation is equivalent to $9 = 17a'b'$, which is impossible since 17 does not divide 9.
3. If $d = 21$, the equation is equivalent to $a'b' = 9$. Because $a', b'$ are relatively prime positive integers, either $a' = 1, b' = 9$ or $a' = 9, b' = 1$. This leads to two solutions $(a, b) = (21, 189)$ and $(a, b) = (189, 21)$.
4. If $d = 63$, the equation is equivalent to $43a'b' = 11$, which is impossible since 43 does not divide 11.

Therefore, the equation has two solutions $(a, b) = (21, 189)$ and $(a, b) = (189, 21)$.

- **Problem 3.** Notice first that if $\lfloor x \rfloor \leq 0$ then

$$\lfloor x^2 \rfloor - 10\lfloor x \rfloor + 24 \geq 24 > 0.$$

This means that if $x$ is a solution to the equation then $\lfloor x \rfloor \geq 1$.

Let $x$ be a solution to the equation, $m = \lfloor x \rfloor \geq 1$, and $r = x - \lfloor x \rfloor \geq 0$. We have

$$0 = \lfloor x^2 \rfloor - 10\lfloor x \rfloor + 24 = \lfloor r(r+2m) \rfloor + (m-4)(m-6).$$

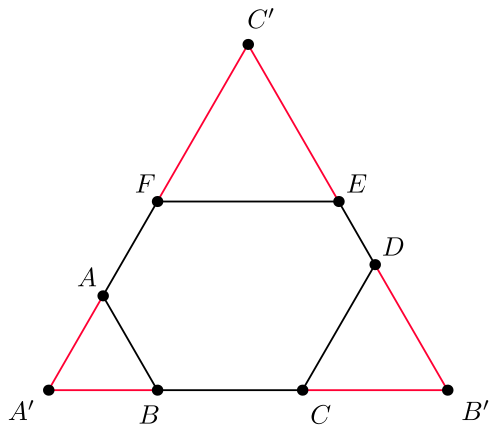
Because $r(r+2m) \geq 0$, we deduce that $(m-4)(m-6) \leq 0$, which is equivalent to $m = 4, 5$, or 6.

1. If $m = 4$, the equation becomes $\lfloor r(r+8) \rfloor = 0$. This is equivalent to $(r+4)^2 - 17 < 0$ and $(r+4)^2 - 16 \geq 0$, and its solutions are $0 \leq r < \sqrt{17} - 4$. This means that the solutions to the equation in this case are $4 \leq x < \sqrt{17}$.
2. If $m = 5$, the equation becomes $\lfloor r(r+10) \rfloor = 1$. This is equivalent to $(r+5)^2 - 27 < 0$ and $(r+5)^2 - 26 \geq 0$, and its solutions are $\sqrt{26} - 5 \leq r < \sqrt{27} - 5$. This means that the solutions to the equation in this case are $\sqrt{26} \leq x < \sqrt{27}$.
3. If $m = 6$, the equation becomes $\lfloor r(r+12) \rfloor = 0$. This is equivalent to $(r+6)^2 - 37 < 0$ and $(r+6)^2 - 36 \geq 0$, and its solutions are $0 \leq r < \sqrt{37} - 6$. This means that the solutions to the equation in this case are $6 \leq x < \sqrt{37}$.

Thus, the set of solutions to this equation is

$$\left[4, \sqrt{17}\right) \cup \left[\sqrt{26}, \sqrt{27}\right) \cup \left[6, \sqrt{37}\right).$$

- **Problem 4.** We extend sides $FA$ and $BC$ to intersect at $A'$, and sides $BC$ and $DE$ to intersect at $B'$, and sides $DE$ and $FA$ to intersect at $C'$.




Triangles $A'B'C'$, $A'BA$, $B'DC$, and $C'FE$ are equilateral with side lengths 11,3,4, and 5 respectively. Therefore, the area of hexagon $ABCDEF$ is

$$\frac{\sqrt{3}}{4}\cdot 11^2 - \frac{\sqrt{3}}{4}\cdot 3^2 - \frac{\sqrt{3}}{4}\cdot 4^2 - \frac{\sqrt{3}}{4}\cdot 5^2 = \frac{71\sqrt{3}}{4}.$$

- **Problem 5.** Notice first that

$$X^4 + 2X^3 + (2+2k)X^2 + (1+2k)X + 2k = (X^2 + X + 1)(X^2 + X + 2k).$$

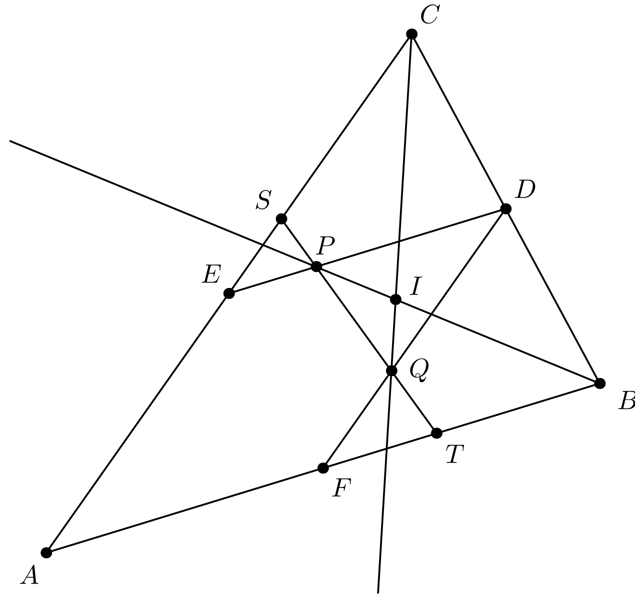
Because the factor $X^2 + X + 1$ has no real roots, we deduce from Vieta relations that $r_1 + r_2 = -1$ and $r_1 r_2 = 2k = -2013$, where $r_1, r_2$ are the real roots of the equation $X^4 + 2X^3 + (2+2k)X^2 + (1+2k)X + 2k = 0$. Therefore,

$$r_1^2 + r_2^2 = (r_1 + r_2)^2 - 2r_1 r_2 = 1 + 2 \times 2013 = 4027.$$

- **Problem 6.** Because sides of triangles $ABC$ and $DEF$ are parallel (homothetic triangles), we have

$$\begin{aligned} \angle BPD &= 180^\circ - \angle EDF - \angle FDB - \tfrac{1}{2}\angle CBA \\ &= 180^\circ - \angle BAC - \angle ACB - \tfrac{1}{2}\angle CBA \\ &= \tfrac{1}{2}\angle CBA = \angle DBP. \end{aligned}$$

We deduce that triangle $DPB$ is isosceles and therefore $DP = DB$.




Similarly, we prove that $DQ = DC$. But $DB = DC$. We conclude that triangle $DPQ$ is isosceles, and therefore, its bisector at $D$ is perpendicular to $PQ$.

But the bisector of triangle $ABC$ at $A$ is parallel to the bisector of triangle $DEF$ at $E$ since the two triangles are homothetic. We deduce that the bisector of triangle $ATS$ at $A$ is perpendicular to $ST$ and therefore $AS = AT$.

- **Problem 7.** Let us associate a 1 to each cell with a black color and a 0 to each cell with a white color. The condition is equivalent to the sum of numbers in each $2 \times 3$ rectangle and in each $3 \times 2$ rectangle is even. Let $a_{i,j}$ be this number at the cell in the $i^{th}$ row and $j^{th}$ column, for $1 \leq i, j \leq 50$.

Consider, for a fixed pair $i, j$, with $1 \leq i \leq 48$ and $1 \leq j \leq 47$, the $3 \times 4$ rectangle:

| | | | |
|---|---|---|---|
| $a_{i,j}$ | $a_{i,j+1}$ | $a_{i,j+2}$ | $a_{i,j+3}$ |
| $a_{i+1,j}$ | $a_{i+1,j+1}$ | $a_{i+1,j+2}$ | $a_{i+1,j+3}$ |
| $a_{i+2,j}$ | $a_{i+2,j+1}$ | $a_{i+2,j+2}$ | $a_{i+2,j+3}$ |

By applying the condition to the two $2\times 3$ rectangles which contain cells from the second and the third rows, we get

$$a_{i+1,j} + a_{i+1,j+1} + a_{i+1,j+2} + a_{i+2,j} + a_{i+2,j+1} + a_{i+2,j+2} \equiv 0 \mod 2,$$

and

$$a_{i+1,j+1} + a_{i+1,j+2} + a_{i+1,j+3} + a_{i+2,j+1} + a_{i+2,j+2} + a_{i+2,j+3} \equiv 0 \mod 2.$$

By applying the condition to all the $3 \times 2$ rectangles, we get

$$a_{i,j} + a_{i,j+1} + a_{i+1,j} + a_{i+1,j+1} + a_{i+2,j} + a_{i+2,j+1} \equiv 0 \mod 2,$$

$$a_{i,j+1} + a_{i,j+2} + a_{i+1,j+1} + a_{i+1,j+2} + a_{i+2,j+1} + a_{i+2,j+2} \equiv 0 \mod 2,$$

and

$$a_{i,j+2} + a_{i,j+3} + a_{i+1,j+2} + a_{i+1,j+3} + a_{i+2,j+2} + a_{i+2,j+3} \equiv 0 \mod 2.$$

By adding these 5 relations and cancelling all even numbers we get

$$a_{i,j} + a_{i,j+3} \equiv 0 \mod 2.$$

This proves that

$$a_{i,j+3} = a_{i,j}.$$

We prove in a similar way that

$$a_{i+3,j} = a_{i,j}.$$

Therefore, it is enough to know the numbers in the $3 \times 3$ rectangle

| | | |
|---|---|---|
| $a_{1,1}$ | $a_{1,2}$ | $a_{1,3}$ |
| $a_{2,1}$ | $a_{2,2}$ | $a_{2,3}$ |
| $a_{3,1}$ | $a_{3,2}$ | $a_{3,3}$ |

to deduce, by periodicity, the numbers in all the other cells.

Applying the condition to the first $2 \times 3$ rectangle, we will get

$$a_{2,3} \equiv a_{1,1} + a_{1,2} + a_{1,3} + a_{2,1} + a_{2,2} \mod 2$$

Applying the condition to the second $2 \times 3$ rectangle and to the second $3 \times 2$ rectangle and adding the two relations, we will get

$$a_{3,1} \equiv a_{2,1} + a_{1,2} + a_{1,3} \mod 2$$

We obtain in a similar way

$$a_{3,2} \equiv a_{2,2} + a_{1,3} + a_{1,1} \mod 2$$

$$a_{3,3} \equiv a_{2,3} + a_{1,1} + a_{1,2} \mod 2$$

Therefore, it is enough to know the 5 numbers

$$a_{1,1}, a_{1,2}, a_{1,3}, a_{2,1}, a_{2,2},$$

to deduce all the numbers in the cells of the $50 \times 50$ chessborad.

Conversely, choose a value in $\{0, 1\}$ for each number $a_{1,1}, a_{1,2}, a_{1,3}, a_{2,1}, a_{2,2}$, and deduce the values in all the other cells of the chessboard. It is easy to check that the condition on the two $2 \times 3$ rectangles and the two $3 \times 2$ rectangles in the $3 \times 3$ rectangle above is satisfied. We deduce, by periodicity, that it is satisfied in all the chessboard. Hence there are $2^5$ ways to color the chessboard.

- **Problem 8.** We prove by induction on $n \geq 2$ that

$$T_n = \frac{1^1 + 3^3 + 5^5 + \cdots + (2^n - 1)^{(2^n - 1)}}{2^n}$$

is an odd integer.

For $n = 2$, $T_2 = \frac{1^1+3^3}{2^2} = 7$ is an odd integer.

Assume that $T_n$ is an odd integer. We have

$$\begin{aligned} T_{n+1} &= \frac{1^1 + 3^3 + 5^5 + \cdots + \left(2^{n+1} - 1\right)^{\left(2^{n+1}-1\right)}}{2^{n+1}} \\ &= T_n + \frac{\sum_{k=1}^{2^{n-1}} \left((2^n + 2k - 1)^{2^n + 2k - 1} - (2k-1)^{2k-1}\right)}{2^{n+1}}. \end{aligned}$$

Therefore, it remains to prove that

$$\frac{\sum_{k=1}^{2^{n-1}} \left((2^n + 2k - 1)^{2^n + 2k - 1} - (2k-1)^{2k-1}\right)}{2^{n+1}}$$

is an even integer, that is

$$\sum_{k=1}^{2^{n-1}} \left((2^n + 2k - 1)^{2^n + 2k - 1} - (2k-1)^{2k-1}\right) \equiv 0 \mod 2^{n+2}.$$

To prove this, we will use the fact that if $l$ is an odd integer, then $l^{2^n} \equiv 1 \mod 2^{n+2}$. This is because each factor in

$$l^{2^n} - 1 = \left(l^{2^{n-1}} + 1\right)\left(l^{2^{n-2}} + 1\right)\cdots\left(l^2 + 1\right)(l+1)(l-1)$$

is even and $(l+1)(l-1)$ is divisible by 8.

Because $n \geq 2$, we have for $1 \leq k \leq 2^{n-1}$

$$\begin{aligned}
(2^n + 2k - 1)^{2^n + 2k - 1} &- (2k-1)^{2k-1} \\
&\equiv (2^n + 2k - 1)^{2k-1} - (2k-1)^{2k-1} \\
&\equiv \sum_{i=0}^{2k-1} \binom{2k-1}{i} 2^{in} (2k-1)^{2k-1-i} - (2k-1)^{2k-1} \\
&\equiv (2k-1) \cdot 2^n \cdot (2k-1)^{2k-2} \\
&\equiv 2^n (2k-1)^{2k-1} \qquad \mod 2^{n+2}.
\end{aligned}$$

Therefore,

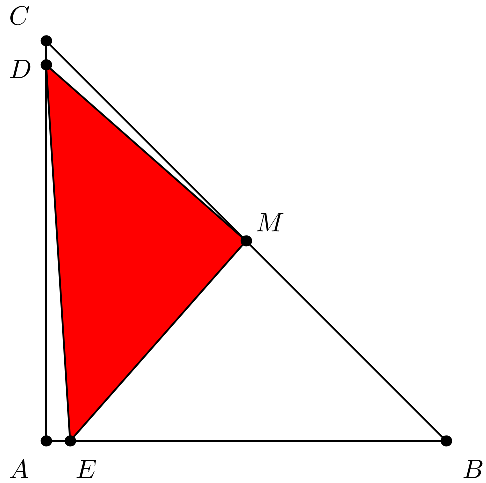
$$\begin{aligned}
&\sum_{k=1}^{2^{n-1}} \left((2^n + 2k - 1)^{2^n + 2k - 1} - (2k-1)^{2k-1}\right) \\
&\qquad \equiv 2^n \sum_{k=1}^{2^{n-1}} (2k-1)^{2k-1} \equiv 2^{2n} T_n \equiv 0 \qquad \mod 2^{n+2}.
\end{aligned}$$

## 7.2 Solutions to the selection test of day II

- **Problem 1.** Because $AEMD$ is cyclic, we have $\angle BEM = \angle ADM$. But $\angle MBE = \angle MAD = 45°$ and $BM = AM$. We deduce that triangles $BME$ and $AMD$ are congruent and therefore $ME = MD$.

  Because $AEMD$ is cyclic, $\angle DME = 90°$. Therefore, the area of triangle $EMD$ is
  $$2 = \frac{1}{2} ME \cdot MD = \frac{1}{2} MD^2,$$

and hence, $MD = ME = 2$.




We have from the cosine law in triangle $AMD$

$$4 = MD^2 = AD^2 + AM^2 - 2AD \cdot AM \cos 45^\circ = AD^2 + \frac{9}{2} - 3AD.$$

Because $AE$ satisfies the same equation as $AD$ above, and $AE < AD$, we deduce that $AD = \frac{3+\sqrt{7}}{2}$ and therefore

$$CD = \frac{3-\sqrt{7}}{2}.$$

- **Problem 2.** Plug in $y = 0$. The functionnal equation becomes

$$f(f(f(x))) = x + f(0),$$

for all $x \in \mathbb{R}$. Since the map $x \mapsto x + f(0)$ is bijective, then so is $f$.

Plug in $y = -x$. The functionnal equation becomes

$$f(f(f(x) - x) - x) = f(-x),$$

for all $x \in \mathbb{R}$. By injectivity of $f$, we can cancel $f$ in both sides and obtain

$$f(f(x) - x) = 0,$$

for all $x \in \mathbb{R}$. By surjectivity of $f$ there exists a real number $a$ such that $f(a) = 0$. Again by cancelling $f$ from both sides we obtain

$$f(x) = x + a,$$

for all $x \in \mathbb{R}$. But $0 = f(a) = 2a$. We deduce that

$$f(x) = x,$$

for all $x \in \mathbb{R}$.

Conversely, we check easily that this function is a solution to the problem.

- **Problem 3.** Let us consider the two possible cases for the integer $x$.

If $x$ is an odd positive integer, we have

$$z^2 \equiv 2^x + 21^y \equiv 2 + 0 \equiv 2 \mod 3.$$

This is impossible since a perfect square is congruent to either 0 or 1 modulo 3.

If $x$ is an even positive integer, write $x = 2x_0$ for a positive integer $x_0$. We have

$$21^y = z^2 - 2^x = (z - 2^{x_0})(z + 2^{x_0}).$$

Assume that one of the two primes 3 or 7 divides both factors $z - 2^{x_0}$ and $z + 2^{x_0}$. This implies that this prime divides their sum $2z$ and therefore $z$. But $2^x = 21^y - z^2$ is neither divisible by 3 nor by 7. This proves that the factors $z - 2^{x_0}$ and $z + 2^{x_0}$ are relatively prime. Because $z - 2^{x_0} < z + 2^{x_0}$, we have two possibilities

a. First possibility is when $z - 2^{x_0} = 1$ and $z + 2^{x_0} = 21^y$. In this case $2^{x_0+1} + 1 = 21^y$. Because $y$ is positive, we have

$$2^{x_0+1} \equiv 6 \mod 7.$$

But this is impossible since $2^{x_0+1}$ is congruent to either $1, 2$ or $4$ modulo 7.

b. Second possibility is when $z - 2^{x_0} = 3^y$ and $z + 2^{x_0} = 7^y$. This is equivalent to $z = 2^{x_0} + 3^y$ and $2^{x_0+1} = 7^y - 3^y$. If $y = 1$, then $x = 2$ and $z = 5$ gives a solution for the problem. If $y \geq 2$, because $7^y - 3^y \equiv 0$

mod 8 only for even number, $y$ must be even. Write $y = 2y_0$ for a positive integer $y_0$. We have $2^{x_0+1} = (7^{y_0} - 3^{y_0})(7^{y_0} + 3^{y_0})$. But $7^{y_0} + 3^{y_0}$ is greater than 4 and is congruent to 2 modulo 4. This means that it is not a power of 2, which is a contradiction.

Hence, the only solution is $(x, y, z) = (2, 1, 5)$.

- **Problem 4.** Notice first that there are no 3 consecutive students with hats of the same color.

If there are two consecutive students with hats of the same color, the next two consecutive students with hats of the same color must be different from the first color. Moreover, the number of students standing between this two pairs of consecutive students with hats of the same colors, must be even since they alternate between the colors of their hats.

We conclude from this that there are three possible cases:

1. First case, when there are no consecutive students with hats of the same color. In this case, the students must alternate the color of their hats and there are precisely 2 possible ways depending on the color of the hat of the first student.
2. Second case, when the position of the first student followed by another student with hat of the same color is odd. In this case, each nonempty subset of the set of pairs $\{(1, 2); (3, 4); (5, 6); (7, 8); (9, 10)\}$ determine the positions of pairs of students with hat of the same color. Because the color of hats of the first pair in this subset determines the color of hats of all the other students, there are $2 \cdot (2^5 - 1)$ ways in which the teacher can place the hats.
3. Third case, when the position of the first student followed by another student with hat of the same color is even. In this case, each nonempty subset of the set of pairs $\{(2, 3); (4, 5); (6, 7); (8, 9)\}$ determine the positions of pairs of students with hat of the same color. Because the color of hats of the first pair in this subset determines the color of hats of all the other students, there are $2 \cdot (2^4 - 1)$ ways in which the teacher can place the hats.

Therefore, the number of ways in which the teacher can place hats is

$$2 + 2 \cdot (2^5 - 1) + 2 \cdot (2^4 - 1) = 94.$$

- **Problem 5.** We prove this by induction on $n$.

For $n = 1, 2, 3$, the numbers 1,34, and 122 are good numbers and all their odd digits are less than 5.

Assume that $a_n$ is an $n$-digit good number and all its odd digits are less than 5.

If $a_n$ has an even digit $2m$, remove this digit and replace it by 4 digits $mmmm$ to obtain an $n+3$-digit number. The sum of squares of the digits of this new number is equal to the sum of squares of the digits of $a_n$ and is therefore a perfect square. Notice that all the odd digits of this new $n+3$-digit good number are less than 5. For example, replace the 4 in the 2-digit good number 34 by 2222 to obtain the 5-digit good number 32222.

If $a_n$ has no even digit, all its digits are less than 5. We can multiply all its digits by 2 to obtain a new $n$-digit good number. Choose one of the new even digits $2m$ remove it and replace it by four digits $mmmm$ to obtain a new $n+3$-digit good number and its odd digits $m$ are less than 5. For example, for the 4-digit good number 3333 multiply all its digit by 2 to obtain the 4-digit good number 6666 and then replace one of the 6 by 3333 to obtain the 7-digit good number 3333666.

This proves by induction that for any positive integer $n$, there exists an $n$-digit good number.

- **Problem 6.** We present for this problem two solutions, one using classical inequalities and the second using geometry.

*First solution.* Notice that, by using AM-GM inequality, we have

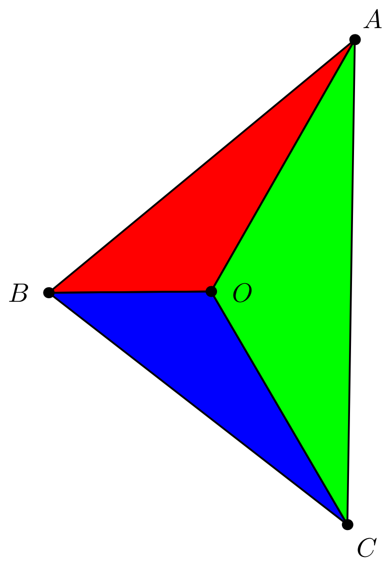
$$a\sqrt{b^2+c^2+bc} = a\sqrt{(b+c)^2-bc} \geq a\sqrt{(b+c)^2-\left(\frac{b+c}{2}\right)^2} = \frac{\sqrt{3}}{2}a(b+c).$$

Applying the same inequality for the two other terms, we deduce that

$$\begin{aligned} a\sqrt{b^2+c^2+bc} \ &+ \ b\sqrt{c^2+a^2+ca} \ + \ c\sqrt{a^2+b^2+ab} \\ &\geq \frac{\sqrt{3}}{2}a(b+c)+\frac{\sqrt{3}}{2}b(c+a)+\frac{\sqrt{3}}{2}c(a+b) \\ &\geq \sqrt{3}(ab+bc+ca) \\ &\geq \sqrt{3}. \end{aligned}$$

The equality holds when $a = b = c = \frac{\sqrt{3}}{3}$.

*Second solution.* Consider four points, $O, A, B$, and $C$, in the plane such that $OA = a, OB = b$, and $OC = c$, and $\angle AOB = \angle BOC = \angle COA = 120°$.




The area of triangle $ABC$ is

$$[ABC] = \frac{1}{2}(OA\cdot OB+OB\cdot OC+OC\cdot OA)\sin 120° = \frac{\sqrt{3}}{4}(ab+bc+ca) = \frac{\sqrt{3}}{4}.$$

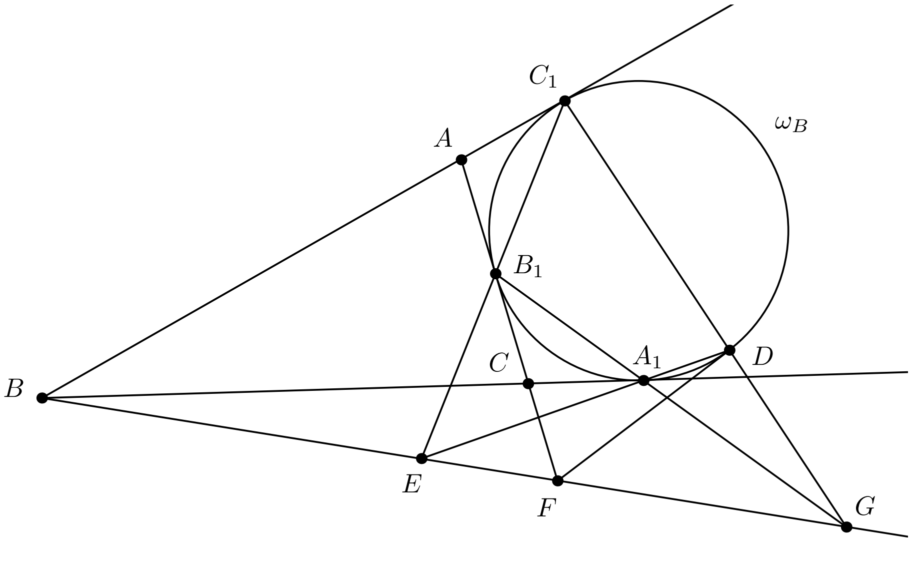
On the other hand, we have

$$a\sqrt{b^2+c^2+bc} = OA \cdot BC \geq 2([AOB]+[COA]).$$

Applying the same inequality for the two other terms, we obtain

$$a\sqrt{b^2+c^2+bc} + \ b\sqrt{c^2+a^2+ca} \ + \ c\sqrt{a^2+b^2+ab}$$
$$\geq 2([AOB]+[COA]) + 2([BOC]+[AOB]) + 2([COA]+[BOC])$$
$$\geq 4[ABC]$$
$$\geq \sqrt{3}.$$

- **Problem 7.** Let $F'$ be the intersection point of the tangents to circle $\omega_B$ at $B_1$ and $D$.




Let $G$ be the intersection point of lines $B_1A_1$ and $DC_1$.

Consider the six cyclic points $C_1, C_1, B_1, A_1, A_1, D$. Because the tangent lines to $\omega_B$ at $C_1$ and $A_1$ intersect at $B$, lines $C_1B_1$ and $A_1D$ intersect at $E$, and lines $B_1A_1$ and $DC_1$ intersect at $G$, from Pascal theorem, the three points $B, E$, and $G$ are collinear.

Consider the six cyclic points $C_1, B_1, B_1, A_1, D, D$. Because lines $C_1B_1$ and $A_1D$ intersect at $E$, the tangent lines to $\omega_B$ at $B_1$ and $D$ intersect at $F'$, and lines $B_1A_1$ and $DC_1$ intersect at $G$, from Pascal theorem, the three points $E, F'$, and $G$ are collinear.

We deduce that $B, E$, and $F'$ are collinear and therefore $F' = F$. This proves that $FD$ is tangent to $\omega_B$ at $D$.

- **Problem 8.** Let $B$ be the number of three-member subsets such that one of the members uses the same language to talk to the two other members and the two other members talk to each other using a different language. Because there are no three-members who use the same language to talk to each other, $A+B$ is the number of all three-member subsets, that is $\binom{101}{3}$. To maximize $A$, is equivalent to minimize $B$.

  Let $X$ be a member, and $x_1, x_2, \ldots, x_{50}$ the number of members with whom $X$ talks in the 50 different languages $l_1, l_2, \ldots, l_{50}$, respectively. We have

  $$x_1 + x_2 + \cdots + x_{50} = 100.$$

  There are

  $$B_X = \binom{x_1}{2} + \binom{x_2}{2} + \cdots + \binom{x_{50}}{2}$$

  three-member subsets each containing $X$ with two other members to whom $X$ talks using the same language. We have, from Cauchy-Schwarz inequality,

  $$B_X = \sum_{i=1}^{50} \frac{x_i(x_i-1)}{2} = \frac{1}{2}\sum_{i=1}^{50} x_i^2 - 50 \geq \frac{\left(\sum_{i=1}^{50} x_i\right)^2}{2 \times 50} - 50 = 50.$$

  Therefore,

  $$B \geq 101 \times 50,$$

  and hence,

  $$A \leq \binom{101}{3} - 101 \times 50 = 161600.$$

  The equality is satisfied when each member $X$ uses each language to talk exactly with two other members, that is

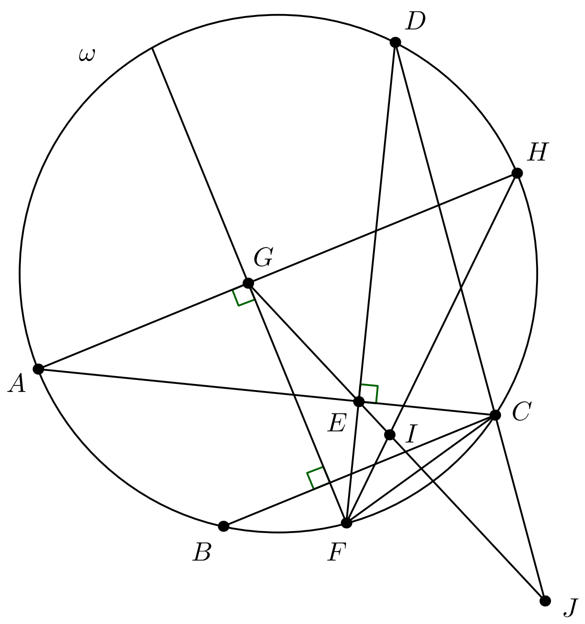
  $$x_1 = x_2 = \cdots = x_{50} = 2.$$

  Assume that $X_1, X_2, \ldots, X_{101}$ are the 101 members, and each member $X_i$, for $i = 1, \ldots, 101$, talks with the members $X_{i+j}$ and $X_{i-j}$ using the language

$l_j$, for $j = 1, \ldots, 50$ (here, the indices are modulo 101). Because $50 = \frac{101-1}{2}$, each member will use each language to talk exactly with two other members and he will not use two different languages to talk to same members. This proves that the maximum possible value of $A$ is 161600.

## 7.3 Solutions to the selection test of day III

- **Problem 1.** To prove that $C, F, I$, and $J$ are concyclic, it is equivalent to prove that $\angle CFI = \angle CJI$ or $\angle CFI + \angle CJI = 180°$, depending on the configuration. We will present here the proof for one configuration. The proof for the other configuration is similar.




Because $AFCH$ is cyclic, we have $\angle CFI = \angle CAH$.

Because $\angle FEA = \angle FGA = 90°$, the quadrilateral $AFEG$ is cyclic, and therefore $\angle CAH = \angle EFG$.

It remains to prove that $\angle EFG = \angle DJG$, which is equivalent to proving that quadrilateral $DGFJ$ is cyclic.

But $\angle EGF = \angle EAF$ since $AFEG$ is cyclic. On the other hand, because $AFCD$ is cyclic, we deduce that $\angle EAF = \angle CDF$. Therefore, $\angle EGF = \angle CDF$, which proves that $DGFJ$ is cyclic.

- **Problem 2.** Let $(a, b, c, n)$ be a quadruplet of positive integers with $a < b < c$ such that each of $a, b, c, a+n, b+n, c+2n$ is a term of the Fibonacci sequence, and let

$$b + n = F_k,$$

for some positive integer $k$. Because $b < b + n$ and $a + n < b + n$, we have $\max\{b, a + n\} \le F_{k-1}$.

Assume $\min\{b, a+n\} \le F_{k-2}$. We have $a+n+b \le F_{k-1}+F_{k-2} = F_k = b+n$, which is impossible since $a > 0$. Therefore,

$$a + n = b = F_{k-1},$$

$$n = (b + n) - b = F_k - F_{k-1} = F_{k-2},$$

and

$$a = (a + n) - n = F_{k-1} - F_{k-2} = F_{k-3}.$$

Let $c + 2n = F_m$. We have $F_k \le c \le F_{m-1}$ and therefore $F_{m-2} \le 2n = 2F_{k-2} \le F_k$.

If $F_{m-2} = F_k$ then $F_{k-2} = F_{k-1} = 1$ and $a = F_{k-3} = 0$ which is impossible. Therefore

$$c = b + n = F_{m-1} = F_k,$$

and

$$c + 2n = F_m = F_{k+1}.$$

But

$$2n = (c + 2n) - c = F_{k+1} - F_k = F_{k-1} = a + n.$$

We deduce that

$$F_{k-3} = a = n = F_{k-2} = 1.$$

Hence, $k = 4$, $(a, b, c, n) = (1, 2, 3, 1)$ and we check easily that

$$a = F_1, b = a + n = F_3, c = b + n = F_4 \quad \text{and} \quad c + 2n = F_5.$$

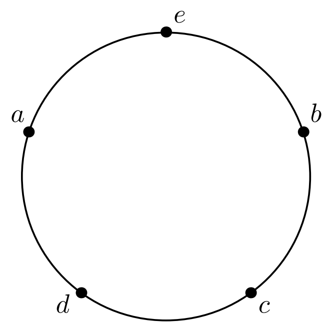
- **Problem 3.** Assume, without loss of generality, that

$$0 \leq a \leq b \leq c \leq d \leq e.$$

Because

$$ea \leq \max\{eb, cd\} \leq ec \leq ed,$$

to get the smallest possible maximum, the best arrangment arround the circle is




In this case, the maximum is given by

$$\max\{eb, cd\}.$$

We have

$$eb \leq e\frac{b+c+d}{3} \leq \frac{e(1-e)}{3} \leq \frac{1}{12},$$

and the equality holds when $a = 0, b = c = d = \frac{1}{6}$, and $e = \frac{1}{2}$.

We have, on the other hand,

$$cd = \sqrt[3]{c^2d^2(cd)} \leq \left(\sqrt[3]{cde}\right)^2 \leq \left(\frac{c+d+e}{3}\right)^2 \leq \frac{1}{9},$$

and the equality holds when $a = b = 0$, and $c = d = e = \frac{1}{3}$.

Therefore, the minimum possible value of $T$ is $\frac{1}{9}$.

- **Problem 4.** Let $n$ be a nonnegative integer. We have

$$f(2n)^2 < f(2n)^2+6f(n)+1 = f(2n+1)^2 < f(2n)^2+6f(2n)+9 = (f(2n)+3)^2.$$

Therefore,

$$f(2n) < f(2n+1) < f(2n) + 3.$$

Assume that $f(2n+1) = f(2n) + 2$. In this case

$$6f(n) + 1 = f(2n+1)^2 - f(2n)^2 = 4f(2n) + 4.$$

This is impossible since the left hand side is odd while the right hand side is even. Therefore $f(2n+1) = f(2n) + 1$.

On the other hand,

$$6f(n) + 1 = f(2n+1)^2 - f(2n)^2 = 2f(2n) + 1.$$

We deduce that $f(2n) = 3f(n)$, and $f(0) = 0$.

Now, let $n \geq 0$ and write $n = \overline{a_1 a_2 \cdots a_k}_{(2)}$ in basis 2. We prove by induction on $k$ that $f(n) = \overline{a_1 a_2 \cdots a_k}_{(3)}$ in basis 3.

For $k = 1$, we have

$$f(\overline{0}_{(2)}) = f(0) = 0 = \overline{0}_{(3)} \quad \text{and} \quad f(\overline{1}_{(2)}) = f(1) = f(0) + 1 = 1 = \overline{1}_{(3)}.$$

Assume this true for $k$. We have

$$\begin{aligned}
f(\overline{a_1 a_2 \cdots a_k a_{k+1}}_{(2)}) &= f(2\overline{a_1 a_2 \cdots a_k}_{(2)} + a_{k+1}) \\
&= f(2\overline{a_1 a_2 \cdots a_k}_{(2)}) + a_{k+1} \\
&= 3f(\overline{a_1 a_2 \cdots a_k}_{(2)}) + a_{k+1} \\
&= 3\overline{a_1 a_2 \cdots a_k}_{(3)} + a_{k+1} \\
&= \overline{a_1 a_2 \cdots a_k a_{k+1}}_{(3)}.
\end{aligned}$$

This completes the induction.

Applying this to 1000, we have

$$1000 = \overline{1101001}_{(3)} = f\left(\overline{1101001}_{(2)}\right) = f(105).$$

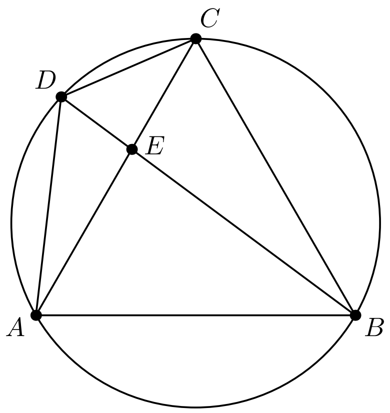
## 7.4 Solutions to the selection test of day IV

- **Problem 1.** Applying Ptolemy relation to the cyclic quadrilateral $ABCD$, we get

$$AB \cdot CD + BC \cdot DA = AC \cdot BD,$$

which simplifies to

$$CD + DA = 25.$$




Let $a = AB = BC = CA$ and $x = AE$.

We have, from similarity of triangles $AED$ and $BEC$, that

$$\frac{DA}{a} = \frac{x}{19}.$$

We have, from similarity of triangles $CED$ and $BEA$, that

$$\frac{CD}{a} = \frac{a-x}{19}.$$

We deduce that

$$\frac{a}{19} = \frac{x}{19} + \frac{a-x}{19} = \frac{DA}{a} + \frac{CD}{a} = \frac{25}{a},$$

and therefore $a = 5\sqrt{19}$.

Applying sine law on the circumcircle of $ABCD$, we obtain

$$\frac{\sin\angle BAD}{25} = \frac{\sin\angle BAC}{5\sqrt{19}} = \frac{\sqrt{57}}{190}.$$

Hence $\sin\angle BAD = \frac{5\sqrt{57}}{38}$, that is

$$\cos\angle BAD = \pm\frac{\sqrt{38^2 - 25\times 57}}{38} = \pm\frac{\sqrt{19}}{38}.$$

Applying cosine law to triangle $ABD$, we obtain

$$25^2 = 25\times 19 + AD^2 \pm 2\times 5\sqrt{19}\times AD\times\frac{\sqrt{19}}{38},$$

and therefore, $AD = 10$ or $15$.

- **Problem 2.** We have $n = c + 7b + 49a = a + 9b + 81c$. This implies that $20(2c - a) = (4a - b)$. Because $0 \le a, b, c \le 6$, either $4a - b = 2c - a = 0$ or $4a - b = 20$ and $2c - a = 1$.

  1. If $4a - b = 2c - a = 0$, then $b = 8c \le 6$ and therefore $a = b = c = 0$ , that is $n = 0$.
  2. If $4a - b = 20$ and $2c - a = 1$, then $a = 5$, or $6$. If $a = 5$ then $b = 0$ and $c = 3$ and $n = 248$. If $a = 6$, then $2c = 7$, which is impossible.

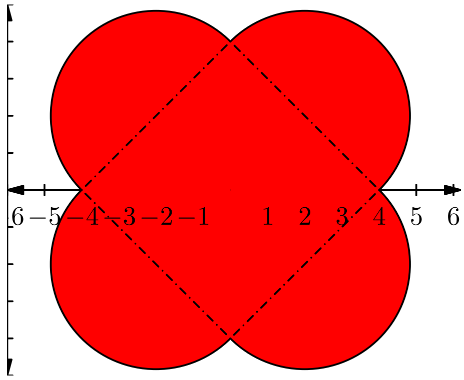
  Therefore, $n = 0$ or $248$.

- **Problem 3.** Notice that, if $(x, y)$ is a solution of the inequality, then $(-x, y), (x, -y)$, and $(-x, -y)$ are all solutions of the inequality. Therfore, we can assume $x, y \ge 0$ and deduce the other solutions by symmetries with respect to $x-$axis and $y-$axis.

  Assume $x, y \ge 0$. The inequality is equivalent to

$$(x - 2)^2 + (y - 2)^2 \le 8.$$

  The points whose coordinates satisfy this inequality are precisely the points in the intersection of the first quadrant with the disk of center $(2, 2)$ and radius $\sqrt{8}$. By applying the above symmetries we obtain the following surface of points whose coordinates satisfy the inequality.




Its area, is the area of a square of side length $4\sqrt{2}$ and four half disks of radius $2\sqrt{2}$, that is $32 + 16\pi$.

- **Problem 4.** Because $589 = 19 \times 31$, we will find all positive integers $n < 589$ such that both 19 and 31 divide $n^2 + n + 1$.

Let $n$ be such an integer. We have

$$0 \equiv n^2 + n + 1 \equiv n^2 + 20n + 1 \equiv (n + 10)^2 - 2^2 \equiv (n + 8)(n + 12) \mod 19.$$

Hence, 19 divides $n^2 + n + 1$ if and only if $n \equiv 7$ or $11 \mod 19$.

On the other hand, we have

$$0 \equiv n^2 + n + 1 \equiv n^2 + 32n + 1 \equiv (n + 16)^2 - 10^2 \equiv (n + 26)(n + 6) \mod 31.$$

Hence, 31 divides $n^2 + n + 1$ if and only if $n \equiv 5$ or $25 \mod 31$.

Using the fact that $8 \times 31 \equiv 1 \mod 19$ and $18 \times 19 \equiv 1 \mod 31$, we consider the following four cases:

1. If $n \equiv 7 \mod 19$ and $n \equiv 5 \mod 31$. There exists an integer $k$ such that

$$n = 7 \times 8 \times 31 + 5 \times 18 \times 19 + k \times 19 \times 31 = 501 + 589(k + 5).$$

But $0 < n < 589$. We deduce that $n = 501$.

2. If $n \equiv 7 \mod 19$ and $n \equiv 25 \mod 31$. There exists an integer $k$ such that

$$n = 7 \times 8 \times 31 + 25 \times 18 \times 19 + k \times 19 \times 31 = 273 + 589(k + 17).$$

But $0 < n < 589$. We deduce that $n = 273$.

3. If $n \equiv 11 \mod 19$ and $n \equiv 5 \mod 31$. There exists an integer $k$ such that

$$n = 11 \times 8 \times 31 + 5 \times 18 \times 19 + k \times 19 \times 31 = 315 + 589(k + 7).$$

But $0 < n < 589$. We deduce that $n = 315$.

4. If $n \equiv 11 \mod 19$ and $n \equiv 25 \mod 31$. There exists an integer $k$ such that

$$n = 11 \times 8 \times 31 + 25 \times 18 \times 19 + k \times 19 \times 31 = 87 + 589(k + 19).$$

But $0 < n < 589$. We deduce that $n = 87$.

Therefore, the possible values for $n$ are 87,273,315, and 501.

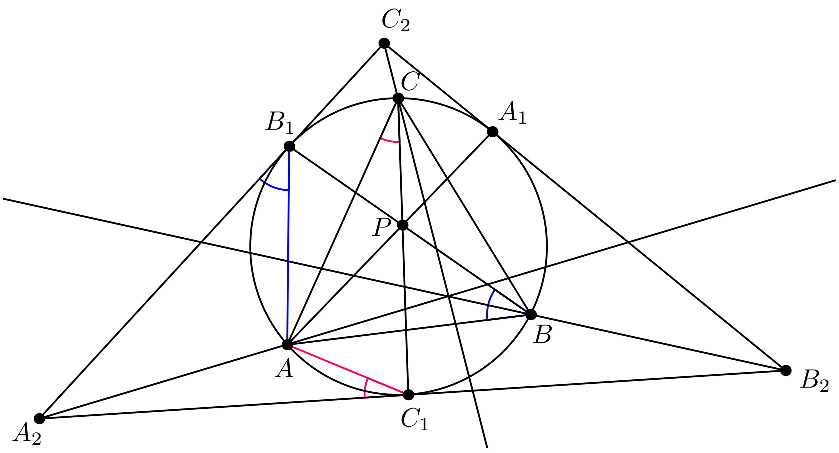
# Chapter 8

# Solutions to the selection tests for the International Mathematical Olympiad 2013

## 8.1 Solutions to the selection test of day I

- **Problem 1.** Let $R$ be the circumradius of triangle $ABC$. We have, using sine law,

$$C_1A = 2R\sin\angle ACP \qquad \text{and} \qquad AB_1 = 2R\sin\angle PBA.$$




On the other hand, because lines $A_2C_1$ and $A_2B_1$ are tangent to $\omega$, we have

$$\angle AC_1A_2 = \angle ACP \qquad \text{and} \qquad \angle A_2B_1A = \angle PBA.$$

Applying sine law to triangles $AA_2C_1$ and $AB_1A_2$, we obtain

$$\frac{\sin\angle AA_2C_1}{AC_1} = \frac{\sin\angle AC_1A_2}{AA_2} \qquad \text{and} \qquad \frac{\sin\angle AA_2B_1}{AB_1} = \frac{\sin\angle A_2B_1A}{AA_2}.$$

Therefore, we have

$$\frac{\sin\angle C_1A_2A}{\sin\angle AA_2B_1} = \frac{AC_1\cdot\sin\angle AC_1A_2}{AB_1\cdot\sin\angle A_2B_1A} = \frac{\sin^2\angle ACP}{\sin^2\angle PBA}.$$

We have, in a similar way

$$\frac{\sin\angle A_1B_2B}{\sin\angle BB_2C_1} = \frac{\sin^2\angle BAP}{\sin^2\angle PCB} \qquad \text{and} \qquad \frac{\sin\angle B_1C_2C}{\sin\angle CC_2A_1} = \frac{\sin^2\angle CBP}{\sin^2\angle PAC}.$$

But, because $AP$, $BP$ and $CP$ are concurrent, we have by trigonometric Ceva

$$\frac{\sin\angle BAP}{\sin\angle PAC}\cdot\frac{\sin\angle CBP}{\sin\angle PBA}\cdot\frac{\sin\angle ACP}{\sin\angle PCB} = 1.$$

We deduce that

$$\frac{\sin\angle C_1A_2A}{\sin\angle AA_2B_1}\cdot\frac{\sin\angle A_1B_2B}{\sin\angle BB_2C_1}\cdot\frac{\sin\angle B_1C_2C}{\sin\angle CC_2A_1} = 1.$$

This proves, by using the reciprocal of trigonometric Ceva, that lines $AA_2, BB_2$ and $CC_2$ are concurrent.

- **Problem 2.** Since $f$ is strictly increasing, we have $f(n) \geq n$ for all $n$ in $S$.

Assume that there exists an integer $n$ in $S$ such that $f(n) > n$. Let $n_0$ be the smallest such $n$ and write $f(n_0) = n_0 + k_0$, for some positive integer $k_0 \geq 1$. Again, since $f$ is strictly increasing, we have $f(n) \geq n + k_0$, for all integers $n \geq n_0$.

Because $k_0 \geq 1$, we have

$$f(n_0 + k_0) \geq n_0 + 2k_0.$$

On the other hand, we have

$$f(n_0 + k_0) = f(f(n_0)) \leq 2f(n_0) - n_0 = n_0 + 2k_0.$$

We deduce that

$$f(n_0 + k_0) = n_0 + 2k_0.$$

Because $f(n_0) = n_0 + k_0$, $f(n_0 + k_0) = (n_0 + k_0) + k_0$ and $f$ is strictly increasing, we deduce that $f(n) = n + k_0$, for all $n_0 \leq n \leq n_0 + k_0$.

We prove by induction on $m$ that for all integer $n$ such that

$$n_0 + mk_0 \leq n \leq n_0 + (m+1)k_0,$$

we have $f(n) = n + k_0$. Hence

$$f(n) = \begin{cases} n + k_0 & \text{if } n \geq n_0, \\ n & \text{otherwise.} \end{cases}$$

Conversely, if $f$ is such a function and $n \geq n_0$ then

$$n + f(f(n)) = n + f(n + k_0) = 2n + 2k_0 = 2f(n).$$

And if $f(n) = n$, the inequality is clearly satisfied.

Therefore, the solutions to this functional inequality are the functions that can be written as

$$f(n) = \begin{cases} n + k_0 & \text{if } n \geq n_0, \\ n & \text{otherwise,} \end{cases}$$

for some $n_0 \in S$ and some nonnegative integer $k_0 \geq 0$.

- **Problem 3.** Assume that we have two correspondents, $R$ from Riyadh and $J$ from Jeddah, who are not connected to each other. Let $\mathcal{J}_R$ be the set of correspondents from Jeddah connected to $R$ and $\mathcal{R}_J$ the set of correspondents from Riyadh connected to $J$. Define the function

$$f : \mathcal{J}_R \longrightarrow \mathcal{R}_J$$

in the following way:

For each correspondent $J' \in \mathcal{J}_R$, since the correspondents $J$ and $J'$ are different, they share a unique correspondent in $\mathcal{R}_J$. Let $f(J')$ be this unique correspondent. So $f$ is well-defined.

Assume that there exist two correspondents $J_1, J_2 \in \mathcal{J}_R$ such that $f(J_1) = f(J_2) = R' \in \mathcal{R}_J$. This means that the correspondent $R'$ from Riyadh shares with the correspondent $R$ two correspondents $J_1, J_2$ from Jeddah. Hence $J_1 = J_2$ and the map $f$ is injective.

Consider a correspondent $R' \in \mathcal{R}_J$ and let $J'$ be the unique correspondent in $\mathcal{J}_R$ that $R'$ shares with $R$. Then $R'$ is the unique correspondent in $\mathcal{R}_J$ that $J'$ shares with $J$. Thus $R' = f(J')$ and $f$ is surjective. This proves that $R$ and $J$ have the same number of correspondents from the other city.

We deduce from this that if the correspondent Amr from Riyadh is not connected to the correspondent Zayd from Jeddah, the correspondent Amr has exactly eight correspondents from Jeddah.

Assume now that the correspondent Amr is connected to the correspondent Zayd from Jeddah. Since there are at least ten correspondents in Riyadh and Zayd is connected only with eight, let $R_1, R_2$ be two correspondents from Riyadh who are not connected to Zayd. Let $J_1$ be the unique correspondent that Amr and $R_1$ share. Let $J_2$ be the unique correspondent that $R_1$ shares with $R_2$. Since $R_1$ is not in contact with Zayd, he has eight correspondents from Jeddah. So there exists at least six correspondents in contact with $R_1$ who are not in contact neither with Amr nor with $R_2$. Let $J$ be one of these correspondents.

The correspondent $R_2$ from Riyadh has exactly eight correspondents from Jeddah since he is not connected to the correspondent Zayd. The correspondent $J$ from Jeddah has exactly eight correspondents from Riyadh since he is not connected to the correspondent $R_2$ from Riyadh. The correspondent Amr from Riyadh has exactly eight correspondents from Jeddah since he is not connected to the correspondent $J$ from Jeddah.

This proves that in all cases the correspondent Amr from Riyadh has exactly eight correspondents from Jeddah.

- **Problem 4.** Assume that it is possible to place the integers $1, 2, \ldots, 2012$ in a circle in such a way that the 2012 products of adjacent pairs of numbers leave pairwise distinct remainders when divided by 2013. Let $a_1, a_2, \ldots, a_{2012}$ be such a reordering of the integers $1, 2, \ldots, 2012$ on the circle.

Because $2013 = 3 \times 11 \times 61$, a number is a multiple of 3 or 11 or 61 if and only if its remainder when divided by 2013 is a multiple of 3 or 11 or 61.

By the pigeonhole principle, there are at least two adjacent numbers in the list $a_1$, $a_2$, $\ldots$, $a_{2012}$ which are not multiple of 3. Make this list starting from these two adjacent numbers and consider the list $b_1, b_2, \ldots, b_{2012}$ of their products, where $b_i = a_i \cdot a_{i+1}$, for $i = 1, \ldots, 2012$ with $a_{2013} = a_1$. Consider $a_{i_1}, a_{i_2}, \ldots, a_{i_{670}}$, all the multiples of 3 with $2 < i_1 < i_2 < \cdots < i_{670}$. It is clear that $b_{i_1}, b_{i_2}, \ldots, b_{i_{670}}$ are all multiples of 3 and that $b_{i_1-1}, b_{i_2-1}, \ldots, b_{i_{670}-1}$ are also all multiples of 3. So their reminders when divided by 2013 are all multiples of 3. But there are only 671 different multiples of 3 between 0 and 2012 included. Therefore $i_{j+1} = i_j + 1$ for all $j = 1, \ldots, 669$. This means that the multiples of 3 in the list $a_1$, $a_2$, $\ldots$, $a_{2012}$ form a block and are not separated by any non multiple of 3.

In a similar way we prove that multiples of 11 form a block in this list and that multiples of 61 form also a block in this list. But since there are common multiples of 3 and 11, their blocks must be connected to each other. For the same reason the 3 blocks must be connected by pairs to each others.

But, in each block, there are numbers which are in none of the two other blocks, for example there are multiples of 3 which are neither multiples of 11 nor multiples of 61 like $3, 6, 9$, and the same thing happens for 11 and 61. So, these numbers are in the middle of each of the blocks and make the blocks intersecting only in their sides. There are also numbers which are not multiples of none of these 3 numbers, like $1, 2, 4$. So, these numbers will prevent two of the three blocks to intersect, and here is the contradiction.

Hence, it is not possible to place the integers $1, 2, \ldots, 2012$ in a circle under the given condition.

## 8.2 Solutions to the selection test of day II

- **Problem 1.** We present for this problem two solutions, one using trigonometric functions and the other using classical inequalities.

**First solution.** The condition $x_1^2 + x_2^2 = y_1^2 + y_2^2 = c^2$, here $c^2 = 2013$, is equivalent to saying that there exist $\alpha, \beta \in \mathbb{R}$ such that

$$x_1 = c\cos\alpha, \quad x_2 = c\sin\alpha, \quad y_1 = c\cos\beta, \quad \text{and} \quad y_2 = c\sin\beta.$$

Therefore

$$\begin{aligned}
S &= (1-x_1)(1-y_1) + (1-x_2)(1-y_2) = 2 - (x_1+x_2+y_1+y_2) + x_1y_1 + x_2y_2 \\
&= 2 - c(\cos\alpha + \sin\alpha + \cos\beta + \sin\beta) + c^2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\
&= 2 - \sqrt{2}c\left(\sin\left(\alpha + \frac{\pi}{4}\right) + \sin\left(\beta + \frac{\pi}{4}\right)\right) + c^2\cos(\alpha - \beta) \\
&= \left(2 - c^2\right) - 2\sqrt{2}c\sin\left(\frac{\alpha+\beta}{2} + \frac{\pi}{4}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + 2c^2\cos^2\left(\frac{\alpha-\beta}{2}\right) \\
&= \left(2 - c^2\right) - 2\sqrt{2}cst + 2c^2t^2,
\end{aligned}$$

where

$$s = \sin\left(\frac{\alpha+\beta}{2} + \frac{\pi}{4}\right) \quad \text{and} \quad t = \cos\left(\frac{\alpha-\beta}{2}\right),$$

are two independent variables, since $\alpha+\beta$ and $\alpha-\beta$ are independent, taking all the real values between $-1$ and $1$ included.

Hence, the maximum of $S$ is equal to $2 + c^2 + 2\sqrt{2}c = 2015 + 2\sqrt{4026}$, and is reached when $s = -t = \pm 1$. That is precisely when $x_1 = x_2 = y_1 = y_2 = -\frac{\sqrt{4026}}{2}$.

For the minimum, notice that

$$S = \left(2 - c^2\right) - s^2 + (\sqrt{2}ct - s)^2$$

Therefore, the minimum of $S$ is equal to $1 - c^2 = -2012$, and is reached when $s = \pm 1$ and $t = \frac{\sqrt{2}s}{2c} = \frac{\pm\sqrt{4026}}{4026}$. That is precisely when

$$x_1 = y_2 = \frac{1+\sqrt{4025}}{2} \quad \text{and} \quad x_2 = y_1 = \frac{1-\sqrt{4025}}{2},$$

or vice-versa.

**Second solution.** For the maximum, we have by Cauchy-Schwartz inequality

$$\begin{aligned} S &= 2 - (x_1 + x_2 + y_1 + y_2) + x_1 y_1 + x_2 y_2 \\ &\le 2 + \sqrt{4 \cdot (x_1^2 + x_2^2 + y_1^2 + y_2^2)} + \sqrt{(x_1^2 + x_2^2)(y_1^2 + y_2^2)} \\ &\le 2015 + 2\sqrt{4026}, \end{aligned}$$

and the equality holds when $x_1 = x_2 = y_1 = y_2 = -\frac{\sqrt{4026}}{2}$.

For the minimum, we have

$$\begin{aligned} S &= 2 - (x_1 + y_1) - (x_2 + y_2) + \frac{(x_1 + y_1)^2}{2} + \frac{(x_2 + y_2)^2}{2} - c^2 \\ &= (1 - c^2) + \frac{(x_1 + y_1 - 1)^2}{2} + \frac{(x_2 + y_2 - 1)^2}{2} \\ &\ge 1 - c^2 = -2012, \end{aligned}$$

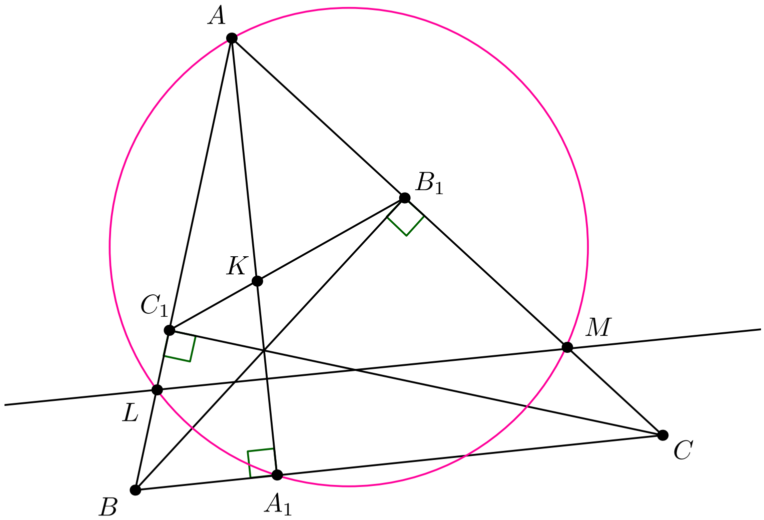
where $c^2 = 2013$.

Thus the minimum of $S$ is $-2012$ and is reached when $y_1 = 1 - x_1$, $y_2 = 1 - x_2$ and $x_1^2 + x_2^2 = y_1^2 + y_2^2 = 2013$. This is equivalent to

$$x_1 = y_2 = \frac{1 + \sqrt{4025}}{2} \quad \text{and} \quad x_2 = y_1 = \frac{1 - \sqrt{4025}}{2},$$

or vice-versa.

- **Problem 2.** Because $LM$ is parallel to $BC$, the problem is equivalent to proving that $\angle AA_1L = \angle AML = \angle ACB$. We present two solutions:




**First solution.** Let $L_1$ be the point on $AB$ such that $\angle AA_1L_1 = \angle ACB$ and let us prove that $L_1 = L$.

Quadrilateral $BCB_1C_1$ is cyclic since $\angle BC_1C = \angle BB_1C = 90°$. Therefore, $\angle B_1C_1A = \angle ACB = \angle AA_1L_1$. We deduce that the quadrilateral $A_1KC_1L_1$ is cyclic. Hence, $\angle L_1KA_1 = \angle L_1C_1A_1$.

Quadrilateral $AC_1A_1C$ is cyclic since $\angle CA_1A = \angle CC_1A = 90°$. We deduce that $\angle L_1C_1A_1 = \angle ACB$. Thus, $\angle L_1KA_1 = \angle AA_1L_1$. This proves that $L_1A_1 = L_1K$, that is $L_1 = L$, the intersection point of the perpendicular bisector of $KA_1$ with $AB$.

**Second solution.** Since $LK = LA_1$, the problem is equivalent to proving that $\angle LKA_1 = \angle ACB$. But $\angle BHA_1 = 90° - \angle A_1BH = \angle ACB$, where $H$ is the orthocenter of triangle $ABC$. Therefore, the problem is equivalent to proving that $LK$ and $BH$ are parallel.

We know that $\frac{AL}{LB} = \frac{AN}{NA_1}$, where $N$ is the midpoint of $KA_1$, since $LM$ and $BC$ are parallel. It remains to prove that $\frac{AK}{KH} = \frac{AN}{NA_1}$.

To simplify the notations let us write $\alpha = \angle BAC$, $\beta = \angle CBA$ and $\gamma = \angle ACB$.

We know that $\angle KC_1A = \gamma$, $\angle HC_1K = 90^\circ - \gamma$, $\angle C_1AK = 90^\circ - \beta$ and $\angle KHC_1 = \beta$. We deduce by applying sine laws on triangles $AC_1K$ and $C_1HK$ that

$$\frac{AK}{KH} = \frac{\sin\angle KC_1A \cdot \sin\angle KHC_1}{\sin\angle HC_1K \cdot \sin\angle C_1AK} = \tan\beta \cdot \tan\gamma.$$

On the other hand, we have $\frac{AN}{NA_1} = \frac{AK}{NA_1} + 1 = 2\frac{AK}{KA_1} + 1$. We also know that $\angle A_1C_1K = 180^\circ - 2\gamma$ and $\angle KA_1C_1 = 90^\circ - \alpha$. We deduce by applying sine laws on triangles $AC_1K$ and $C_1A_1K$ that

$$\frac{AK}{KA_1} = \frac{\sin\angle KC_1A \cdot \sin\angle KA_1C_1}{\sin\angle A_1C_1K \cdot \sin\angle C_1AK} = \frac{\sin\gamma \cdot \cos\alpha}{\sin 2\gamma \cdot \cos\beta} = \frac{\cos\alpha}{2\cos\beta \cdot \cos\gamma}.$$

We deduce that

$$\frac{AN}{NA_1} = \frac{\cos\alpha}{\cos\beta \cdot \cos\gamma} + 1 = \frac{\cos\beta \cdot \cos\gamma - \cos(\beta+\gamma)}{\cos\beta \cdot \cos\gamma} = \tan\beta \cdot \tan\gamma = \frac{AK}{KH}.$$

- **Problem 3.** Let $n$ be a positive integer. Consider $D_1$, the set of all divisors of $n$ with unit digit 3, and $D_2$ the set of all the other divisors of $n$.

If the unit digit of $n$ is different from 9, consider the map $\delta : D_1 \longrightarrow D_2$ defined by $\delta(d) = \frac{n}{d}$ for all $d \in D_1$. If $d \equiv 3 \mod 10$ and $\delta(d) \equiv 3 \mod 10$ for some $d \in D_1$, then $n = d \cdot \delta(d) \equiv 9 \mod 10$ which is a contradiction. Then the map $\delta$ is well defined. Clearly, the map $\delta$ is an injective map. Therefore the cardinality of $D_1$ is less than or equal to the cardinality of $D_2$. Hence $p \leq 50$.

If the unit digit of $n$ is equal to 9, the integer $n$ is neither divisible by 2 nor by 5. This means that all prime divisors of $n$ have unit digit equal to 1, 3, 5, or 9.

If all prime divisors of $n$ have unit digits equal to 1 or 9, all divisors of $n$ have unit digits equal to 1 or 9. Therefore $p = 0$.

If the integer $n$ has a prime divisor $p$ with unit digit 3 or 7, consider the map $\delta : D_1 \longrightarrow D_2$ defined for $d \in D_1$ by $\delta(d) = \frac{d}{p}$ if $p$ divides $d$ and by $\delta(d) = pd$ otherwise. We can see easily in both cases, that the unit digit of $\delta(d)$ is equal to 1 or 9. Hence $\delta$ is well defined.

Assume that there exist $d_1, d_2 \in D_1$ such that $\delta(d_1) = \delta(d_2)$. It is clear that if both $d_1, d_2$ are divisible by $p$ or both $d_1, d_2$ are not divisible by $p$, we have $d_1 = d_2$. Assume that $d_1$ is divisble by $p$ and $d_2$ is not divisible by $p$. We have $\frac{d_1}{p} = d_2 p$, which is equivalent to $d_1 = p^2 d_2$. Because $d_1 \equiv d_2 \equiv 3 \mod 10$, we deduce that $p^2 \equiv 1 \mod 10$, which contradicts the fact that $p^2 \equiv -1 \mod 10$ since $p \equiv 3, 7 \mod 10$. Hence $\delta$ is injective and therefore the cardinality of $D_1$ is less than or equal to the cardinality of $D_2$. Thus $p \leq 50$.

Notice that for $n = 3$, its divisors are $1, 3$ and $p = 50$. This proves that the maximum possible value of $p$ is 50.

- **Problem 4.** We will construct such a sequence by induction:

Define $a_1 = 1$ and $a_2 = 2$. In this case we have $b_1 = |a_1 - a_2| = 1$.

Assume that $a_1, a_2, \ldots, a_{2n}$ are defined such that there is no positive integer which occurs at least twice neither in the finite sequence $a_1, a_2, \ldots, a_{2n}$ nor in the finite sequence $b_1 = |a_1 - a_2|, b_2 = |a_2 - a_3|, \ldots, b_{2n-1} = |a_{2n-1} - a_{2n}|$.

Let $M_n$ be the maximum element of the set of integers $\{a_1, a_2, \ldots, a_{2n}\}$, $c_n$ the maximum of the integers $k$ such that $\{1, 2, \ldots, k\} \subseteq \{a_1, a_2, \ldots, a_{2n}\}$, and $d_n$ the maximum of the integers $k$ such that $\{1, 2, \ldots, k\} \subseteq \{b_1, b_2, \ldots, b_{2n-1}\}$. Notice that for all $1 \leq i \leq 2n - 1$, we have

$$b_i = |a_i - a_{i+1}| \leq \max_{1 \leq j \leq 2n} a_j - \min_{1 \leq j \leq 2n} a_j = M_n - 1.$$

If $c_n < d_n$ define $a_{2n+1} = 2M_n + c_n + 1$ and $a_{2n+2} = c_n + 1$. If $c_n \geq d_n$ define $a_{2n+1} = 2M_n + 1$ and $a_{2n+2} = 2M_n + d_n + 2$.

It is clear that none of the two possible values for $a_{2n+1}$ and for $a_{2n+2}$ occur in the finite sequence $a_1, a_2, \ldots, a_{2n}$.

In the first case, we have

$$b_{2n} = 2M_n + c_n + 1 - a_{2n} \geq M_n + c_n + 1 > M_n - 1 \geq b_i,$$

and

$$b_{2n+1} = 2M_n > M_n - 1 \geq b_i,$$

for all $1 \leq i \leq 2n-1$, and $b_{2n+1} \neq b_{2n}$ since $a_{2n} \neq c_n + 1$.

In the second case, we have

$$b_{2n} = 2M_n + 1 - a_{2n} \geq M_n + 1 > M_n - 1 \geq b_i,$$

and

$$b_{2n+1} = d_n + 1 \neq b_i,$$

for all $1 \leq i \leq 2n-1$, and $b_{2n+1} = d_n + 1 \leq M_n < M_n + 1 \leq b_{2n}$.

Therefore, there is no positive integer which occurs at least twice in the finite sequence $a_1, a_2, \ldots, a_{2n+2}$ or in the finite sequence $b_1, b_2, \ldots, b_{2n+1}$.

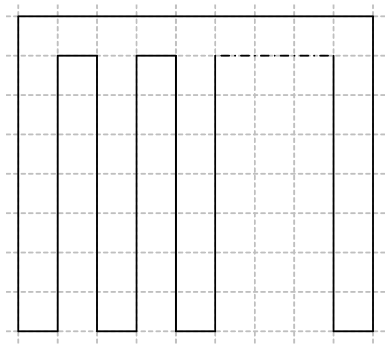
This proves that there is no positive integer which occurs at least twice in the infinite sequence $a_1, a_2, \ldots, a_n, \ldots$ or in the infinite sequence $b_1, b_2, \ldots, b_n, \ldots$.

Notice moreover, that in the first case $c_{n+1} \geq c_n + 1$ and in the second case $d_{n+1} \geq d_n + 1$. Therefore, if $e_n = \min\{c_n, d_n\}$ then $e_{n+2} \geq e_n + 1$. But $c_1 = 2$ and $d_1 = 1$. We deduce that $e_{2n} \geq n$ for all integer $n$. Therefore, any positive integer $n$ occurs in both finite sequences $a_1, a_2, \ldots, a_{4n}$ and $b_1, b_2, \ldots, b_{4n-1}$.

This proves that any positive integer $n$ occurs exactly once in each infinite sequence $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$.

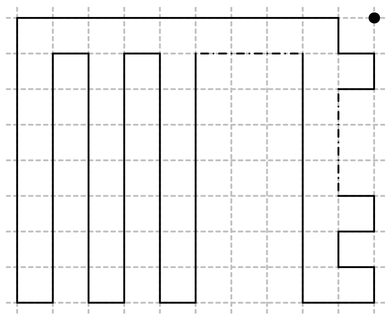
## 8.3 Solutions to the selection test of day III

- **Problem 1.** If $m$ is even, Adel can draw the following closed path which passes through all the dots of his grid. Therefore, the maximum possible value of $k$ is $mn$.



If $n$ is even, Adel can draw a similar closed path obtained by symmetry with respect to the first diagonal. Therefore, the maximum possible value of $k$ is $mn$.

If $mn$ is odd, Adel can draw the following closed path which passes through $mn - 1$ dots of his grid:



It remains to prove that this is the maximal possible value of $k$. For this, assign black color to all dots of the grid of coordinates $(a, b)$ with $a + b$ odd and white color to all dots with $a + b$ even. Any consecutive dots in a path $(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)$ that Adel can draw have different colors. Therefore, the colors of the first and last dots in Adel's path describe the parity of the length of the path. Since the path of Adel is closed, it will start and end with the same color and therefore, its length is even. Therefore $k$ is even. This proves that the maximum possible value of $k$ is $mn - 1$.

- **Problem 2.** We present for this problem two solutions.

**First solution.** Notice that we have

$$\begin{aligned}(a_1-n)^2+\cdots+(a_n-n)^2 &= (a_1^2+\cdots+a_n^2)-2n(a_1+\cdots+a_n)+n^3\\ &\le n^3+1-2n^3+n^3=1.\end{aligned}$$

Therefore, there are two cases:

The first case is when $a_1=a_2=\cdots=a_n=n$. This is a solution since it satisfies both inequalities.

The second case is when there exists $1\le i_0\le n$ such that $|a_{i_0}-n|=1$ and $a_i=n$ for all $1\le i\le n$ with $i\neq i_0$.

If $a_{i_0}=n-1$, the first inequality becomes $n^2-1=a_1+\cdots+a_n\ge n^2$ which is impossible.

If $a_{i_0}=n+1$, the second inequality becomes $n^3+2n+1=a_1^2+\cdots+a_n^2\le n^3+1$ which is also impossible.

Hence, the only solution to both inequalities is given by $a_1=a_2=\cdots=a_n=n$.

**Second solution.** We have by Cauchy-Schwartz inequality

$$(n^2+1)^2>n^4+n\ge n(a_1^2+\cdots+a_n^2)\ge(a_1+\cdots+a_n)^2\ge n^4.$$

We deduce that

$$a_1+\cdots+a_n=n^2,$$

and

$$n^3+1\ge a_1^2+\cdots+a_n^2\ge n^3,$$

and the right hand side equality occurs if and only if $a_1=\cdots=a_n=n$.

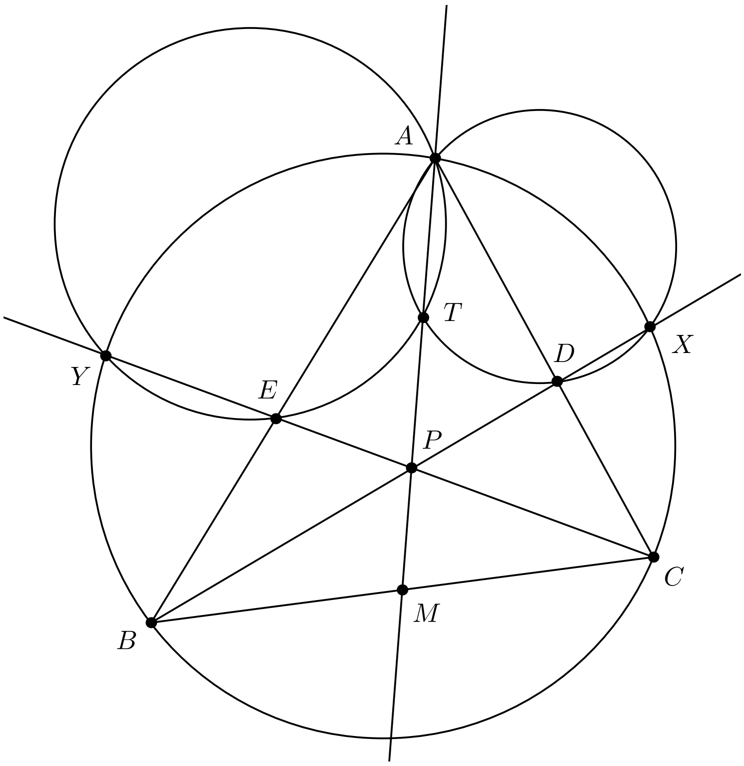
Now, assume, looking for a contradiction, that $a_1^2+\cdots+a_n^2=n^3+1$. Since a number and all his powers have the same parity, we deduce that

$$n\equiv n^2\equiv a_1+\cdots+a_n\equiv a_1^2+\cdots+a_n^2\equiv n^3+1\equiv n+1 \mod 2,$$

which is impossible.

Hence, the only solution to both inequalities is given by $a_1=a_2=\cdots=a_n=n$.

- **Problem 3.** By Applying Ceva to the concurrent cevians $AM$, $BD$ and $CE$, we obtain
$$\frac{AE}{EB} = \frac{AD}{DC} \cdot \frac{CM}{MB} = \frac{AD}{DC}.$$
We deduce from Thales theorem that segments $ED$ and $BC$ are parallel.




Since quadrilateral $BCXY$ is cyclic, we have $\angle BXY = \angle BCY = \angle CED$. We deduce that quadrilateral $EDXY$ is cyclic and therefore

$$PD \cdot PX = PE \cdot PY.$$

Hence, point $P$ lies on the radical axis of circumcircles of triangles $AXD$ and $AYE$ which passes trough $A$.

If these two circles are tangent to $AP$ at $A$ then

$$\angle MAC = \angle AXB = \angle ACB, \quad \text{and} \quad \angle BAM = \angle CYA = \angle CBA,$$

which implies that triangle $ABC$ is a right triangle at $A$ and this is a contradiction.

We conclude that the two circles intersect in a second point $T \neq A$ on line $AM$.

- **Problem 4.** Suppose there is some value of $n$ such that $p(n) \neq \pm 1$. Let $q$ be a prime divisor of $p(n)$. Because $q = (n+q) - q$ divides $p(n+q) - p(n)$, we deduce that $q$ divides $p(n+q)$. Therefore, $q$ divides both $2^n - 1$ and $2^{n+q} - 1$, which implies in particular that $q$ is an odd prime. We have

$$1 \equiv 2^{n+q} \equiv 2^n \cdot 2^q \equiv 2^q \equiv 2 \mod q,$$

by Fermat's little theorem. This is a contradiction.
Hence, for all values of $n$, $p(n) = \pm 1$. Since $p$ is a polynomial and takes infinitely many times the same value, either 1 or $-1$, it is constant. This proves that the only polynomials with integer coefficients which have the required property are $p(X) = 1$ and $p(X) = -1$.