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On a property of Toeplitz operators on Bergman space with a logarithmic weight

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Abstract
An operator T on a Hilbert space is hyponormal if T*T-TT* is positive. In this work we consider hyponormality of Toeplitz operators on the Bergman space with a logarithmic weight. Under a smoothness assumption we give a necessary condition when the symbol is of the form $f + \overline{g}$ with f, g analytic on the unit disk. We also find a sufficient condition when f is a monomial and g a polynomial.

Keywords Toeplitz operator · Weighted Bergman spaces · Hyponormality · Positive matrices

Mathematics Subject Classification Primary 47B35 · 47B20; Secondary 15B48

1 Introduction

A bounded operator T, on a Hilbert space, is said to be hyponormal, if $T^*T - TT^*$ is positive. Let U denote the unit disk in the complex plane. We consider the Hilbert space of analytic functions on U such that $\int_U |f(z)|^2 d\mu(z) < \infty$, where $d\mu(z) = \frac{2}{\pi}|\log|z||dA(z)$, and $dA(z)$ is the Lebesgue measure on the unit disk. When f is analytic on U, we have $f = \sum a_n z^n$ and $\int_U |f(z)|^2 d\mu(z) = \sum \frac{1}{(n+1)^2}|a_n|^2$. Denote by $L^2_{a,w}$ such a Hilbert space. Its orthonormal basis is given by $\{(n+1)z^n, n \geq 0\}$. The Toeplitz operator is defined by $T_f(k) = P(fk)$, where f, is a bounded and measurable function on the disk, k is in $L^2_{a,w}$, and P is the orthogonal projection of $L^2(U, d\mu)$ on $L^2_{a,w}$. Hankel operators are defined by $H_f(k) = (I - P)(fk)$, for f, and k as before. Basic material on unweighted Bergman spaces ($d\mu = dA$) can be found in [2,3,10]. Hyponormality of Toeplitz operators on unweighted

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Bergman spaces was first considered in [1,9]. The Hardy space and hyponormality on the Hardy space can be found respectively in [6] and [4,5]. Assuming that f' is in the Hardy space H^2, we give a necessary condition for the hyponormality of $T_{f+\overline{g}}$, where, f and g are bounded analytic on U. We also give sufficient conditions, when f is a monomial, and g a polynomial. We start by stating some basic properties of Toeplitz operators, the proofs of which are straightforward.

2 General properties of Toeplitz operators

We assume f, g are in $L^\infty(U, d\mu)$. Then we have

1. $T_{f+g} = T_f + T_g$.
2. $T_f^* = T_{\overline{f}}$.
3. $T_{\overline{f}} T_g = T_{\overline{f}g}$ if f or g analytic on U.

The use of these properties leads to a description of hyponormality, in more than one form. Douglas lemma is used to get one of these forms [7].

Proposition 1 *Let f, g be bounded and analytic on U. The following are equivalent*

(i) $T_{f+\overline{g}}$ *is hyponormal.*
(ii) $T_{\overline{g}} T_g - T_g T_{\overline{g}} \leq T_{\overline{f}} T_f - T_f T_{\overline{f}}$.
(iii) $H_{\overline{g}}^* H_{\overline{g}} \leq H_{\overline{f}}^* H_{\overline{f}}$.
(iv) $||(I-P)(\overline{g}k)|| \leq ||(I-P)(\overline{f}k)||$ *for any k in* $L^2_{\alpha,w}$.
(v) $||\overline{g}k||^2 - ||P(\overline{g}k)||^2 \leq ||\overline{f}k||^2 - ||P(\overline{f}k)||^2$ *for any k in* $L^2_{\alpha,w}$.
(vi) $H_{\overline{g}} = K H_{\overline{f}}$ *where K is of norm less than or equal to one.*

The following lemma will be needed in the sequel.

Lemma 1 *For s and t integers we have:*

$$P(\overline{z}^t z^s) = \begin{cases} z^{s-t} \frac{(s-t+1)^2}{(s+1)^2}, & if\ s \geq t \\ 0, & if\ s < t \end{cases}.$$

3 The necessary condition

We now prove a computational lemma.

Lemma 2 *Let $f = \sum a_n z^n$ be bounded and analytic on U. The matrix of $H_{\overline{f}}^* H_{\overline{f}}$ in the orthonormal basis $\{(n+1)z^n, n \geq 0\}$ is given by:*

$$\zeta_{i,j} = \sum_{m \geq j-i,\ m \geq 0} a_{m+i-j} \overline{a_m} \frac{(i+1)(j+1)}{(i+m+1)^2} - \sum_{i-j \leq m \leq i,\ m \geq 0} a_m \overline{a_{m+j-i}} \frac{(i-m+1)^2}{(i+1)(j+1)}.$$

Proof Since $\bar{f} f (j+1) z^j = \sum_{n,m=0}^{\infty} (j+1)\, a_n \overline{a_m} \overline{z^m} z^{n+j}$, we get

$$\begin{aligned} P(\bar{f} f (j+1) z^j) &= \sum_{m \geq 0,\ p \geq m,\ p \geq j}^{\infty} (j+1) \overline{a_m}\, a_{p-j} \frac{(p-m+1)^2}{(p+1)^2} z^{p-m} \\ &= \sum_{m+n \geq j,\ m \geq 0,\ n \geq 0} \overline{a_m}\, a_{m+n-j} \frac{(j+1)(n+1)^2}{(m+n+1)^2} z^n \end{aligned}$$

thus

$$< P(\bar{f} f (j+1) z^j), (i+1)\, z^i >= \sum_{m+i \geq j,\ m \geq 0} a_{m+i-j}\, \overline{a_m}\, \frac{(j+1)(i+1)}{(i+m+1)^2}.$$

Similarly we show

$$< T_f\ T_{\bar{f}}\ (j+1) z^j\ , (i+1) z^i >= \sum_{i-j \leq m \leq i,\ m \geq 0} a_m \overline{a_{m+j-i}} \frac{(i-m+1)^2}{(i+1)(j+1)}.$$

The result follows. □

Lemma 3 *Let f be a bounded and analytic function in U, such that $f' \in H^2$. Let $(\sigma_{i,j})$ be the matrix of the Hardy space Toeplitz operator $T_{|f'|^2}$. Then $n^2 \zeta_{i+n,j+n} \to 2\sigma_{i,j}$ as $n \to \infty$.*

Proof From the previous lemma we get

$$\zeta_{i+n,i+n+p} = \sum_{m \geq p} a_{m-p} \overline{a_m}\, \frac{(i+n+1)(i+n+p+1)}{(i+n+m+1)^2} - \sum_{0 \leq m \leq i+n} a_m \overline{a_{m+p}}\, \frac{(i+n-m+1)^2}{(i+n+1)(i+n+p+1)}.$$

Set $m = p + l$ in the first sum, and $m = l$ in the second. Also put $A_{n,i,p} = (i+n+1)(i+n+p+1)$

$$\begin{aligned}
\zeta_{i+n,i+n+p} &= \sum_{l \leq i+n} a_l \overline{a_{l+p}}\, \frac{A^2_{n,i,p} - (i+n+p+l+1)^2 (i+n-l+1)^2}{(i+n+p+l+1)^2 A_{n,i,p}} \\
&\quad + \sum_{l > i+n} a_l\ \overline{a_{l+p}}\, \frac{(i+n+1)(i+n+p+1)}{(i+n+p+l+1)^2} \\
&= \sum_{l \leq i+n} l(l+p)\, a_l \overline{a_{l+p}}\, \frac{A_{n,i,p} + (i+n+p+l+1)(i+n-l+1)}{(i+n+p+l+1)^2 A_{n,i,p}} \\
&\quad + \sum_{l > i+n} a_l\ \overline{a_{l+p}}\, \frac{A_{n,i,p}}{(i+n+p+l+1)^2}
\end{aligned}$$

thus we can write with obvious notations

$$n^2 \zeta_{i+n,i+n+p} = \sum_{l \leq i+n} l(l+p)\, a_l \overline{a_{l+p}} n^2 C_{l,i,n,p} + \sum_{l > i+n} a_l\ \overline{a_{l+p}} n^2 D_{l,i,n,p}.$$

It is easy to see that $\sum_{0 \leq m \leq i+n} l(l+p) a_l \overline{a_{l+p}} n^2 C_{l,i,n,p} = \int_0^\infty h_n(l) d\upsilon(l)$ where υ is the counting measure, with

$$\begin{aligned}
h_n(l) &= \chi_{\{0,\ldots,i+n\}}(l) l(l+p)\, a_l \overline{a_{l+p}}\, \frac{n^2 [A_{n,i,p} + (i+n+p+l+1)(i+n-l+1)]}{(i+n+p+l+1)^2 A_{n,i,p}} \\
&\longrightarrow 2l(l+p)\, a_l \overline{a_{l+p}} \quad \text{as } n \to \infty.
\end{aligned}$$

We also have

$$|h_n(l)| \leq l^2 |a_l|^2 + (p+l)^2 |a_{p+l}|^2 = M(l)$$

Since $f' \in H^2$, we have $\int_0^\infty M(l)d\upsilon(l) < \infty$. So by the dominated convergence theorem we have

$$\lim_{n\to\infty}\int_0^\infty h_n(l)d\upsilon(l) = 2\sum_0^\infty l(l+p)\, a_l\overline{a_{l+p}}.$$

For $l > i+n$ we have

$$|a_l\ \overline{a_{l+p}}\ \frac{n^2 A_{n,i,p}}{(i+n+p+l+1)^2}| \le \frac{1}{2}(l^2|a_l|^2\ + (l+p)^2|a_{l+p}|^2).$$

Applying the dominated convergence theorem we deduce that

$$\sum_{l>i+n} a_l\ \overline{a_{l+p}} n^2 D_{l,i,n,p} = \sum_{l>i+n} a_l\ \overline{a_{l+p}}\ \frac{n^2 A_{n,i,p}}{(i+n+p+l+1)^2} \to 0$$

as $n \to \infty$. This leads us to

$$\lim_{n\to\infty} n^2\zeta_{i+n,i+n+p} = 2\sum_{l=0}^\infty l(l+p)\, a_l\overline{a}_{l+p} = 2\sigma_{i,i+p}. \tag{1}$$

□

We obtain the following theorem

Theorem 1 *Let f and g be bounded analytic functions on U, such that $f' \in H^2$. If $T_{f+\overline{g}}$ is hyponormal on $L^2_{a,w}$, then $g' \in H^2$ and $|g'| \le |f'|$ a.e on ∂U.*

Proof Let $(\eta_{i,j})$ denote the matrix of $H^*_{\overline{g}}H_{\overline{g}}$ in the orthonormal basis $\{(n+1)z^n, n \ge 0\}$. Using the notation of the previous lemma and setting $g = \sum b_n z^n$ we have

$$n^2\zeta_{i+n,i+n} = \sum_{l\le i+n} l^2\ |a_l|^2 n^2 C_{l,i,n,0} + \sum_{l>i+n} |a_l|^2 n^2 D_{l,i,n,0},$$
$$n^2\eta_{i+n,i+n} = \sum_{l\le i+n} l^2\ b_l\overline{b_l} n^2 C_{l,i,n,0} + \sum_{l>i+n} |b_l|^2\ n^2 D_{l,i,n,0}.$$

Hyponormality of $T_{f+\overline{g}}$ leads to the inequality

$$\sum_{l\le i+n} l^2\ |b_l|^2 n^2 C_{l,i,n,0} \le n^2\zeta_{i+n,i+n}.$$

Set $s_n(l) = \chi_{\{0,\ldots i+n\}}(l) l^2\ |b_l|^2 n^2 C_{l,i,n,0}$. We have:

$$s_n(l) \longrightarrow 2l^2\ |b_l|^2 \quad \text{as } n \to \infty.$$

Using Fatou's lemma and (1) we get

$$2\sum_{l\ge 0} l^2\ |b_l|^2 \le \sum_{l\ge 0} 2l^2|a_l|^2.$$

Since $f' \in H^2$, the right hand side of the above inequality is finite. Thus the left hand side is finite and $g' \in H^2$. If $(\Psi_{i,j})$ denotes the matrix of $H^*_{\overline{f}}H_{\overline{f}} - H^*_{\overline{g}}H_{\overline{g}}$, and $(\Pi_{i,j})$ denotes the matrix of the Hardy space operator $T_{|f'|^2-|g'|^2}$, from the previous lemma we have $n^2\Psi_{i+n.j+n} \longrightarrow 2\Pi_{i,j}$. From the assumption of hyponormality and a property of Toeplitz forms [8], we obtain $|g'| \le |f'|$ a.e on ∂U. □

4 The sufficient condition

In what follows we take $f = z^q$, q a positive integer, and we give sufficient conditions for the hyponormality of $T_{z^q+\overline{g}}$, where g is a polynomial. We will need the following lemma:

Lemma 4 *The matrix of $H^*_{\overline{z^q}} H_{\overline{z^q}}$ in the orthonormal basis $\{(n+1)z^n, n \geq 0\}$ is diagonal and is given by:* $d_i = \begin{cases} \frac{(i+1)^2}{(i+q+1)^2}, & \text{if } i < q \\ \frac{(i+1)^2}{(i+q+1)^2} - \frac{(i-q+1)^2}{(i+1)^2}, & \text{if } i \geq q \end{cases}.$

Proposition 2 *Let $p > q$ then $T_{z^q+\alpha\overline{z^p}}$ is hyponormal if and only if $|\alpha| \leq \frac{q}{p}$.*

Proof Hyponormality of $T_{z^q+\alpha\overline{z^p}}$ is equivalent to the inequality

$$|\alpha^2| H^*_{\overline{z^p}} H_{\overline{z^p}} \leq H^*_{\overline{z^q}} H_{\overline{z^q}},$$

which is equivalent to the following inequalities:

$$|\alpha|^2 \frac{(i+1)^2}{(i+p+1)^2} \leq \frac{(i+1)^2}{(i+q+1)^2} \quad i < q \tag{2}$$

$$|\alpha|^2 \frac{(i+1)^2}{(i+p+1)^2} \leq \frac{(i+1)^2}{(i+q+1)^2} - \frac{(i-q+1)^2}{(i+1)^2} \quad q \leq i < p \tag{3}$$

$$|\alpha|^2 \left(\frac{(i+1)^2}{(i+p+1)^2} - \frac{(i-p+1)^2}{(i+1)^2} \right) \leq \left(\frac{(i+1)^2}{(i+q+1)^2} - \frac{(i-q+1)^2}{(i+1)^2} \right) \quad p \leq i \tag{4}$$

It is clear that (2) is equivalent to $|\alpha| \leq \min\left\{\frac{i+p+1}{i+q+1},\ i < q\right\}$. Since $\frac{i+p+1}{i+q+1}$ decreases with i, the minimum is assumed at $i = q-1$. Thus (2) is equivalent to $|\alpha| \leq \frac{p+q}{2q}$.
Similarly, (3) is equivalent to

$$|\alpha|^2 \leq \min\left\{ \frac{(i+p+1)^2}{(i+q+1)^2} - \frac{(i+p+1)^2(i-q+1)^2}{(i+1)^4}, q \leq i < p \right\}.$$

Set $\omega_1(i) = \frac{(i+p+1)^2}{(i+q+1)^2} - \frac{(i+p+1)^2(i-q+1)^2}{(i+1)^4}$, then

$$\omega_1(i) = \frac{(i+p+1)^2(2q^2((i+1)^2 - q^4)}{(i+q+1)^2(i+1)^4}.$$

Using logarithmic differentiation, we can easily verify that, ω is decreasing if $q < p$. Thus $\min\{\omega(i), q \leq i < p\} = \omega_1(p-1) = \frac{4q^2(2p^2-q^2)}{p^2(p+q)^2}$ and (3) is equivalent to $|\alpha| \leq \frac{2q\sqrt{2p^2-q^2}}{p(p+q)}$. A computation shows that inequality (4) is equivalent to $|\alpha|^2 \leq \min\{\frac{q^2}{p^2} \frac{(i+p+1)^2(2(i+1)^2-q^2)}{(i+q+1)^2(2(i+1)^2-p^2)}, p \leq i$. Using logarithmic differentiation, we can verify that $\omega_2(i) = \frac{(i+p+1)^2(2(i+1)^2-q^2)}{(i+q+1)^2(2(i+1)^2-p^2)}$ decreases with i. We deduce that inequality (4) is equivalent to $|\alpha| \leq \frac{q}{p}$. It is easy to see that, $\frac{q}{p} \leq \frac{2q\sqrt{2p^2-q^2}}{p(p+q)}$, and $\frac{q}{p} \leq \frac{p+q}{2q}$ for $q < p$. This completes the proof. □

Remark 1 The previous result obviously holds also, when $p = q$.

Proposition 3 *Let p and q be positive integers, such that $p < q$. Then $T_{z^q+\alpha\overline{z^p}}$ is hyponormal if and only if $|\alpha| \leq \frac{p+1}{q+1}$.*

Proof As in the proof of the previous of proposition, hyponormality of $T_{z^q+\alpha\overline{z^p}}$ is equivalent to the following three inequalities

$$|\alpha|^2 \frac{(i+1)^2}{(i+p+1)^2} \leq \frac{(i+1)^2}{(i+q+1)^2} \quad i < p \tag{5}$$

$$|\alpha|^2 \left(\frac{(i+1)^2}{(i+p+1)^2} - \frac{(i-p+1)^2}{(i+1)^2} \right) \leq \frac{(i+1)^2}{(i+q+1)^2} \quad p \leq i < q \tag{6}$$

$$|\alpha|^2 \left(\frac{(i+1)^2}{(i+p+1)^2} - \frac{(i-p+1)^2}{(i+1)^2} \right) \leq \left(\frac{(i+1)^2}{(i+q+1)^2} - \frac{(i-q+1)^2}{(i+1)^2} \right) \quad q \leq i \tag{7}$$

The same method, as the one used in the proof of the previous proposition, leads to the following: (5) is equivalent to $|\alpha| \leq \frac{p+1}{q+1}$, inequality (6) is equivalent to the inequality $|\alpha| \leq \min\left\{ \frac{(i+1)^2(i+p+1)}{(i+q+1)\sqrt{2p^2(i+1)^2-p^4}}, p \leq i < q \right\} = A_{p,q} = \frac{(p+1)^2(2p+1)}{p(p+q+1)\sqrt{p^2+4p+2}}$, while inequality (7) is equivalent to $|\alpha| \leq \min\left\{ \frac{q}{p}\frac{i+p+1}{i+q+1}\frac{\sqrt{2(i+1)^2-q^2}}{\sqrt{2(i+1)^2-p^2}}, q \leq i \right\} = B_{p,q}$ $= \frac{q(q+p+1)\sqrt{q^2+4q+2}}{p(2q+1)\sqrt{2q^2+4q+2-p^2}}$.

It is easy to see that $\frac{2p+1}{p+q+1} \geq \frac{p+1}{q+1}$, and that $\frac{(p+1)^2}{p\sqrt{p^2+4p+2}} \geq 1$. Thus $A_{p,q} \geq \frac{p+1}{q+1}$.

We also have $q\sqrt{q^2+4q+2} \geq p\sqrt{2q^2+4q+2-p^2}$, and $\frac{p+q+1}{2q+1} \geq \frac{p+1}{q+1}$, which leads to $B_{p,q} \geq \frac{p+1}{q+1}$. This completes the proof. □

In what follows, we give a sufficient condition for the hyponormality of $T_{z^q+\overline{g}}$, where g is a polynomial. Denote by B_1 the unit ball of $(L^2_{a,w})^{\perp}$. We need to introduce the following set

Definition 1 For $f \in L^2_{a,w}$, set $\Lambda_f = \{g \in L^2_{a,w}, \sup\{| < \bar{g}k, u > |, u \in B_1\} \leq \sup\{| < \bar{f}k, u > |, u \in B_1\}$ for any $k \in H^\infty\}$.

By the density of H^∞ in $L^2_{a,w}$, we see that, $g \in \Lambda_f$ is equivalent to $T_{f+\overline{g}}$ is hyponormal. Some of the properties of Λ_f are listed in the following proposition:

Proposition 4 *Let $f \in L^2_{a,w}$, the following holds:*

(i) *Λ_f is convex and balanced.*
(ii) *If $g \in \Lambda_f$ and c is a constant then $g + c \in \Lambda_f$.*
(iii) *$f \in \Lambda_f$.*
(iv) *Λ_f is weakly closed.*

Proof We show only *(iv)*, the other properties being easy to verify. Assume (g_α) is a net in Λ_f, such that $g_\alpha \to g$, and $u_0 \in B_1$. Then $\lim | < \overline{g_\alpha}k, u_0 > | = | < \bar{g}k, u_0 > |$ and $| < \overline{g_\alpha}k, u_0 > | \leq \sup\{| < \bar{f}\ k, u > |, u \in B_1\}$ and thus $| < \bar{g}k, u_0 > | \leq \sup\{| < \bar{f}\ k, u > |, u \in B_1\}$. We get $\sup\{| < \bar{g}\ k, u > |, u \in B_1\} \leq \sup\{| < \bar{f}\ k, u > |, u \in B_1\}$ for any k in H^∞. □

Using this proposition we obtain the following result

Theorem 2 *Let (λ_n) be a sequence of complex numbers such that $\sum |\lambda_n| \leq 1$, then the operator $T_{z^q+\sum_0^q \lambda_n \frac{n+1}{q+1}\overline{z^n}+\sum_{q+1}^\infty \frac{q}{n}\lambda_n\overline{z^n}}$ is hyponormal.*

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