1- Let V be vector space. Then $W = \{0\}$ is vector subspace of V. Our goal is to find The basis of W and $Dim(W)$. For that :

Let $W' = \{\lambda v : \lambda \in \mathbb{R}\}$ where $v \neq 0$. It is easy to show that $W' \triangledown V$. Clearly $S = \{v\}$ spans and Linearly indep for W'. Hence $Dim(W') = 1$.

Now, we have $\{0\} \triangledown \underbrace{W'}_{Dim = 1} \triangledown V$.

Therefore, $Dim(\{0\}) = 0$, and hence $basis(\{0\}) = \phi$ ■

** Don't forget : If $W \triangledown V$ (proper subspace), then $Dim(W) < Dim(V)$.

If $W \triangledown V$ and $Dim(W) = Dim(V)$ then $W = V$.

** Also, DON'T forget : For $S = \{u_1, u_2\}$, S is Linear indep iff $u_1 \neq \lambda u_2$ for all $\lambda \in \mathbb{R}$.

for example : In $\mathbb{R}^2$, $S = \{(1,2), (3,4)\}$ is Linearly indep- because $(1,2) \neq \lambda(3,4)$ for all $\lambda \in \mathbb{R}$) ■

(Q) Let $AX = B$ and $AX = B'$ be two systems with the same Coficients matrix. If $AX = B$ has unique Solution and $AX = B'$ has no-solution. Show that $S = \{B, B'\}$ is Linearly indep ?

<u>Sol</u>: Suppose the converse, $S = \{B, B'\}$ is Linearly dep- $\Leftrightarrow \lambda B = B'$. So, $AX = B' \Leftrightarrow$ $AX = \lambda B \Leftrightarrow$ The system has unique Solution because $AX = B$ has unique Solution (contradiction) $\Rightarrow$ $\{B, B'\}$ is L. indep ■

(Q) Let $S = \{u_1, u_2, u_3, u_4, u_5\}$ be a set in vector space V where $u_2 - 2u_3 = 5u_1 + 2u_4$. Show that S is L. Dep ?

<u>Sol</u> : Clear that $u_2 = 5u_1 + 2u_4 + 2u_3 + 0u_5$. Therefore, u_2 is written by other Vectors in $S \Rightarrow S$ is L. Dep ■

(2) (Q) Let $V = \mathbb{R}^3$ and $B = \{u_1, u_2, u_3\}$ be a basis. If $B' = \{v_1, v_2, v_3\}$ where $u_1 = v_2 - v_3$, $u_2 = 2v_1 + v_2$ and $u_3 = v_1 - v_3 + 2v_2$. Show that B' is a basis of $\mathbb{R}^3$?

Sol: (1) $|B'| = 3 = \text{Dim}(\mathbb{R}^3)$.

(2) Take any $v \in \mathbb{R}^3 \Rightarrow v = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3$ because B is basis

$$\Rightarrow v = \lambda_1 (v_2 - v_3) + \lambda_2 (2v_1 + v_2) + \lambda_3 (v_1 - v_3 + 2v_2)$$

$$= \underbrace{(2\lambda_2 + \lambda_3)}_{\in \mathbb{R}} v_1 + \underbrace{(\lambda_1 + \lambda_2 + 2\lambda_3)}_{\in \mathbb{R}} v_2 + \underbrace{(-\lambda_1 - \lambda_3)}_{\in \mathbb{R}} v_3$$

So, $\{v_1, v_2, v_3\}$ spans $\mathbb{R}^3$. From (1), (2), $\{v_1, v_2, v_3\}$ is a basis to $\mathbb{R}^3$ ■

How to find the basis

(1) Let $S = \{u_1, u_2, \dots, u_n\}$ generators of $W \trianglelefteq V$ (which means $W = \text{span}(S)$). To find basis:

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} 1 & & \\ 0 & 1 & \cdots \\ 0 & 0 & \end{bmatrix}$$

The colomns with (1) leaders are formed the basis.

[So, if we knew generators, we could find the basis] ■

(2) We know that if S is L-indep which is not basis, then there exists basis $B \supseteq S$. For example $S = \{(1,3,2), (2,1,0)\}$ L-indep but not basis (why?) To find basis $B \supseteq S$, Put

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} 1 & & \cdots & \\ 0 & & & \\ 0 & & & \end{bmatrix}$$

add standered basis

then colomns with (1) leaders are basis ■

③ (3) basis of vector subspace $W = span(S)$. For example:

$W = \{(2a-b, a+b, a+3b) : a,b \in \mathbb{R}\} \triangleleft \mathbb{R}^3$

because $(2a-b, a+b, a+3b) = (2a, a, a) + (-b, b, 3b) = a(1,1,1) + b(-1,1,3)$. So,

$W = span(S)$ where $S = \{(1,1,1), (-1,1,3)\}$.

Notice that $(1,1,1) \neq \lambda(-1,1,3)$ for all $\lambda \in \mathbb{R}$. Hence, S is L-indep $\Rightarrow$ S basis to W ∎

(4) basis of the space of solutions of homogenous system $AX=0$, for example:

Let $S = \left\{ \begin{bmatrix} t/2 \\ t \\ s \\ \frac{t+s}{3} \end{bmatrix} ; t,s \in \mathbb{R} \right\}$ be the set of sol- of $AX=0$. To find the basis:

(1) find the generators:

$$\begin{bmatrix} t/2 \\ t \\ s \\ \frac{t+s}{3} \end{bmatrix} = \begin{bmatrix} t/2 \\ t \\ 0 \\ t/3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ s \\ s/3 \end{bmatrix} = t \begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 1/3 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1/3 \end{bmatrix}$$

So, the set of generators $\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1/3 \end{bmatrix} \right\}$

(2) find the basis from the set of generators:

since $\begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 1/3 \end{bmatrix} \neq \lambda \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1/3 \end{bmatrix}$ for all $\lambda \in \mathbb{R}$. Therefore

the set of <u>two</u> generators is L-indep. Hence

$\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1/3 \end{bmatrix} \right\}$ is basis of the space of solutions of $AX=0$ ∎

④ Remark Let A be a matrix.

* Col (A) = Coloumn space = the space generated by coloumns.
* Row (A) = Row space = the space generated by rows.
* N(A) = null (A) = the space of solutions of $AX = 0$

★ Rank (A) = Dim (Col(A)) = Dim (Row(A))

To find basis of Col(A): Do REF for A, and then coloumns with (1) Leader are basis.

To find basis of Row(A): Do REF for A, and then rows with (1) leader are basis

★ nullity (A) = Dim (N(A)) (Notice that we have provided How to get basis of set of sol of $AX = 0$)

★ Rules:

(1) Nullity (A) + Rank (A) = number of columns of A

(2) Rank (A) = Rank (A^t)

(Q) Let $[v]_B = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ where $v = (1, 0, -1)$.
If $B = \{u, (1,-1,1), (0,1,3)\}$. Find u?

Sol $v = 1u + 2(1,-1,1) - (0,1,3)$

$(1, 0, -1) = (x, y, z) + (2, -2, 1) + (0, -1, -3)$
$= (x+2, y-3, z-2)$

$\Rightarrow x = -1 \wedge y = 3 \wedge z = 1$

Hence $u = (-1, 3, 1)$ ■