Contents

1 Measure Theory

1 Review on Riemann Integral

1.1 Definition of the Riemann Integral

Definition 1.1
A finite ordered set $\sigma = \{x_0, \ldots, x_n\}$ *is called a partition of the interval* $[a,b]$ *if* $a = x_0 < \ldots < x_n = b$. *The interval* $[x_j, x_{j+1}]$ *is called the* j^{th} *subinterval of* σ.

Definition 1.2
Let $f\colon [a,b] \longrightarrow \mathbb{R}$ *be a bounded function. Define*

$$M_j = \operatorname*{Sup}_{x\in[x_j,x_{j+1}]} f(x), \qquad m_j = \inf_{x\in[x_j,x_{j+1}]} f(x),$$

$$S(f,\sigma) = \sum_{j=0}^{n-1} M_j(x_{j+1} - x_j) \tag{1.1}$$

and

$$s(f,\sigma) = \sum_{j=0}^{n-1} m_j(x_{j+1} - x_j). \tag{1.2}$$

$S(f,\sigma)$ *and* $s(f,\sigma)$ *are called respectively the upper sum and the lower sum of* f *on the partition* σ. *Note that* $s(f,\sigma) \leq S(f,\sigma)$.

Definition 1.3
We say that a partition σ_1 *is finer than the partition* σ_2 *if as sets* $\sigma_2 \subset \sigma_1$.

Proposition 1.4
If σ_1 *is finer than* σ_2 *and* $f\colon [a,b] \longrightarrow \mathbb{R}$ *is a bounded function, then*

$$s(f,\sigma_2) \leq s(f,\sigma_1) \leq S(f,\sigma_1) \leq S(f,\sigma_2) \tag{1.3}$$

Proof .
By induction, it suffices to prove the equation 1.3 for $\sigma_1 = \sigma_2 \cup \{\alpha\}$, with $\alpha \in]x_j, x_{j+1}[$. We remark that:

$$M'_j = \operatorname*{Sup}_{x\in[x_j,\alpha]} f(x) \le M_j, \qquad M''_j = \operatorname*{Sup}_{x\in[\alpha,x_{j+1}]} f(x) \le M_j,$$

$$M_j \ge M'_j = \operatorname*{Sup}_{x\in[x_j,\alpha]} f(x), \qquad M_j \ge M''_j = \operatorname*{Sup}_{x\in[\alpha,x_{j+1}]} f(x).$$

$$m_j \le m'_j = \inf_{x\in[x_j,\alpha]} f(x) \qquad \text{and} \quad m_j \le m''_j = \inf_{x\in[\alpha,x_{j+1}]} f(x).$$

Then

$$\begin{aligned} S(f,\sigma_1) &= \sum_{k=1}^{j-1} M_k(x_{k+1}-x_k)) + M'_j(\alpha - x_j) + M''_j(x_{j+1}-\alpha) + \sum_{k=j+1}^{n-1} M_k(x_{k+1}-x_k)) \\ &\le S(f,\sigma_2). \end{aligned}$$

and

$$\begin{aligned} s(f,\sigma_1) &= \sum_{k=1}^{j-1} m_k(x_{k+1}-x_k)) + m'_j(\alpha - x_j) + m''_j(x_{j+1}-\alpha) + \sum_{k=j+1}^{n-1} m_k(x_{k+1}-x_k)) \\ &\ge s(f,\sigma_2). \end{aligned}$$

Proposition 1.5
If $f\colon [a,b] \longrightarrow \mathbb{R}$ is a bounded function and σ_1, σ_2 are two partitions of the interval $[a,b]$, then $s(f,\sigma_1) \le S(f,\sigma_2)$.

Proof .
$s(f,\sigma_1) \le s(f,\sigma_1 \cup \sigma_2) \le S(f,\sigma_2)$.

Definition 1.6
Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a bounded function. If we denote $K([a,b])$ the set of partitions of $[a,b]$, then we define the upper integral of f on the interval $[a,b]$ by:

$$S(f) = \inf_{\sigma\in K([a,b])} S(f,\sigma)$$

and the lower integral of f on the interval $[a,b]$ by:

$$s(f) = \operatorname*{Sup}_{\sigma\in K([a,b])} s(f,\sigma)$$

Definition 1.7
Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a bounded function. We say that f is Riemann integrable on the interval $[a,b]$ if $S(f) = s(f)$.
If f is Riemann integrable on the interval $[a,b]$, we denote $\displaystyle\int_a^b f(x)dx = S(f) = s(f)$ which called the integral of f on the interval $[a,b]$.
The set of Riemann integrable functions on the interval $[a,b]$ is denoted by $\mathscr{R}([a,b])$.

Examples .

1. If $\sigma = \{x_0 = a, \dots, x_n = b\}$ is a partition of the interval $[a, b]$ and $f : [a, b] \longrightarrow \mathbb{R}$ the function defined by $f(x) = c_j$ on the interval $[x_j, x_{j+1}[$ for $j = 0, \dots, n-1$ and $f(b) = 0$, then f is Riemann integrable on $[a, b]$ and $\displaystyle\int_a^b f(x)dx = \sum_{j=0}^{n-1}(x_{j+1} - x_j)c_j$.
2. Let $f = \chi_{\mathbb{Q}\cap[0,1]}$ defined on $[0, 1]$ and let $\sigma = \{x_0 = 0, \dots, x_n = 1\}$ any partition of the interval $[0, 1]$. Then $S(f, \sigma) = 1$ and $s(f, \sigma) = 0$. Hence f is not Riemann integrable on $[0, 1]$.

1.2 Criterions of Integrability

Theorem 1.8 *(Riemann's criterion)*
Let $f : [a, b] \longrightarrow \mathbb{R}$ be a bounded function. The following statements are equivalent
i) f is Riemann-integrable.
ii) $\forall \varepsilon > 0$; there exists a partition σ such that $S(f, \sigma) - s(f, \sigma) \le \varepsilon$.

Proof .
NC: If $S(f) = s(f)$, then $\forall\ \varepsilon > 0$, there exists a partition σ such that $0 \le s(f) - s(f, \sigma) \le \frac{\varepsilon}{2}$ and there exists a partition σ' such that $0 \le S(f, \sigma') - S(f) \le \frac{\varepsilon}{2}$. Then $0 \le S(f, \sigma \cup \sigma') - S(f) \le S(f, \sigma') - S(f) \le \frac{\varepsilon}{2}$. In the same way $0 \le s(f) - s(f, \sigma \cup \sigma') \le s(f) - s(f, \sigma) \le \frac{\varepsilon}{2}$. It follows that $S(f, \sigma \cup \sigma') - s(f, \sigma \cup \sigma') \le \varepsilon$.
SC: $s(f, \sigma) \le s(f) \le S(f, \sigma)$ and $s(f, \sigma) \le S(f) \le S(f, \sigma)$, then $0 \le S(f) - s(f) \le S(f, \sigma) - s(f, \sigma) \le \varepsilon$, for all $\varepsilon > 0$. It follows that $S(f) = s(f)$.

□

Definition 1.9
If $\sigma = \{x_0, \dots, x_n\}$ is a partition of the interval $[a, b]$, we define the norm of σ by:

$$||\sigma|| = \operatorname*{Sup}_{0\le j\le n-1} x_{j+1} - x_j.$$

Theorem 1.10 *(Darboux's criterion)*
Let $f : [a, b] \longrightarrow \mathbb{R}$ be a bounded function. The following statements are equivalent
i) f is Riemann-integrable.
ii) For all $\varepsilon > 0$; there exists $\delta > 0$ such that for all partition of the interval $[a, b]$ such that if $||\sigma|| \le \delta$ then $S(f, \sigma) - s(f, \sigma) \le \varepsilon$.

Proof .
From the theorem (1.8) the sufficient condition is obvious.
NC: assume that f is not constant. We know that there exists a partition $\sigma = \{x_0, \dots, x_n\}$ such that $S(f, \sigma) - s(f, \sigma) \le \varepsilon$. We denote $M = O(f, A) = \operatorname{Sup}_{x\in[a,b]} f(x) - \inf_{x\in[a,b]} f(x)$ called the oscilation of f on the interval $[a, b]$. Let $\alpha_1 = \dfrac{\varepsilon}{nM}$, $\alpha_2 = \displaystyle\inf_{0\le j\le n-1}(x_{j+1} - x_j)$ and $\alpha = \min(\alpha_1, \alpha_2)$.

Let $\sigma' = (y_0 = a, \dots, y_m = b)$ a partition of $[a,b]$ of norm $||\sigma'|| < \alpha$. There exists at most n intervals $]y_{j-1}, y_j[$ which contain some points x_j. The others are contained in the intervals $]x_{k-1}, x_k[$. We denote

$$\begin{aligned} M'_j &= \operatorname*{Sup}_{x\in]y_j,y_{j+1}[} f(x), \quad M_j = \operatorname*{Sup}_{x\in]x_j,x_{j+1}[} f(x), \\ m'_j &= \inf_{x\in]y_j,y_{j+1}[} f(x) \quad \text{and } m_j = \inf_{x\in]x_j,x_{j+1}[} f(x). \end{aligned}$$

$$\begin{aligned} D(f,\sigma') - d(f,\sigma') &= \sum_{]y_j,y_{j+1}[\subset]x_i,x_{i+1}[} (y_{j+1}-y_j)(M'_j - m'_j) \\ &+ \sum_{x_i \in]y_j,y_{j+1}[} (y_{j+1}-y_j)(M'_j - m'_j) \end{aligned}$$

It follows that

$$\begin{aligned} D(f,\sigma') - d(f,\sigma') &\le \sum_{i=0}^{n-1} (x_{i+1}-x_i)(M_i - m_i) + n\alpha M \\ &= D(f,\sigma) - d(f,\sigma) + n\alpha M \le 2\varepsilon. \end{aligned}$$

□

Definition 1.11
Let $\sigma = \{x_0, \dots, x_n\}$ be a partition of the interval $[a,b]$. We say that $\alpha = \{\alpha_0, \dots, \alpha_{n-1}\}$ is a mark of σ if $\forall 0 \le j \le n-1$, $\alpha_j \in [x_j, x_{j+1}]$.
We define

$$S(f,\sigma,\alpha) = \sum_{j=0}^{n-1} f(\alpha_j)(x_{j+1}-x_j)$$

called the Riemann sum of f on σ with respect to the mark α.

As particular case, if f is Riemann integrable on the interval $[a,b]$, the sequence S_n defined by:

$$S_n = \frac{b-a}{n} \sum_{k=1}^{n} f(a + k\frac{b-a}{n})$$

converges to $\int_a^b f(x)dx$. (S_n is called a Riemann sum of f on the interval $[a,b]$).

1.3 Properties of the Riemann Integrals

Properties .

i) Linearity: $\int_a^b \alpha(f+\beta g)(x)dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx.$

ii) If $f \geq 0$, then $\int_a^b f(x)dx \geq 0$.

iii) If $f \leq g$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

iv) $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$.

v) If $m \leq f(x) \leq M$, for all $x \in [a,b]$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

vi) $\forall c \in]a,b[$; f is Riemann integrable on $[a,b]$ if and only if f is Riemann integrable on $[a,c]$ and f Riemann integrable on $[c,b]$ and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

(This identity is called the Chasles identity)

Proof .

We prove only the property vi), the others the other properties are left to the reader. Assume that f is Riemann integrable on the interval $[a,b]$, then $\forall\ \varepsilon > 0$, there exists a partition σ of $[a,b]$ such that $S(f,\sigma) - s(f,\sigma) \leq \varepsilon$. Let $\sigma' = \sigma \cup \{c\}$; then $S(f,\sigma') - s(f,\sigma') \leq S(f,\sigma) - s(f,\sigma) \leq \varepsilon$. We write $\sigma' = \sigma_1 \cup \sigma_2$, with σ_1 a partition of $[a,c]$ with points of σ' contained in $[a,c]$ and σ_2 a partition of $[c,b]$ with points of σ' $[c,b]$. It follows that $S(f,\sigma_1) - s(f,\sigma_1) \leq \varepsilon$ and $S(f,\sigma_2) - s(f,\sigma_2) \leq \varepsilon$. Then f is Riemann integrable on $[a,c]$ and on $[c,b]$.

If f is Riemann integrable on $[a,c]$ and on $[c,b]$, then $\forall\ \varepsilon > 0$, there exists a partition σ_1 of $[a,c]$ and a partition σ_2 of $[c,b]$ such that $S(f,\sigma_1) - s(f,\sigma_1) \leq \varepsilon$ and $S(f,\sigma_2) - s(f,\sigma_2) \leq \varepsilon$. We put $\sigma = \sigma_1 \cup \sigma_2$. σ is a partition of the interval $[a,b]$ and $S(f,\sigma) - s(f,\sigma) \leq 2\varepsilon$, which proves that f is Riemann integrable on the interval $[a,b]$.

We prove now that

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx.$$

We put $I = \int_a^b f(x)\ dx$, $I_1 = \int_a^c f(x)\ dx$ and $I_2 = \int_c^b f(x)\ dx$.

$\forall\ \varepsilon > 0$, there exists $\alpha > 0$ such that for all partitions σ of $[a,b]$, σ_1 of $[a,c]$ and σ_2 of $[c,b]$, with ($|\sigma| < \alpha$, $|\sigma_1| < \alpha$ and $|\sigma_2| < \alpha$ we have:

$$|S(f,\sigma) - I| \leq \varepsilon, \qquad |S(f,\sigma_1) - I_1| \leq \varepsilon$$

and

$$|S(f,\sigma_2) - I_2| \leq \varepsilon.$$

We take the partition $\sigma' = \sigma_1 \cup \sigma_2$, then $|\sigma'| < \alpha$ and $|S(f,\sigma') - I| \leq \varepsilon$. in the same way $|S(f,\sigma') - I_1 - I_2| \leq |S(f,\sigma_1) - I_1| + |S(f,\sigma_2) - I_2| \leq 2\varepsilon$. Then $I = I_1 + I_2$. □

Remark .
If $b < a$, we denote $\displaystyle\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$.

Theorem 1.12
Let $f\colon [a,b] \longrightarrow [c,d]$ be a Riemann integrable function and let $\varphi\colon [c,d] \longrightarrow \mathbb{R}$ be a continuous function. Then $\varphi \circ f$ is Riemann integrable.

Proof .
Let $\varepsilon > 0$, we which construct a partition $\sigma = (x_0 = a, x_1, \ldots, x_n = b)$ of the interval $[a,b]$ such that: $S(\varphi \circ f, \sigma) - s(\varphi \circ f, \sigma) < \varepsilon$.
the function φ is uniformly continuous on $[c,d]$ and bounded, then there exists $M > 0$ such that $|\varphi(x)| \leq M$, $\forall x \in [c,d]$ and if $\varepsilon' = \dfrac{\varepsilon}{2M + (b-a)}$, there exists $0 < \alpha < \varepsilon'$ such that for $|x - y| < \alpha$, $|\varphi(x) - \varphi(y)| \leq \varepsilon'$, for $x, y \in [c,d]$.
As f is Riemann integrable on the interval $[a,b]$, there exists a partition $\sigma = (x_0 = a, x_1, \ldots, x_n = b)$ of $[a,b]$ such that:

$$S(f,\sigma) - s(f,\sigma) < \alpha^2. \tag{1.4}$$

Let $M_j = \text{Sup}\{f(x);\ x \in [x_j, x_{j+1}]\}$, $m_j = \inf\{f(x);\ x \in [x_j, x_{j+1}]\}$, $\tilde{M}_j = \text{Sup}\{\varphi \circ f(x);\ x \in [x_j, x_{j+1}]\}$, $\tilde{m}_j = \inf\{\varphi \circ f(x);\ x \in [x_j, x_{j+1}]\}$.
we denote $J_1 = \{0 \leq j \leq n-1;\ M_j - m_j < \alpha$ and $J_2 = \{0 \leq j \leq n-1;\ M_j - m_j \geq \alpha$.
If $j \in J_1$, then from the uniform continuity of $\varphi \circ f$, we have $|\varphi \circ f(x) - \varphi \circ f(y)| < \varepsilon'$ for all $x, y \in [x_j, x_{j+1}]$, which gieves that $\tilde{M}_j - \tilde{m}_j \leq \varepsilon'$, then

$$\sum_{j \in J_1} (\tilde{M}_j - \tilde{m}_j)(x_{j+1} - x_j) \leq \varepsilon'(b-a). \tag{1.5}$$

It follows from the equation 1.4,

$$\alpha^2 > \sum_{j \in J_2} (M_j - m_j)(x_{j+1} - x_j) \geq \alpha \sum_{j \in J_2} (x_{j+1} - x_j).$$

Then $\displaystyle\sum_{j \in J_2} (x_{j+1} - x_j) < \alpha < \varepsilon'$ and as $\tilde{M}_j - \tilde{m}_j \leq 2M$, we have:

$$\sum_{j \in J_2} (\tilde{M}_j - \tilde{m}_j)(x_{j+1} - x_j) \leq 2M \sum_{j \in J_2} (x_{j+1} - x_j) < 2M\varepsilon'. \tag{1.6}$$

It follows from (1.5) and (1.6) that

$$D(\varphi \circ f, \sigma) - d(\varphi \circ f, \sigma) = \sum_{j=0}^{n-1} (\tilde{M}_j - \tilde{m}_j)(x_{j+1} - x_j) \leq \varepsilon'((b-a) + 2M) = \varepsilon.$$

$\square$

Theorem 1.13
Let $f\colon [a,b] \longrightarrow [c,d]$ be a Riemann integrable function, then the function F defined by

$$F(x) = \int_a^x f(t)dt$$

is continuous.
If f is continuous in the point c, then F is differentiable in c and $F'(c) = f(c)$.

Theorem 1.14 *(The fundamental theorem of calculus)*
Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a differentiable function and f' is Riemann integrable, then

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Proof .
Let $\sigma = \{x_0, \ldots, x_n\}$ be any partition of $[a,b]$, By the Mean-Value Theorem applied to f on $[x_{j-l}, x_j]$, there is $c_j \in [x_{j-l}, x_j]$ such that $f(x_j) - f(x_{j-1}) = f'(c_j)(x_j - x_{j-1})$. Thus

$$\sum_{j=1}^{n} f'(c_j)(x_j - x_{j-1}) = \sum_{j=1}^{n} f(x_j) - f(x_{j-1}) = f(b) - f(a).$$

The sum $\sum_{j=1}^{n} f'(c_j)(x_j - x_{j-1}) = S(f, \sigma, w)$, with $w = (c_1, \ldots, c_n)$ the mark on the partition σ given by the Mean-Value Theorem. Let new a sequence of partition σ_m of [a, b], each marked in this fashion and such that $||\sigma_n||$ converges to zero. As f' is Riemann integrable, the sequence $S(f, \sigma_m, w_m)$ converges to $\int_a^b f'(x)dx$, then

$$\int_a^b f'(x)dx = f(b) - f(a).$$

$\square$

2 Algebra and $\sigma-$Algebra

2.1 Elementarily Operations on Sets

In all that follow, X will denote a nonempty set. We denote by $\mathscr{P}(X)$ the collection of subsets of X. If A and B are in $\mathscr{P}(X)$, we put: $A \setminus B := \{x \in A \text{ and } x \notin B\} = A \cap B^c$. $A \Delta B = (A \setminus B) \bigcup (B \setminus A)$ called symmetric difference of B from A, and if $A = X$, $X \setminus B = B^c$

We show easily prove that

$$A \setminus B = A \setminus (A \cap B) = (A \cup B) \setminus B, \qquad A\Delta B = (A \cup B) \setminus (A \cap B).$$
$$(A \setminus B) \cap (C \setminus D) = (A \cap C) \setminus (B \cup D), \qquad (A \cap B)\Delta(A \cap C) = A \cap (B\Delta C)$$

Definition 2.1 *Characteristic functions of sets —*
For any subset $A \in \mathscr{P}(X)$; we denote χ_A the characteristic function (or the indicator function) of A defined by $\chi_A(x) = 1;\ \forall\ x \in A$ and $\chi_A(x) = 0;\ \forall\ x \notin A$.

Properties .
All the operations on sets can be translated easily in term of characteristic functions of sets by the correspondence: $A \longrightarrow \chi_A$ when $A \in \mathscr{P}(X)$. We have the following relations:

1. $A \subset B \iff \chi_A \leq \chi_B$.
2. $C = A \cap B \iff \chi_C = \chi_A.\chi_B$.
3. $B = A^c \iff \chi_B = 1 - \chi_A$.
4. $C = A \cup B \iff \chi_C = \chi_A + \chi_B - \chi_A.\chi_B$.
5. $C = A \setminus B \iff \chi_C = \chi_A(1 - \chi_B)$.
6. $C = A\Delta B \iff \chi_C =\mid \chi_A - \chi_B \mid$.
7. If $(A_n)_{n\in\mathbb{N}}$ is a sequence of subsets of X, then

$$\chi_{\bigcap_n A_n} = \inf_n \chi_{\{\bigcap_{p\leq n} A_p\}} = \lim_{n\to+\infty} \prod_{k=1}^{n} \chi_{A_k}.$$

$$\chi_{\bigcup_n A_n} = \operatorname{Sup}_n \chi_{\{\bigcup_{p\leq n} A_p\}} = \lim_{n\to+\infty} \chi_{\{\bigcup_{p\leq n} A_p\}}.$$

8. If $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ are two sequences of subsets of X, then

$$(\bigcup_{n=1}^{+\infty} A_n)\Delta(\bigcup_{n=1}^{+\infty} B_n) \subset \bigcup_{n=1}^{+\infty}(A_n\Delta B_n).$$

Definition 2.2
A family of subsets of X indexed by the set of indexes I, is a mapping $j \longmapsto X(j)$ from I in $\mathscr{P}(X)$. We denote $X(j) = X_j$ and the family is denoted by $(X_j)_{j\in I}$.

1. *The family $(X_j)_{j\in I}$ is called finite (resp countable) if I is finite (resp countable).*
2. *A family $(X_j)_j$, is called pairwise disjoint (or simply disjoints) if $X_j \cap X_k = \emptyset$, $\forall j \neq k$.*

Definition 2.3

1. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of real functions on X. We define*

$$(\lim\ \mathrm{Sup})_{n\to+\infty}\ f_n\ =\overline{\lim}_{n\to+\infty}f_n\ =\ \inf_n\ \mathrm{Sup}\ \{f_m; m\geq n\}$$

and

$$(\lim\ \inf)_{n\to+\infty}\ f_n\ =\underline{\lim}_{n\to+\infty}f_n\ =\ \mathrm{Sup}_n\ \inf\ \{f_m; m\geq n\}.$$

These two limits are always exist and can take the values $\pm\infty$.

2. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of subsets of X. We define*

$$\overline{\lim}_{n\to+\infty}A_n=\bigcap_{n=1}^{+\infty}\bigcup_{m=n}^{+\infty}A_m \text{ and } \underline{\lim}_{n\to+\infty}A_n=\bigcup_{n=1}^{+\infty}\bigcap_{m=n}^{+\infty}A_m.$$

$\overline{\lim}_{n\to+\infty}A_n$ *(or $\limsup\limits_{n\to+\infty}A_n$) is called the limit superior and* $\underline{\lim}_{n\to+\infty}A_n$ *(or $\liminf\limits_{n\to+\infty}A_n$) is called the limit inferior.*

Note that $(\bigcup\limits_{m=n}^{+\infty}A_m)_n$ is a decreasing sequence of subsets of X and t follows that $\lim\limits_{n\to+\infty}\bigcup\limits_{m=n}^{+\infty}A_m=\bigcap\limits_{n=1}^{+\infty}\bigcup\limits_{m=n}^{+\infty}A_m$ exists. Similarly $(\bigcap\limits_{m=n}^{+\infty}A_m)_n$ is an increasing sequence of subsets of X and this implies that $\lim\limits_{n\to+\infty}\bigcap\limits_{m=n}^{+\infty}A_m=\bigcup\limits_{n=1}^{+\infty}\bigcap\limits_{m=n}^{+\infty}A_m$ exists.

The interpretation is that $\limsup_n A_n$ contains those elements of X that occur "infinitely often" in the sets A_n, and $\liminf_n A_n$ contains those elements that occur in all except finitely many of the sets A_n.

Remarks .

1. If the sequence $(f_n)_{n\in\mathbb{N}}$ converges to the function f; then $\overline{\lim}_{n\to+\infty}f_n=\underline{\lim}_{n\to+\infty}f_n=f$.

2. $\overline{\lim}_{n\to+\infty}A_n$ is the set of the elements of X which are in an infinite sets of A_n. Thus

$$\overline{\lim}_{n\to+\infty}A_n\ =\ \{x\in X:\sum_{n=1}^{\infty}\chi_{A_n}(x)\ =+\infty\}.$$

3. $\underline{\lim}_{n\to+\infty}A_n$ is the set of elements of X which are in all the A_n except a finite number and thus

$$\underline{\lim}_{n\to+\infty}A_n=\{x\in X:\sum_{n=1}^{\infty}\chi_{A_n^c}(x)\ <+\infty\}.$$

4. $\underline{\lim}_{n\to+\infty} A_n \subset \overline{\lim}_{n\to+\infty} A_n$.

5. $\chi_{\overline{\lim}_{n\to+\infty} A_n} = \overline{\lim}_{n\to+\infty} \chi_{A_n}$.

6. $\chi_{\underline{\lim}_{n\to+\infty} A_n} = \underline{\lim}_{n\to+\infty} \chi_{A_n}$.

Example .
Let $X = \mathbb{R}$ and let a sequence $(A_n)_n$ of subsets of $\mathbb{R}$ be defined by $A_{2n+1} = [0, \frac{1}{2n+1}]$, and $A_{2n} = [0, 2n]$. Then

$$\underline{\lim}_{n\to+\infty} A_n = \{x \in X;\ x \in A_n \text{for all but finitely many } n \in \mathbb{N}\} = \{0\}$$

and

$$\overline{\lim}_{n\to+\infty} A_n = \{x \in X;\ x \in A_n \text{for infinitely many } n \in \mathbb{N}\} = [0, \infty[.$$

2.2 General Properties of $\sigma-$Algebra

Definition 2.4
Let $\mathscr{A}$ be a collection of subsets of X. $\mathscr{A}$ is called an algebra or a field if:

1. *$X \in \mathscr{A}$;*

2. *(Closure under complement) if $A \in \mathscr{A}$, then $A^c \in \mathscr{A}$;*

3. *(Closure under finite intersection) if $A_1, \dots, A_n \in \mathscr{A}$, then $\bigcap_{j=1}^{n} A_j \in \mathscr{A}$.*
   $\mathscr{A}$ is called a $\sigma-$algebra or a $\sigma-$field if in addition

4. *(Closure under countable intersection) if $(A_j)_{j\in\mathbb{N}}$ are in $\mathscr{A}$, then $\bigcap_{j=1}^{+\infty} A_j \in \mathcal{A}$.*

If $\mathscr{A}$ is a $\sigma-$algebra, the pair $(X, \mathscr{A})$ is called a **measurable space***, and the subsets in $\mathscr{A}$ are called the measurable sets.*

Remarks .
By complementarity

1. If $\mathscr{A}$ is an algebra, then $\emptyset \in \mathscr{A}$.

2. (Closure under finite union) If $\mathscr{A}$ is an algebra and $A_1, \dots, A_n \in \mathscr{A}$,, then $\displaystyle\bigcup_{j=1}^{n} A_j \in \mathscr{A}$.

3. (Closure under countable union) If $\mathscr{A}$ is a $\sigma-$algebra and $(A_j)_{j\in\mathbb{N}}$ in $\mathscr{A}$, then $\displaystyle\bigcup_{j=1}^{+\infty} A_j \in \mathscr{A}$.

2.3 Examples

Example 1 :
$\mathscr{A} = \{\emptyset, X\}$ is an algebra and a $\sigma-$algebra. This is the smallest $\sigma-$algebra in $\mathscr{P}(X)$.

Example 2 :
$\mathscr{A} = \mathscr{P}(X)$ is an algebra and a $\sigma-$algebra. This is the largest $\sigma-$algebra in $\mathscr{P}(X)$.

Example 3 :
Let $\mathscr{F} = \{A, B, C\}$ be a partition of X. The set

$$\mathscr{A} = \{\emptyset, X, A, B, C, A \cup B = C^c, A \cup C = B^c, B \cup C = A^c\}.$$

is a $\sigma-$algebra.

Example 4 :

1. Let $X = \mathbb{R}$ and $\mathscr{A}$ the collection of subsets A of X such that either A or A^c is countable or $\emptyset$. $\mathscr{A}$ is a $\sigma-$algebra. In fact let $(A_j)_{j\in\mathbb{N}}$ be a sequence of elements of $\mathscr{A}$.

   If there exists p such that A_p is countable, then $\cap_{j=1}^{+\infty} A_j \subset A_p$ is countable and $\cap_{j=1}^{+\infty} A_j \in \mathscr{A}$.

   If every A_j is not countable, then all A_k^c are countable, and then $\cup_{j=1}^{+\infty} A_j^c$ is a countable subset of $\mathbb{R}$ and then $\cap_{j=1}^{+\infty} A_j \in \mathscr{A}$.

2. Let X be an infinite set and let $\mathscr{A}$ the collection of subsets A of X such that either A or A^c is finite, then $\mathscr{A}$ is an algebra but it is not a σ-algebra.

2.4 $\sigma-$Algebra Generated by a Subset $P \subset \mathscr{P}(X)$

Definition 2.5
Let X be a non empty set and $\mathscr{A}_1$, $\mathscr{A}_2$ two $\sigma-$algebras on X. We say that $\mathscr{A}_1$ is finer then $\mathscr{A}_2$ if any element of $\mathscr{A}_1$ is an element of $\mathscr{A}_2$. In this case we write $\mathscr{A}_1 \subset \mathscr{A}_2$.

Remark .
Any intersection of algebras (resp $\sigma-$ algebra) is an algebra (resp $\sigma-$ algebra).

Definition 2.6
Let X be a non empty set and $\mathcal{B} \subset \mathscr{P}(X)$. There exists a smallest algebra (resp $\sigma-$algebra) denoted by $\mathcal{A}$ ($\mathcal{B}$), (resp $\sigma(\mathcal{B})$) that contain $\mathcal{B}$. This algebra (resp $\sigma-$algebra) is called the algebra (resp $\sigma-$algebra) generated by $\mathcal{B}$.
$\mathcal{A}(\mathcal{B})$ (resp $\sigma(\mathcal{B})$) is the intersection of all the algebras on X (resp $\sigma-$algebra) containing $\mathcal{B}$. So this is the smallest algebra (resp $\sigma-$algebra) which contains $\mathcal{B}$.

Example 1 :
Let A be a subset of X with $A \neq \emptyset$ and $A \neq X$. The $\sigma-$algebra generated by $\{A\}$ is $\{\emptyset, X, A, A^c\}$.

Example 2 :
Let X be a non empty set and $(P_j)_{j\in J}$ is a finite partition of X. The algebra generated by (P_j) is constituted by the subsets of the form $\bigcup_{j\in I} P_j$, where $I \in \mathscr{P}(J)$, and the mapping

$$I \longmapsto \bigcup_{j\in I} P_j$$

is an isomorphism of $\mathscr{P}(J)$ in the algebra.
We remark that if J contains n elements, then the algebra contains 2^n elements.

Exercise .
Let X be an arbitrary nonempty set, and let $\mathscr{A}$ be the family of all subsets $A \subset X$ such that either A or $X \setminus A$ is countable. Show that $\mathscr{A}$ is the σ-algebra generated by the singleton sets $S = \{\{x\}; x \in X\}$.

Exercise .
Let X be a non empty set and $\mathcal{C} \subset \mathscr{P}(X)$. We define successively the sets:
$\mathcal{C}_1 = \{\emptyset\} \cup \{X\} \cup \{A, A^c;\ A \in \mathcal{C}\}$,
$\mathcal{C}_2$ constituted by the finite intersections of elements of $\mathcal{C}_1$,
$\mathcal{C}_3$ constituted by the finite union of elements of $\mathcal{C}_2$ that are disjoints.
Prove that $\mathcal{C}_3$ is the algebra generated by $\mathscr{C}$.

2.5 Borelian $\sigma-$Algebra in $\mathbb{R}$

If $X = \mathbb{R}$ and $\mathscr{B}$ is the $\sigma-$algebra generated by the family $\{[a, b[; (a, b) \in \mathbb{R}^2\}$. This $\sigma-$algebra is denoted by $\mathscr{B}_{\mathbb{R}}$ and called the $\sigma-$algebra of Borel subsets on $\mathbb{R}$. ($\mathscr{B}_{\mathbb{R}}$ contains all open and closed subsets of $\mathbb{R}$.) Every element of $\mathscr{B}_{\mathbb{R}}$ is called a Borel subset of $\mathbb{R}$.
We can prove easily that
$\mathscr{B}_{\mathbb{R}}$ is generated by $\{[a, b[; (a, b) \in \mathbb{R}^2\}$,
$\mathscr{B}_{\mathbb{R}}$ is generated by the family of open subsets in $\mathbb{R}$,
$\mathscr{B}_{\mathbb{R}}$ is generated by the family of closed subsets in $\mathbb{R}$,
$\mathscr{B}_{\mathbb{R}}$ is generated by $\{]a, +\infty[;\ a \in \mathbb{R}\}$,
$\mathscr{B}_{\mathbb{R}}$ is generated by $\{] - \infty, a];\ a \in \mathbb{R}\}$,

2.6 Borelian $\sigma-$Algebra in a Topological Space

Let X be a topological space and $\mathcal{A}$ be the family of the open subsets of X. Let $\mathscr{B}$ be the $\sigma-$algebra generated by the family $\mathcal{A}$. Then $\mathscr{B}$ is called the $\sigma-$algebra of Borel subsets on X and denoted by $\mathscr{B}_X$. All open and closed subsets of X are Borel subsets.
The family of the closed subsets of X generates $\mathscr{B}_X$.

2.7 Product of $\sigma-$Algebras

Definition 2.7

Let $(X_1, \mathscr{A}_1)$ and $(X_2, \mathscr{A}_2)$ be two measurable spaces. We denote by X the cartesian product $X_1 \times X_2$. A subset $R = A_1 \times A_2$ of $X_1 \times X_2$ is called a rectangle with $A_1 \in \mathscr{A}_1$ and $A_2 \in \mathscr{A}_2$. We denote by $\mathcal{R}$ the set of all rectangles in X. The product $\sigma-$algebra of $\mathscr{A}_1$ and $\mathscr{A}_2$ on X is the $\sigma-$algebra generated by $\mathcal{R}$ and will be denoted by $\mathscr{A}_1 \otimes \mathscr{A}_2$.

Remarks .
In the same way if $(X_j, \mathscr{A}_j)$, $j = 1, \dots, n$ are n measurable spaces, we define the $\sigma-$algebra $\otimes_{j=1}^n \mathscr{A}_j$ on the space $X = \prod_{j=1}^n X_j$, and for the remainder of this course, we provide the product space X with this $\sigma-$algebra.

2.8 Pull back of a $\sigma-$Algebra

Let X and X' two non empty sets, and let $f : X \longrightarrow X'$ a mapping. Let $\mathscr{B}$ be a family of subsets of X'. We define

$$f^{-1}(\mathscr{B}) = \{f^{-1}(A);\ A \in \mathscr{B}\}$$

Proposition 2.8
If $\mathscr{B}$ is a $\sigma-$algebra on X', then $f^{-1}(\mathscr{B})$ is a $\sigma-$algebra on X called the pull back of $\mathscr{B}$ by f.

Proof .
We have $f^{-1}(X') = X$ and $\bigcup_j f^{-1}(A_j) = f^{-1}(\bigcup_j A_j)$ and $(f^{-1}(A))^c = f^{-1}(A'^c)$. □

If X is a subset of X' and f is an injection of X into X', then the pull back of a $\sigma-$algebra on X' is called the **trace** of this $\sigma-$algebra on X.

Proposition 2.9
Let X and X' be two non empty sets and $f : X \longrightarrow X'$ a mapping. Let $\mathcal{B}$ be a family of subsets of X' and $\mathscr{B}$ the $\sigma-$algebra generated by $\mathcal{B}$. Then $f^{-1}(\mathscr{B})$ is the $\sigma-$algebra generated by $f^{-1}(\mathcal{B})$.

Proof .
If we denote by $\sigma(\mathcal{A})$ the $\sigma-$algebra generated by an arbitrary subset $\mathcal{A}$ of $\mathscr{P}(X)$, then we must prove that $f^{-1}(\sigma(\mathcal{B})) = \sigma(f^{-1}(\mathcal{B}))$.
As $f^{-1}(\mathcal{B}) \subset f^{-1}(\sigma(\mathcal{B}))$, then $\sigma(f^{-1}(\mathcal{B})) \subset f^{-1}(\sigma(\mathcal{B})) = f^{-1}(\mathscr{B})$.
We shall prove the inverse inclusion in the particular case when f is surjective (onto).
Let $\mathscr{A}$ be a $\sigma-$algebra on X such that $f^{-1}(\mathcal{B}) \subset \mathscr{A} \subset f^{-1}(\mathscr{B})$. Let $\mathscr{B}_1 = f(\mathscr{A}) = \{f(A); A \in \mathscr{A}\}$. The family $\mathscr{B}_1$ is closed under countable union and as f is surjective (onto) and $\mathscr{A}$ contains X then $X' \in \mathscr{B}_1$.
Let proving now that $\mathscr{B}_1$ is closed under complementarity.
For $K \in \mathscr{B}_1$, there exists $H \in \mathscr{A}$ such that $K = f(H)$. As $H \in f^{-1}(\mathscr{B})$, there exists $L \in \mathscr{B}$ such that $H = f^{-1}(L)$. Thus $K = f(f^{-1}(L))$ with $L \in \mathscr{B}$. We deduce that $K^c = f(f^{-1}(L^c))$ and as $f^{-1}(L^c) = (f^{-1}(L))^c = H^c \in \mathscr{A}$, we conclude that $K^c = f(Z)$, with $Z = H^c \in \mathscr{A}$.

It results that $\mathscr{B}_1$ is a $\sigma-$algebra. So $\mathcal{B} \subset \mathscr{B}_1 \subset \mathscr{B}$, and as $\mathscr{B}$ is the $\sigma-$algebra generated by $\mathcal{B}$, we deduce that $\mathscr{B}_1 = \mathscr{B}$.
(Let $Y \in \mathscr{B}$ then $Y \in \mathscr{B}_1$, there exists thus $Z \in \mathscr{A}$ such that $Z = f^{-1}(Y) \Rightarrow f^{-1}(Y) \in \mathscr{A}$, for any $Y \in \mathscr{B}$ where $f^{-1}(\mathscr{B}) \subset \mathscr{A}$.)
Assume now that f is injective.
We can identify X as a subset of X' and f is the canonical injection of $X \longrightarrow X'$. Let $\mathscr{A}$ be a $\sigma-$algebra such that $f^{-1}(\mathcal{B}) \subset \mathscr{A} \subset f^{-1}(\mathscr{B})$. We put

$$\mathscr{B}_1 = \{C \in \mathscr{P}(X'); C \cap X \in \mathscr{A}\}.$$

$\mathscr{B}_1$ is a $\sigma-$algebra which contain $\mathcal{B}$. So $\mathscr{B}_1 \supset \mathscr{B}$. Thus $f^{-1}(\mathscr{B}_1) \supset f^{-1}(\mathscr{B})$. The result is deduced easily.
In the general case: we put $Y = f(X)$. Let $f_1: X \longrightarrow Y$ be the mapping defined by f. Let f_2 be the canonical injection of Y into X'. $f = f_2 \circ f_1$ with f_1 surjective (onto) and f_2 injective. Let $A = f^{-1}(\mathcal{B})$ and $\mathscr{A} = f^{-1}(\mathscr{B})$. Thus $\mathscr{A} = f_1^{-1}(f_2^{-1}(\mathscr{B}))$.
From the previous result, $\sigma(f^{-1}(\mathcal{B})) = f_2^{-1}(\mathscr{B})$ is a $\sigma-$algebra generated by $f_2^{-1}(\mathcal{B})$ and $f_1^{-1}(\sigma(f^{-1}(\mathcal{B})))$ is generated by $f_1^{-1}(f_2^{-1}(\mathcal{B}))$. □

3 Measures

We wish define a non-negative set function called a measure μ on $\mathscr{P}(\mathbb{R})$ which satisfies the following conditions:

i) μ is defined on $\mathscr{P}(\mathbb{R})$

ii) For any interval I, $\mu(I) = \ell(I)$

iii) If $(E_n)_{n\in\mathbb{N}}$ is a disjoint sequence of $\mathscr{P}(\mathbb{R})$, $(E_j \cap E_k = \emptyset,\ \forall j \neq k)$, then $\mu(\bigcup_{j=1}^{+\infty} E_j) = \sum_{j=1}^{+\infty} \mu(E_j)$ (countable additivity)

iv) μ is invariant under translation, in the sens that $\mu(E+x) = \mu(E)$, $\forall x \in \mathbb{R}$ and $\forall E \subset \mathbb{R}$.

So we can not find this function defined on all $\mathscr{P}(\mathbb{R})$, but we can define this function on special subsets of $\mathscr{P}(\mathbb{R})$. (See Halmos [?])

3.1 Generalities on Measures

Definition 3.1
Let $(X, \mathscr{A})$ be a measurable space. A measure (or a positive measure) on X is a function $\mu: \mathscr{A} \to [0, \infty]$ such that:

1. $\mu(\emptyset) = 0$;
2. *(Countable additivity:) For any disjoint sequence $(A_j)_j \in \mathscr{A}$,*

$$\mu(\cup_{j=1}^{+\infty} A_j) = \sum_{j=1}^{+\infty} \mu(A_j). \tag{3.7}$$

(We mention that the term countably additive set function μ indicates that μ satisfies (3.7). We shall also use the term σ-additive set function.)

The set $(X, \mathscr{A}, \mu)$ will be called a measure space.

Examples .

1. Let X be any non empty set and let $\mathscr{A} = \mathscr{P}(X)$. For $A \in \mathscr{A}$, we define $\mu(A)$ the number of elements in A if A is finite and equal to $+\infty$ if not. μ is then a measure on $\mathscr{A}$. This measure is called the counting measure.

2. $\delta_x(A) = 1$ if $x \in A$ and 0 otherwise. The measure δ_x is called the point mass at x or the Dirac measure on x.

3. Let μ defined on $\mathscr{P}(\mathbb{R})$ by:

$$\mu(A) = \begin{cases} 0 & \text{if } A \text{ is finite} \\ \infty & \text{otherwise} \end{cases}$$

μ is finite additive but not countably additive since $\mathbb{N} = \bigcup_{j=1}^{+\infty} \{j\}$, but $\mu(\mathbb{N}) = +\infty \neq \displaystyle\sum_{j=1}^{+\infty} \mu(\{j\}) = 0$. Then μ is not a measure.

Theorem 3.2
Let μ be a measure on the measurable space $(X, \mathscr{A})$. It has the following basic properties:

1. *μ is finitely additive: For any finite subsets $A_1, \ldots, A_n \in \mathscr{A}$ of disjoints elements of $\mathscr{A}$, $\mu(\cup_{j=1}^n A_j) = \displaystyle\sum_{j=1}^{n} \mu(A_j)$.*

2. *μ is monotone: If $A, B \in \mathscr{A}$ with $A \subset B$, then $\mu(A) \leq \mu(B)$.*

3. *μ is countably subadditive: If $(A_j)_{j\in\mathbb{N}} \in \mathscr{A}$ and $A = \cup_{j=1}^{+\infty} A_j$, then*

$$\mu(A) \leq \sum_{j=1}^{+\infty} \mu(A_j).$$

4. *(Continuity from below:) If $(A_j)_j$ is an increasing sequence in $\mathscr{A}$, and $A = \cup_{j=1}^{+\infty} A_j$, then $\mu(A) = \lim_{n\to+\infty} \mu(A_n)$.*

5. *μ is subtractive: If $A, B \in \mathscr{A}$ and $A \subset B$ and $\mu(B) < +\infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$. ($\mu(A) < \infty$ suffices).*

6. *(Continuity from above:) If $(A_j)_j$ is a decreasing sequence in $\mathscr{A}$ with $\mu(A_1) < \infty$, then $\mu(A) = \lim_{n\to+\infty} \mu(A_n)$, with $A = \cap_{j=1}^{+\infty} A_j$.*

Proof .

1. This property is obvious.

2. $B = A \cup (B \setminus A)$, then $\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$. We use property property 2) of the measure definition.

3. Let $B_1 = A_1$, and $B_n = A_n \setminus \bigcup_{j=1}^{n-1} B_j$, for $n \geq 2$. The sequence $(B_n)_{n\in\mathbb{N}}$ are disjoints and $\bigcup_{n=1}^{+\infty} B_n = \bigcup_{n=1}^{+\infty} A_n$. So $\mu(A) = \sum_{n=1}^{+\infty} \mu(B_n) \leq \sum_{n=1}^{+\infty} \mu(A_n)$.

4. Define $(B_n)_{n\in\mathbb{N}}$ as in 3). Since $\bigcup_{j=1}^{n} A_j = \bigcup_{j=1}^{n} B_j$, then

$$\begin{aligned} \mu(A) = \mu(\bigcup_{n=1}^{+\infty} A_n) &= \mu(\bigcup_{n=1}^{+\infty} B_n) = \sum_{n=1}^{+\infty} \mu(B_n) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu(B_j) \\ &= \lim_{n\to\infty} \mu(\bigcup_{j=1}^{n} B_j) = \lim_{n\to\infty} \mu(\bigcup_{j=1}^{n} A_j). \end{aligned}$$

5. $\mu(B \setminus A) + \mu(A) = \mu(B)$. If $\mu(A) < \infty$ then $\mu(B \setminus A) = \mu(B) - \mu(A)$.

6. Apply 3) to the sequence $(A_1 \setminus A_j)_j$.

Remark . (Exercise)
It is easy to prove that μ is a measure on the measurable space $(X, \mathscr{B})$ if and only if:

i) $\mu(\emptyset) = 0$
ii) $\mu(A \cup B) = \mu(A) + \mu(B)$, if $A \cap B = \emptyset$.
iii) If $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence of the σ-algebra $\mathscr{B}$, then

$$\mu(\bigcup_{n=1}^{+\infty} A_n) = \operatorname{Sup}_n \mu(A_n).$$

Definition 3.3

1. *We say that the measure μ is* **finite** *if $\mu(X) < +\infty$.*

2. *We say that the measure μ is* **σ-finite** *if there exists an increasing sequence $(A_j)_j$ of measurable subsets of finite measure and $\cup_{j=1}^{+\infty} A_j = X$.*

3. *A probability measure is a measure on $(X, \mathscr{A})$ is a measure such that $\mu(X) = 1$. In this case the $\sigma-$algebra $\mathscr{A}$ is called the space of events.*

3.2 Properties of Measures

Let $(X, \mathscr{B})$ be a measurable space. We denote by $\mathscr{M}(X, \mathscr{B})$ or $\mathscr{M}(X)$ the set of measures on the measurable space $(X, \mathscr{B})$. We have the following properties:

1. The set $\mathscr{M}(X)$ is a convex cone. If μ_1 and μ_2 are in $\mathscr{M}(X)$ and $\lambda \in \mathbb{R}^+$, then $\mu_1 + \mu_2$, $\lambda\mu_1$ are measures.

   We order the set $\mathscr{M}(X)$ by the relationship
   $$\mu_1 \leq \mu_2 \iff \mu_1(A) \leq \mu_2(A);\ \forall A \in \mathscr{B}.$$

2. If $(\mu_n)_{n\in\mathbb{N}}$ is an increasing sequence of measures, then the mapping $\mu\colon \mathscr{B} \longrightarrow [0, +\infty]$ defined by $\mu(A) = \lim\limits_{n\to+\infty} \mu_n(A) = \operatorname{Sup}\limits_{n} \mu_n(A)$ for any $A \in \mathscr{B}$ is a measure on X.

   It is clear that $\mu(\emptyset) = 0 = \lim\limits_{n\to+\infty} \mu_n(\emptyset)$, and if A, B are two disjoints elements of $\mathscr{B}$, we have
   $$\mu(A \cup B) = \lim_{n\to+\infty} \mu_n(A) + \lim_{n\to+\infty} \mu_n(B) = \mu(A) + \mu(B).$$

   Let now (A_n) be an increasing sequence of $\mathscr{B}$ and $A = \bigcup_n A_n$. We have $\mu_j(A_n) \leq \mu(A_n) \leq \mu(A)$. Then
   $$\mu_j(A) = \lim_{n\to+\infty} \mu_j(A_n) \leq \lim_{n\to+\infty} \mu(A_n) \leq \mu(A)$$
   and
   $$\mu(A) = \lim_{j\to+\infty} \mu_j(A) \leq \lim_{n\to+\infty} \mu(A_n) \leq \mu(A).$$
   Then $\mu(A) = \lim\limits_{n\to+\infty} \mu(A_n)$.

4 Complete Measure Spaces

Definition 4.1
Let $(X, \mathscr{B}, \mu)$ be a measure space. A subset A of X is called **a null set or a negligible set** *if A is contained in a measurable subset of measure zero.*

Example .
Let $(X, \mathscr{B})$ be a measurable space such that $\forall x \in X$; $\{x\} \in \mathscr{B}$. If we take $\mu = \delta_a$, with $a \in X$; then every subset $A \subset \mathscr{B}$ such that $a \notin A$, is a null set.

Remarks .
We denote by $\mathscr{N}$ the set of null sets. We have:

1. $\emptyset \in \mathscr{N}$.

2. Any subset of a null set is a null set. If $A \subset B$ and $B \in \mathscr{N}$, then there is an $C \in \mathscr{B}$ such that $\mu C = 0$ and $B \subset C$; now $A \subset C$.

3. A countable union of null sets is a null set. If $(A_n)_n$ is any sequence in $\mathscr{N}$. For each $n \in \mathbb{N}$ choose an $B_n \in \mathscr{B}$ such that $A_n \subset B_n$ and $\mu(B_n) = 0$. Now $B = \bigcup_{n\in\mathbb{N}} B_n \in \mathscr{B}$ and $\bigcup_{n\in\mathbb{N}} A_n \subset \bigcup_{n\in\mathbb{N}} B_n$, and $\mu(\bigcup_{n\in\mathbb{N}} B_n) \leq \sum_{n=0}^{\infty} \mu B_n$, so $\mu(\bigcup_{n\in\mathbb{N}} B_n) = 0$.

Definition 4.2
If $P(x)$ is some assertion applicable to numbers x of the set X, we say that

$$P(x) \textit{ for almost every } x \in X \quad \text{or } P(x) \text{ a.e. } (x)$$

or

$$P(x) \text{ for } \mu - \textit{almost every } x, \quad P(x)\, \mu - \text{a.e.}(x),$$

to mean that

$$\{x \in X;\ P(x) \ \ \textit{is false}\}$$

is a null set.

Definition 4.3
A measure space $(X, \mathscr{B}, \mu)$ is said to be complete if any null set is measurable ($\mathscr{N} \subset \mathscr{B}$), we say that the measure μ is complete.

Theorem 4.4
Let $(X, \mathscr{B}, \mu)$ be a measure space, and let $\mathscr{N}$ be the set of the null sets of X. Let $\mathscr{B}' = \{A \cup B;\ A \in \mathscr{B} \text{ and } B \in \mathscr{N}\}$. $\mathscr{B}'$ is a σ-algebra on X and there exists a unique measure μ' which extends the measure μ on the σ-algebra $\mathscr{B}'$. The measure space $(X, \mathscr{B}', \mu')$ is complete.

Proof .
Let prove now that $\mathscr{B}'$ is a σ-algebra.
$\mathscr{B}'$ is evidently closed under countable union. It suffices to prove that it is closed under complementarity. Let $A' = A \cup N$ be an element of $\mathscr{B}'$. As N is a null set there exists a subset B of $\mathscr{B} \cap \mathscr{N}$ and $N \subset B$. We have

$$A'^c = (A \cup N)^c = (A \cup B)^c \cup (B \setminus (A \cup N)).$$

It follows then that A'^c is an element of $\mathscr{B}'$.
If the measure μ' exists it is unique. In fact we must have $\mu'(N) = 0$ for any $N \in \mathscr{N}$, thus if $A' = A \cup N$ is an element of $\mathscr{B}'$ we shall have $\mu'(A') = \mu(A)$.
To show that μ' is a mapping on $\mathscr{B}'$, we must show that if $A_1 \cup N_1 = A_2 \cup N_2$ with $A_1, A_2 \in \mathscr{B}$ and $N_1, N_2 \in \mathscr{N}$, then $\mu(A_1) = \mu(A_2)$. So we have $A_1 \setminus A_2 \in N_2$, then it is a null set. If $B = A_1 \cap A_2$, then $A_1 = B \cup (A_1 \setminus A_2)$ and $\mu(B) = \mu(A_1)$. In the same way we shall have $\mu(B) = \mu(A_2)$, then $\mu(A_1) = \mu(A_2)$.

Let we prove now that μ' defines a measure on the σ-algebra $\mathscr{B}'$. If $(A'_n)_{n\in\mathbb{N}}$ be a sequence of disjoint elements of $\mathscr{B}'$, with $A'_n = A_n \cup N_n$, $A_n \in \mathscr{B}$ and $N_n \in \mathcal{N}$; $\forall n \in \mathbb{N}$. We have

$$\mu'(\bigcup_{n=1}^{+\infty} A'_n) = \mu'\Big((\bigcup_{n=1}^{+\infty} A_n) \cup (\bigcup_{n=1}^{+\infty} N_n)\Big) = \mu(\bigcup_{n=1}^{+\infty} A_n) = \sum_{n=1}^{+\infty} \mu(A_n) = \sum_{n=1}^{+\infty} \mu'(A'_n).$$

Finally the measure space $(X, \mathscr{B}', \mu')$ is complete because the μ'-null sets are elements of $\mathscr{N}$. It is evident that μ' is the smallest complete extention of the measure μ. $\square$

5 Outer Measure

Definition 5.1
Let X be a nonempty set. An outer measure μ^* on X is a mapping $\mu^*\colon \mathscr{P}(X) \longrightarrow [0, \infty]$ which fulfills the following axioms:

i) $\mu^*(\emptyset) = 0$.

ii) If $(A_n)_{n\in\mathbb{N}}$ is a sequence of subsets of X, then

$$\mu^*(\bigcup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} \mu^*(A_n).$$

iii) μ^* is increasing (i.e. $\mu^*(A) \le \mu^*(B)$ if $A \subset B$).

Example .
Any measure on $\mathscr{P}(X)$ is an outer measure.

Definition 5.2
Let X be a set and μ^* be an outer measure on X. A subset A of X is called μ^*-measurable if

$$\forall B \subset X; \quad \mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c)$$

Now we introduce the most important method of constructing measures.

Theorem 5.3 *(Caratheodory's construction)*
Let X be a non empty set and μ^* be an outer measure on X. Then the set $\mathscr{B}'$ of the μ^*-measurable subsets is a σ-algebra on X and the restriction of μ^* on $\mathscr{B}'$ denoted $\mu^*|_{\mathscr{B}'}$ is a complete measure.

Proof .

i) $\emptyset$ is μ^*-measurable. $(\mu^*(B \cap \emptyset) + \mu^*(B \cap \emptyset^c) = \mu^*(\emptyset) + \mu^*(B) = \mu^*(B))$.

ii) Let A be a μ^*-measurable set and let B a subset of X. It follows from the definition of the outer measure that $\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c)$, then A^c is μ^*-measurable.

iii) Let $A, B \in \mathscr{B}'$ and E a subset of X. As A is a measurable subset, we have

$$\begin{aligned}\mu^*(E\cap(A\cup B)) &= \mu^*(E\cap(A\cup B)\cap A)+\mu^*(E\cap(A\cup B)\cap A^c)\\ &= \mu^*(E\cap A)+\mu^*(E\cap B\cap A^c) \qquad (5.8)\end{aligned}$$

$$\begin{aligned}\mu^*(E\cap(A\cup B))+\mu^*(E\cap(A\cup B)^c) &= \mu^*(E\cap A)+\mu^*(E\cap B\cap A^c)+\mu^*(E\cap A^c\cap B^c)\\ &= \mu^*(E\cap A)+\mu^*(E\cap A^c)=\mu^*(E). \qquad (5.9)\end{aligned}$$

Then $A\cup B$ is in $\mathscr{B}'$.

iv) Let A_1, A_2 be two disjoint elements of $\mathscr{B}'$, B a subset of X and $E = B\cap(A_1\cup A_2)$. As $E\cap(A_1\cup A_2)^c=\emptyset$, we use the relationship given in iii) for the subset E, we will have:

$$\mu^*(E\cap(A_1\cup A_2))+\mu^*(E\cap(A_1\cup A_2)^c)=\mu^*(E\cap A_1)+\mu^*(E\cap A_1^c)$$

$$=\mu^*(B\cap A_1)+\mu^*(B\cap A_2).$$

Then

$$\mu^*(B\cap(A_1\cup A_2))=\mu^*(B\cap A_1)+\mu^*(B\cap A_2).$$

Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of disjoint elements of $\mathscr{B}'$, then we have

$$\begin{aligned}\mu^*(B) &= \mu^*(B\cap\bigcup_{j=1}^{n}A_j)+\mu^*(B\cap(\bigcup_{j=1}^{n}A_j)^c)\\ &\geq \mu^*(B\cap\bigcup_{j=1}^{n}A_j)+\mu^*(B\cap(\bigcup_{j=1}^{\infty}A_j)^c)\\ &\geq \sum_{j=1}^{n}\mu^*(B\cap A_j)+\mu^*(B\cap(\bigcup_{j=1}^{\infty}A_j)^c).\end{aligned}$$

Then

$$\mu^*(B)\geq\sum_{j=1}^{\infty}\mu^*(B\cap A_j)+\mu^*(B\cap(\bigcup_{j=1}^{\infty}A_j)^c)\geq\mu^*(B\cap\bigcup_{j=1}^{\infty}A_j)+\mu^*(B\cap(\bigcup_{j=1}^{\infty}A_j)^c).$$

The other inequality results from the property ii) of the outer measure μ^*.
To finish the proof we take a sequence $(B_n)_{n\in\mathbb{N}}$ of $\mathscr{B}'$, and put $A_1 = B_1$, $A_n = B_n\setminus\bigcup_{j=1}^{n-1}B_j$. We have $\bigcup_{n=1}^{\infty}A_n=\bigcup_{n=1}^{\infty}B_n$. Thus $\mathscr{B}'$ is a σ-algebra.
It is evident that the restriction of μ^* on $\mathscr{B}'$ is a measure.
It remains to show that the measure μ^* is complete. To prove this fact it suffices to prove that any null set A is measurable. If A is a null set, then there exist an element

$B \in \mathscr{B}'$ such that $A \subset B$ and $\mu^*(B) = 0$. Let E be a subset of X, then $\mu^*(E \cap A) = 0$ and

$$\mu^*(E) \geq \mu^*(E \cap A^c) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

The other inequality results from the definition of the outer measure μ^*. Thus A is μ^*-measurable.

Exercise .
Let $(X, \mathscr{B}, \mu)$ be a measure space. We define the mapping $\mu^*\colon \mathscr{P}(X) \longrightarrow [0, +\infty]$ by

$$\mu^*(A) = \inf\{\sum_{j=1}^{\infty} \mu\ (A_j);\ A \subset \cup_{j=1}^{\infty} A_j \text{ and } A_j \in \mathscr{B}\}. \tag{5.10}$$

Show that μ^* is an outer measure and any $\mu-$measurable set is μ^*-measurable and the restriction of μ^* on $\mathscr{B}$ is equal to the measure μ.

Solution .
It is easy to prove that $\mu^*(\emptyset) = 0$ and μ^* is increasing.
Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of subsets of X. We want to prove that $\mu^*(\cup_{n=1}^{+\infty} A_n) \leq \displaystyle\sum_{n=1}^{+\infty} \mu^*(A_n)$. If there exists A_n such that $\mu^*(A_n) = +\infty$, then the inequality is trivial.
Assume now that $\forall n \in \mathbb{N}$; $\mu^*(A_n) < +\infty$.
For every $n \in \mathbb{N}$, and for every $\varepsilon > 0$, there exists a sequence $(A_{n,j})_j \in \mathscr{B}$, such that $\mu^*(A_n) \geq \sum_{j=1}^{+\infty} \mu(A_{n,j}) - \frac{\varepsilon}{2^n}$. Then the sequence $(A_{n,j})_{j,n \in \mathbb{N}}$ is a covering of the set $A = \displaystyle\bigcup_{j=1}^{+\infty} A_n$ and $\displaystyle\sum_{n=1}^{+\infty}\sum_{j=1}^{+\infty} \mu(A_{n,j}) \leq \sum_{n=1}^{+\infty} \mu^*(A_n) + \varepsilon$. Then $\mu^*(A) \leq \sum_{n=1}^{+\infty} \mu^*(A_n) + \varepsilon$, for all $\varepsilon > 0$ and so $\mu^*(A) \leq \displaystyle\sum_{n=1}^{+\infty} \mu^*(A_n)$. Then μ^* is an outer measure.
Let now proving that $\mu^* = \mu$ on $\mathscr{B}$.
If $A \in \mathscr{B}$, then $\mu^*(A) \leq \mu(A)$, and if $\mu^*(A) = +\infty$ then $\mu^*(A) = \mu(A)$.
Assume now that $\mu^*(A) < +\infty$, then for every $\varepsilon > 0$, there exists $(A_n)_{n \in \mathbb{N}}$ a covering of A in $\mathscr{B}$ and $\mu^*(A) \geq \displaystyle\sum_{n=1}^{+\infty} \mu(A_n) - \varepsilon$. As $\mu(A) \leq \displaystyle\sum_{n=1}^{+\infty} \mu(A_n)$, then $\mu(A) \leq \mu^*(A) + \varepsilon$ for every $\varepsilon > 0$. It result that $\mu(A) = \mu^*(A), \forall A \in \mathscr{B}$.
Let now proving that any $\mu-$measurable set is μ^*-measurable.
If $A \in \mathscr{B}$ and $B \subset X$. From the definition of the outer measure μ^*, we have $\mu^*(B) \leq \mu^*(B \cap A) + \mu^*(B \cap A^c)$. Then if $\mu^*(B) = +\infty$ we have the desired equality. Assume now that $\mu^*(B) < +\infty$. Then for every $\varepsilon > 0$, there exists a covering $(B_n)_{n \in \mathbb{N}}$ of B in $\mathscr{B}$ and $\mu^*(B) \geq \displaystyle\sum_{n=1}^{+\infty} \mu(B_n) - \varepsilon$. As μ is a measure $\mu(A \cap B_n) + \mu(A^c \cap B_n) = \mu(B_n)$, then $\mu^*(B) \geq \displaystyle\sum_{n=1}^{+\infty} \mu(B_n \cap A) + \sum_{n=1}^{+\infty} \mu(B_n \cap A^c) - \varepsilon \geq \mu^*(B \cap A) + \mu^*(B \cap A^c) - \varepsilon$.
Then $\mu^*(B) \geq \mu^*(B \cap A) + \mu^*(B \cap A^c)$. Then $\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c)$ and A is μ^* measurable.

Theorem 5.4
Let $(X, \mathscr{B}, \mu)$ be a measure space and μ sigma-finite measure. Let μ^* the outer measure defined on $\mathscr{P}(X)$ by $\mu^*(A) = \inf\{\sum_j \mu\ (A_j);\ A \subset \cup_j A_j \text{ and } A_j \in \mathscr{B}\}$. We denote by $\hat{\mathscr{B}}$ the complete σ-algebra and $\mathscr{B}_0$ the σ-algebra of the μ^*-measurable sets. Then $\hat{\mathscr{B}} = \mathscr{B}_0$.

Proof .
According to the previous exercise $\mathscr{B} \subset \mathscr{B}_0$. Let A be a null set, there exists a measurable set B such that $A \subset B$ and $\mu(B) = 0$. Let E be a subset of X; $\mu^*(E \cap A) \leq \mu(B) = 0$ and $\mu^*(E \cap A^c) \leq \mu^*(E)$ then $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$ and $\hat{\mathscr{B}} \subset \mathscr{B}_0$. Let $A \in \mathscr{B}_0$, assume that $\mu^*(A) < +\infty$, there exists a sequence $(A_{j,n})$ of $\mathscr{B}$ such that $A \subset \bigcup_j A_{j,n}$ and $\sum_j \mu(A_{j,n}) \leq \mu^*(A) + 1/n$. We denote $B_n = \bigcup_{j=1}^{\infty} A_{j,n}$. $B_n \supset A$ and $\mu(B_n) \leq \mu^*(A) + 1/n$. Let $B = \bigcap_n B_n$, $B \in \mathscr{B}$; $A \subset B \Rightarrow \mu^*(A) \leq \mu(B)$, and we have $\mu(B) \leq \mu(B_n) \leq \mu^*(A) + 1/n, \forall n \Rightarrow \mu(B) \leq \mu^*(A) \Rightarrow \mu(B) = \mu^*(A) \Rightarrow \mu^*(B \setminus A) = 0$, because $\mu^*(A) < \infty$. Then $A = B \setminus (B \setminus A) = B \cap (B \setminus A)^c$. $(B \setminus A)$ is a null set then it is in the σ-algebra $\hat{\mathscr{B}}$ and in the same way for B, then $A \in \hat{\mathscr{B}}$. If $\mu^*(A) = +\infty$. Since μ is σ-finite, there exists a sequence $(E_n)_{n\in\mathbb{N}}$ of measurable sets such that $\mu(E_n) < +\infty$ and $\bigcup_{n=1}^{+\infty} E_n = X$. Then any $A \in \mathscr{B}_0$ is written as

$$A = \bigcup_{n=1}^{+\infty} A_n, \quad A_n \in \mathscr{B}_0, \ \textit{and}\ \mu^*(A_n) < +\infty.$$

Then $A_n \in \hat{\mathscr{B}}$ and $A \in \hat{\mathscr{B}}$.

5.1 Monotone Class and $\sigma-$Algebra

Definition 5.5
A collection of sets $\mathcal{M}$ is called a **monotone class** *if for any monotone sequence $(A_n)_{n\in\mathbb{N}}$ of $\mathcal{M}$; $\lim\limits_{n\to+\infty} A_n \in \mathcal{M}$.*

Examples .

1. Any $\sigma-$algebra is a monotone class.
2. An arbitrary intersection of monotone classes is a monotone class.
3. If $A \subset X$, the intersection of all monotone classes that contain A is called the monotone class generated by A and denoted by $\mathscr{M}(A)$.

Theorem 5.6
Let $\mathcal{A}$ be an algebra of X. We denote by $\mathscr{M}(\mathcal{A})$ the monotone class generated by $\mathcal{A}$, and by $\sigma(\mathcal{A})$ the $\sigma-$algebra generated by $\mathcal{A}$. Then $\mathscr{M}(\mathcal{A}) = \sigma(\mathcal{A})$.

Proof .

It follows from the above remark that $\sigma(\mathcal{A})$ is a monotone class, as $\sigma(\mathcal{A})$ contains $\mathcal{A}$, then $\sigma(\mathcal{A})$ contains the smallest monotone class containing $\mathcal{A}$ thus $\sigma(\mathcal{A}) \supset \mathscr{M}(\mathcal{A})$.
For proving that $\sigma(\mathcal{A}) \subset \mathscr{M}(\mathcal{A})$, we define for every subset S of X the set $\tilde{S}$ by:

$$\tilde{S} = \{T \in \mathscr{P}(X);\ S \cup T, S \setminus T \text{ and } T \setminus S \in \mathscr{M}(\mathcal{A})\}.$$

This definition is symmetric with respect to S and T, then $S \in \tilde{T} \iff T \in \tilde{S}$. We want to prove that $\tilde{S}$ is a monotone class if it exists.
If $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence of $\tilde{S}$; $(S \cup A_n)_{n\in\mathbb{N}}$ is a increasing sequence of $\mathscr{M}(\mathcal{A})$, the same for the sequence $(A_n \setminus S)_{n\in\mathbb{N}}$, the sequence $(S \setminus A_n)_{n\in\mathbb{N}}$ is a decreasing sequence of $\mathscr{M}(\mathcal{A})$. Then the limit of the sequences are in $\mathscr{M}(\mathcal{A})$.
Let $A \in \mathcal{A}$, then $\forall\ B \in \mathcal{A}$, $B \in \tilde{A}$, then $\tilde{A}$ is a monotone class containing $\mathcal{A}$, then $\tilde{A} \supset \mathscr{M}(\mathcal{A})$. So $\forall\ S \in \mathscr{M}(\mathcal{A})$, $S \in \tilde{A}$ for any $A \in \mathcal{A}$, and so $A \in \tilde{S}$, then $\mathcal{A} \subset \tilde{S}$; $\forall S \in \mathscr{M}(\mathcal{A})$. As $\tilde{S}$ is a monotone class then $\mathscr{M}(\mathcal{A}) \subset \tilde{S}$.
We prove that:
$\forall\ S, S' \in \mathscr{M}(\mathcal{A})$, $S \setminus S'$, $S' \setminus S$, $S \cup S' \in \mathscr{M}(\mathcal{A})$. If we take $S' = X$, we find that $S^c \in \mathscr{M}(\mathcal{A})$, in this way $\mathscr{M}(\mathcal{A})$ is an algebra. The result can be deduced from the following lemma.

Lemma 5.7
Let $\mathscr{M}$ be an algebra closed under increasing limit, (i.e. if $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence of $\mathscr{M}$ then the limit of A_n is in $\mathscr{M}$), then $\mathscr{M}$ is a $\sigma-$algebra.

Proof .
Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of $\mathscr{M}$. Consider $B_n = \bigcup_{1\le j\le n} A_j$, the sequence B_n is increasing in $\mathscr{M}$ and $\cup_n A_n = \cup_n B_n \in \mathscr{M}$.

We end this paragraph with a property of measure that we need in the construction of Lebesgue measure.

Theorem 5.8
Let μ_1 and μ_2 be two positive measures on a measurable space $(X, \mathscr{B})$. Assume that there exists a class $\mathscr{C}$ of measurable subsets such that:

a) $\mathscr{C}$ is closed under finite intersection and that the σ-algebra generated by $\mathscr{C}$ is equal to $\mathscr{B}$.

b) There exists an increasing sequence $(E_n)_{n\in\mathbb{N}}$ in $\mathscr{C}$ such that $\lim_{n\to+\infty} E_n = X$.

c) $\mu_1(C) = \mu_2(C) < +\infty$, for any $C \in \mathscr{C}$.

Then $\mu_1 = \mu_2$.

Proof .
We suppose in the first case that $\mu_1(X) = \mu_2(X) < +\infty$.
Let $\mathscr{A} = \{A \in \mathscr{B};\ \mu_1(A) = \mu_2(A)\}$. By hypothesis $X \in \mathscr{C}$ and $\mathscr{C} \subset \mathscr{A}$. It is easy to prove that $\mathscr{A}$ is a monotone class. (If $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence of $\mathscr{A}$, then $\mu_1(A_n) = \mu_2(A_n)$ for all n, and then

$$\mu_1(\bigcup_{n=1}^{+\infty} A_n) = \mu_2(\bigcup_{n=1}^{+\infty} A_n) = \mu_1(\lim A_n) = \mu_2(\lim A_n).$$

If $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence of $\mathscr{A}$, then $\mu_1(A_n) = \mu_2(A_n)$ for all n, as $\mu_1(X) = \mu_2(X) < +\infty$, then $\mu_1(\bigcap_{n=1}^{+\infty} A_n) = \mu_2(\bigcap_{n=1}^{+\infty} A_n)$.)
$\mathscr{A}$ is a $\sigma-$algebra. (If $A, B \in \mathscr{A}$ with $A \subset B$, then $\mu_1(B \setminus A) = \mu_1(B) - \mu_1(A) = \mu_2(B) - \mu_2(A) = \mu_2(B \setminus A)$ and so $B \setminus A \in \mathscr{A}$. We use the fact that μ_1, μ_2 are finite and $\mu_1(X) = \mu_2(X)$). Then $\sigma(\mathscr{C}) = \mathscr{B} \subset \mathscr{A}$ and $\mathscr{A} = \mathscr{B}$ and $\mu_1 = \mu_2$.
In the general case we take $\mu_{j,n}$ the restriction of μ_j on E_n for all $n \in \mathbb{N}$. From the first case $\mu_{1,n} = \mu_{2,n}$, which gives $\mu_1 = \mu_2$, because $\mu_j = \lim\limits_{n\to+\infty} \mu_{j,n}$; $j = 1, 2$.

□

6 Lebesgue Measure on $\mathbb{R}$

Theorem 6.1
There exists only and only one measure λ on $\mathscr{B}_{\mathbb{R}}$ satisfying:
i) λ is invariant under translation. (i.e. $\forall x \in \mathbb{R}$, $\forall A \in \mathscr{B}_{\mathbb{R}}$; $\lambda(x+A) = \lambda(A)$).
ii) $\lambda([0, 1]) = 1$.

Proof .
Uniqueness: Assume that there exists two measures μ and ν on $\mathscr{B}_{\mathbb{R}}$ satisfying (i) and (ii) then $\nu[0, 1/n[\leq 1/n \;\Rightarrow \nu\{0\} = 0$ and then any finite set or countable set is a null set and all the intervals $[a, b]$, $]a, b]$, $[a, b[$ and $]a, b[$ have the same measure and equal to $b-a$. (We treat the case of a and b are rationals and then we take the limit.) We denote by $\mathscr{E}$ the set of finite union of intervals of $\mathbb{R}$ of the form $[a, b[$; $a, b \in \mathbb{R}$. The set $\mathscr{E}$ is closed under finite intersection and $\mathbb{R} = \bigcup_n [-n, n[$. Then we shall have $\mu = \nu$ on $\mathscr{E}$. It follows from the unicity theorem 4.4 that μ and ν are equal on $\mathscr{B}_{\mathbb{R}}$.
Existence: Define for any subset A of $\mathbb{R}$

$$\mu^*(A) = \inf_{\mathscr{R}} \sum_{I\in\mathscr{R}} \mathscr{L}(I).$$

$\mathscr{R}$ describes the whole of finite or countable coverings of A by open intervals, and $\mathscr{L}(I)$ is the length of I.
We first prove that for any interval I of $\mathbb{R}$, $\mu^*(I) = \mathscr{L}(I)$.
If a and b are the endpoints of I and $\varepsilon > 0$, then $I \subset]a-\varepsilon, b+\varepsilon[$ and $\mu^*(I) \leq \mathscr{L}(I)+2\varepsilon$. It follows that $\mu^*(I) \leq \mathscr{L}(I)$.
Conversely let $(I_k)_k$ be an open covering of I, then $[a+\varepsilon, b-\varepsilon] \subset \cup_k I_k$. As $[a+\varepsilon, b-\varepsilon]$ is compact, there exist a finite sub-covering $(I_k)_{1\leq k\leq n}$ such that $[a+\varepsilon, b-\varepsilon] \subset \cup_{k=1}^n I_k$. It results that $b - a - 2\varepsilon \leq \sum_{k=1}^{n} \mathscr{L}(I_k) \leq \sum_{k=1}^{+\infty} \mathscr{L}(I_k)$. Thus $b - a - 2\varepsilon \leq \mu^*(I)$ for any $\varepsilon > 0$ and then $\mathscr{L}(I) = \mu^*(I)$.
Let Ω be an open set of $\mathbb{R}$ and let $(I_n)_{n\in\mathbb{N}}$ be the connected components of Ω, then $\mu^*(\Omega) = \sum_{n=1}^{\infty} \mathscr{L}(I_n)$. In fact from the definition of μ^*

$$\mu^*(\Omega) \leq \sum_{n=1}^{\infty} \mathscr{L}(I_n). \tag{6.11}$$

Conversely let $(J_k)_k$ be a covering of Ω by open intervals, we have $I_n = \bigcup_k J_k \cap I_n$. It results that $\sum_{n=1}^{+\infty} \mathscr{L}(I_n) \leq \sum_{n=1}^{+\infty}\sum_{k=1}^{+\infty} \mathscr{L}(I_n \cap J_k) = \sum_{k=1}^{+\infty}\sum_{n=1}^{+\infty} \mathscr{L}(I_n \cap J_k)$. In the other hand the intervals $(I_n)_n$ are disjoints, then for any m, $\bigcup_{n=1}^{m}(J_k \cap I_n) \subset J_k$ and for all $m \in \mathbb{N}$; $\sum_{n=1}^{m} \mathscr{L}(J_k \cap I_n) \leq \mathscr{L}(J_k)$. It results that $\sum_{n=1}^{+\infty} \mathscr{L}(I_n \cap J_k) \leq \sum_{k=1}^{+\infty} \mathscr{L}(J_k)$.

Then

$$\sum_{n=1}^{+\infty} \mathscr{L}(I_n) \leq \mu^*(\Omega). \tag{6.12}$$

So relations (6.11) and (6.12) gives that $\mu^*(\Omega) \leq \sum_{n=1}^{\infty} \mathscr{L}(I_n)$.

We deduce that if $(\omega_n)_{n\in\mathbb{N}}$ is a sequence of open sets, then $\mu^*(\bigcup_n \omega_n) \leq \sum_{n=1}^{+\infty} \mu^*(\omega_n)$.

In fact if $(I_{n,k})_k$ are the connected components of ω_n, we have: $\mu^*(\omega_n) = \sum_{k=1}^{+\infty} \mathscr{L}(I_{n,k})$ and

$$\mu^*(\bigcup_{n=1}^{+\infty} \omega_n) = \mu^*(\bigcup_{n,k=1}^{+\infty} I_{n,k}) \leq \sum_{n,k=1}^{+\infty} \mathcal{L}(I_{n,k}) = \sum_{n=1}^{+\infty}\sum_{k=1}^{+\infty} \mathcal{L}(I_{n,k}) = \sum_{n=1}^{+\infty} \mu^*(\omega_n).$$

Let now prove that for any subset $A \subset \mathbb{R}$, $\mu^*(A) = \inf_{O \text{ open} \supset A} \mu^*(O)$. If (I_n) be a finite or countable covering of A by open intervals. Put $\omega = \bigcup_{n=1}^{+\infty} I_n$, then $\mu^*(A) \leq \mu^*(\omega) \leq \sum_{n=1}^{+\infty} \mathscr{L}(I_n)$. We deduce that μ^* is an outer measure on $\mathscr{P}(\mathbb{R})$; in fact:

i) $\mu^*(\emptyset) = 0$.

ii) If $A \subset B$, then $\mu^*(A) = \inf_{\omega(open)\supset A} \mu^*(\omega) \leq \inf_{\omega(open)\supset B} \mu^*(\omega) = \mu^*(B)$.

iii) If $(A_n)_{n\in\mathbb{N}}$ is a sequence of subsets of $\mathbb{R}$. Our goal is to prove that

$$\mu^*(\cup_n A_n) \leq \sum_n \mu^*(A_n). \tag{6.13}$$

If there exists n_0 such that $\mu^*(A_{n_0}) = +\infty$, the inequality (6.13) is trivially fulfilled. Assume now that $\mu^*(A_n) < +\infty$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$, for any $n \in \mathbb{N}$ there exists an open set ω_n containing A_n such that $\mu^*(\omega_n) \leq \mu^*(A_n) + \frac{\varepsilon}{2^n}$.

$$\mu^*(\cup_{n=1}^{+\infty} A_n) \leq \mu^*(\cup_{n=1}^{+\infty}\omega_n) \leq \sum_{n=1}^{+\infty} \mu^*(\omega_n) \leq \sum_{n=1}^{+\infty} \mu^*(A_n) + \sum_{n=1}^{+\infty} \frac{\varepsilon}{2^n} = \sum_{n=1}^{+\infty} \mu^*(A_n) + \varepsilon \tag{6.14}$$

for any $\varepsilon > 0$, thus $\mu^*(\cup_{n=1}^{+\infty} A_n) \leq \sum_{n=1}^{+\infty} \mu^*(A_n)$.
According to the theorem 5.3 the set of the μ^*-measurable subsets is a σ-algebra $\mathscr{L}$ on $\mathbb{R}$ and $\mu^*|_{\mathscr{L}}$ is a complete measure. This σ-algebra is called the **Lebesgue σ-algebra**, and the elements of $\mathscr{L}$ are called the **Lebesgue measurable sets**. We will note $\mathscr{B}^*_{\mathbb{R}}$ this σ-algebra.

Proposition 6.2
Any Borelian subset is Lebesgue measurable.

Proof .
It suffices to show that $\forall a \in \mathbb{R}$, $]a, +\infty[\in \mathscr{L}$. Let E be a subset of $\mathbb{R}$. our goal is to prove that

$$\mu^*(E) = \mu^*(E\cap]a, +\infty[) + \mu^*(E\cap] - \infty, a]). \tag{6.15}$$

The inequality $\mu^*(E) \leq \mu^*(E\cap]a, +\infty[) + \mu^*(E\cap] - \infty, a])$ results from the fact that μ^* is an outer measure. For the other inequality the result is trivial if $\mu^*(E) = +\infty$. Assume that $\mu^*(E) < +\infty$. Let $\varepsilon > 0$ there exists an open set $\Omega_\varepsilon \supset E$ such that : $\mu^*(\Omega_\varepsilon) \leq \mu^*(E) + \varepsilon$. Assume in the first time that $a \notin \Omega_\varepsilon$.

$$\mu^*(\Omega_\varepsilon) = \sum_{I\in\mathcal{C}} \mathscr{L}(I) = \sum_{I\in\mathcal{C}\cap]a,+\infty[} \mathscr{L}(I) + \sum_{I\in\mathcal{C}\cap]-\infty,a[} \mathscr{L}(I) \tag{6.16}$$

with $\mathcal{C}$ the set of the connected components of Ω_ε. Then it results that

$$\mu^*(\Omega_\varepsilon) = \mu^*(\Omega_\varepsilon\cap]a, +\infty[) + \mu^*(\Omega_\varepsilon\cap] - \infty, a[) \geq \mu^*(E\cap]a, +\infty[) + \mu^*(E\cap] - \infty, a[).$$

Then $\mu^*(E) \geq \mu^*(E\cap]a, +\infty[) + \mu^*(E\cap] - \infty, a])$.
If now $a \in \Omega_\varepsilon$, let $\Omega'_\varepsilon = \Omega_\varepsilon \setminus \{a\}$. According to the first remark $\mu^*(\Omega'_\varepsilon) = \mu^*(\Omega_\varepsilon)$. □
This which ends the proof of the theorem in taking $\lambda = \mu^*$. The measure λ on $\mathscr{B}^*_{\mathbb{R}}$ is called the **Lebesgue measure** on $\mathbb{R}$. □

Proposition 6.3
Let $\mathscr{B}^*_{\mathbb{R}}$ the Lebesgue σ-algebra on $\mathbb{R}$, then $\forall\ A \in \mathscr{B}^*_{\mathbb{R}}$

$$\lambda(A) = \inf_{\omega\ open\supset A} \lambda(\omega)$$

$$\lambda(A) = \operatorname{Sup}_{K\ compact\subset A} \lambda(K).$$

We say that the measure λ is regular.

Proof .
If A is bounded, there exists $n \in \mathbb{N}$ such that $A \subset [-n, n]$. Let $\varepsilon > 0$, the set $[-n, n] \setminus A$ is measurable, then there exists an open set $\omega \supset ([-n, n] \setminus A)$ such that

$$\lambda(\omega) \leq \lambda([-n, n] \setminus A) + \varepsilon = \lambda[-n, n] - \lambda(A) + \varepsilon$$

because $\lambda([-n,n]\setminus A)=\inf_{\omega\ open\supset([-n,n]\setminus A)}\lambda(\omega)$.
Let $K=[-n,n]\cap\omega^c$. K is a compact in A.

$$2n=\lambda[-n,n]=\lambda([-n,n]\cap\omega^c)+\lambda([-n,n]\cap\omega)\leq\lambda(K)+\varepsilon+\lambda[-n,n]-\lambda(A).$$

Then $\lambda(A)\leq\lambda(K)+\varepsilon$ and $\lambda(A)=\text{Sup}_{K\ compact\subset A}\lambda(K)$.
If A is not bounded, then $\forall n\in\mathbb{N}$ there exists a compact $K_n\subset[-n,n]\cap A$ such that

$$\lambda(K_n)\geq\lambda([-n,n]\cap A)-1/n$$

then

$$\underset{K\ compact\subset A}{\text{Sup}}\lambda(K)\geq\underset{n}{\text{Sup}}(\lambda(K_n))\geq\lim_{n\to+\infty}(\lambda([-n,n]\cap A)-1/n)=\lambda(A)$$

7 Measurable Functions

Let X and Y be two nonempty sets. We showed in the previous section 2.9 that the pull back of a σ-algebra by a mapping $f\colon X\longrightarrow Y$ is a σ-algebra of X.

Definition 7.1
If $(X,\mathscr{A})$ and $(Y,\mathscr{B})$ are two measurable spaces. A mapping $f\colon X\longrightarrow Y$ is called measurable if the σ-algebra $f^{-1}(\mathscr{B})\subset\mathscr{A}$.

Theorem 7.2
Let $(X,\mathscr{A})$ and $(Y,\mathscr{B})$ be two measurable spaces, and suppose that $\mathcal{B}$ generates the σ-algebra $\mathscr{B}$. A function $f\colon X\to Y$ is measurable if and only if for every subset V in the generator set $\mathcal{B}$, its pre-image $f^{-1}(V)$ is in $\mathscr{A}$.

Proof .
The sufficient condition is just the definition of measurability.

For the "if" direction, define
$\mathcal{H}=\{V\in\mathscr{B}\colon f^{-1}(V)\in\mathscr{A}\}$.
It is easily verified that $\mathcal{H}$ is a σ-algebra, since the operation of taking the inverse image commutes with the set operations of union, intersection and complement.

By hypothesis, $\mathcal{B}\subseteq\mathcal{H}$. Therefore, $\sigma(\mathcal{B})\subseteq\sigma(\mathcal{H})$. But $\mathscr{B}=\sigma(\mathcal{B})$ by the definition of $\mathcal{B}$, and $\mathcal{H}=\sigma(\mathcal{H})$ since $\mathcal{H}$ is a σ-algebra. This means that $f^{-1}(V)\in\mathscr{A}$ for every $V\in\mathscr{B}$. □

Remark .
To show that a mapping $f\colon X\longrightarrow Y$ is measurable; it suffices to give a set $\mathcal{C}$ which generates $\mathscr{B}$ and such that $f^{-1}(\mathcal{C})\subset\mathscr{A}$.

Proposition 7.3
Let $(X,\mathscr{A})$ be a measurable space and let $f\colon X\longrightarrow\mathbb{R}$ (or in $\overline{\mathbb{R}}$) a mapping. Then f is measurable, if one of the following conditions is fulfilled:

1. $\forall a \in \mathbb{R}\ \{x \in X;\ f(x) \geq a\} \in \mathscr{A}$.

2. $\forall a \in \mathbb{R}\ \{x \in X;\ f(x) < a\} \in \mathscr{A}$.

3. $\forall a \in \mathbb{R}\ \{x \in X;\ f(x) \leq a\} \in \mathscr{A}$.

4. $\forall a, b \in \mathbb{R}\ \{x \in X;\ a < f(x) < b\} \in \mathscr{A}$.

5. $\forall a, b \in \mathbb{R}\ \{x \in X;\ a \leq f(x) < b\} \in \mathscr{A}$.

The space $\mathbb{R}$ *(resp* $\overline{\mathbb{R}}$*) is equipped with the Borel* σ*-algebra* $\mathscr{B}_{\mathbb{R}}$ *(resp* $\mathscr{B}_{\overline{\mathbb{R}}}$*).*

We take the measurable spaces $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ *and* $(\overline{\mathbb{R}}, \mathscr{B}_{\overline{\mathbb{R}}})$*.*

Proof .
Let taking for example the measurable space $(\overline{\mathbb{R}}, \mathscr{B}_{\overline{\mathbb{R}}})$. As $\{x \in \overline{\mathbb{R}};\ f(x) < a\} = f^{-1}([-\infty, a[) \in \mathscr{A}$. The first condition of the proposition is still written $f^{-1}\{\mathcal{C}\} \subset \mathscr{A}$, where $\mathcal{C}$ is the class of the intervals $[-\infty, a[$ of $\overline{\mathbb{R}}$, with $a \in \mathbb{R}$. To show that f is measurable it suffices to show that the σ-algebra generated by $\mathcal{C}$ is the Borelian σ-algebra of $\overline{\mathbb{R}}$. It is easy to show that the open intervals of $\overline{\mathbb{R}}$ are in the σ-algebra generated by $\mathcal{C}$.
Let $\mathcal{T}$ the $\sigma-$algebra generated by $\mathcal{C}$. By complementarity $[a, +\infty] \in \mathcal{T}$, and $[a, b[\in \mathcal{T}$, $\forall a, b \in \mathbb{R}$, because $[a, b[= [a, +\infty] \cap [-\infty, b[$. And $]a, b[= \bigcup_{n=1}^{+\infty} [a + \frac{1}{n}, b[\in \mathcal{T}$. And for the same way $]a, +\infty] = \bigcup_{n=1}^{+\infty} [a + \frac{1}{n}, +\infty]$. Then $\mathcal{T}$ contains all the open sets of X and then $\mathcal{T} = \mathscr{B}_{\overline{\mathbb{R}}}$. □

Particular Case .
Let X and Y two topological spaces and let $\mathscr{B}_X$ and $\mathscr{B}_Y$ the Borelian $\sigma-$algebras on X and Y respectively. Then every continuous function is measurable.
X and Y two topological spaces and let $\mathscr{B}_X$ and $\mathscr{B}_Y$ the Borelian $\sigma-$algebras on X and Y respectively. Then every measurable function $f\colon X \longrightarrow Y$ is called a Borelian function.

Proposition 7.4
Let $(X_0, \mathscr{B}_0)$, $(X_1, \mathscr{B}_1)$ *and* $(X_2, \mathscr{B}_2)$ *three measurable spaces. Let* $f_1\colon X_0 \longrightarrow X_1$ *and* $f_2\colon X_1 \longrightarrow X_2$ *two measurable mappings, then the mapping* $f_2 \circ f_1$ *is measurable.*

The proposition results from the fact that

$$(f_2 \circ f_1)^{-1}(\mathscr{B}_2) = f_1^{-1}(f_2^{-1}(\mathscr{B}_2)) \subset f_1^{-1}(\mathscr{B}_1) \subset \mathscr{B}_0.$$

Proposition 7.5
Let $(X, \mathscr{B})$ *and* $(X_j, \mathscr{B}_j)$, $j = 1, \ldots, n$ $(n+1)$ *measurable spaces, and let* $f\colon X \longrightarrow \prod_{j=1}^{n} X_j$, *a mapping* $f = (f_1, \ldots, f_n)$. *Then* f *is measurable if and only if each partial mapping* $f_j\colon X \longrightarrow X_j$ *is measurable.*

Proof .
We remark that if p_j is the natural projection $p_j\colon \prod_{k=1}^n X_k \longrightarrow X_j$, $p_j^{-1}(A_j) = X_1 \times X_2 ... \times A_j \times \ldots \times X_n$, which is measurable if A_j is measurable. Then p_j is a measurable mapping.
The partial mappings $f_j = p_j \circ f$ are measurable if f is measurable. Let now suppose that f_j, $j = 1, \ldots, n$ are measurable. Let $A_1 \times ... \times A_n$ be a rectangle in $\prod_{k=1}^n X_k$, then

$$f^{-1}(A_1 \times ... \times A_n) = f^{-1}(\bigcap_{j=1}^n p_j^{-1}(A_j)) = \bigcap_{j=1}^n f^{-1}(p_j^{-1}(A_j)) = \bigcap_{j=1}^n f_j^{-1}(A_j).$$

Then f is measurable. □

Corollary 7.6
Let $(X, \mathscr{B})$ be a measurable space, f and g are two measurable functions on X with values in $\mathbb{R}$ or $\overline{\mathbb{R}}$. Let $F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a continuous function. Then the function $h = F(f, g)$ is a measurable function.

Proof .
The mapping (f, g) is measurable on X with values in $\mathbb{R}^2$ and F is measurable thus h is measurable on X. □

Corollary 7.7
Let $(X, \mathscr{B})$, $(Y, \mathscr{B}')$ and $(Z, \mathscr{T})$ three measurable spaces and let $f\colon X \times Y \longrightarrow Z$ a mapping. Then for any $a \in X$ (resp $b \in Y$), the partial mapping $f(a, .)$ (resp $f(., b)$) is measurable.

Proof .
Let us fix an element $a \in X$. The mapping $g\colon Y \longrightarrow X \times Y$, defined by $g(y) = (a, y)$ is measurable from the previous proposition. $f(a, .) = f \circ g$ this which shows the corollary. □

Corollary 7.8
Let $(X_1, \mathscr{B}_1), \ldots, (X_n \mathscr{B}_n)$, n measurable spaces, $f_j\colon X_j \longrightarrow \overline{\mathbb{R}}$, $j = 1, \ldots, n$ and $f\colon \prod_{j=1}^n X_j \longrightarrow \overline{\mathbb{R}}$ defined by $f(x_1, \ldots, x_n) = f_1(x_1) \ldots f_n(x_n)$. Assume that $f_j \not\equiv 0$. Then f is measurable if and only if the functions $f_1, \ldots, f_n$ are measurable.

Proof .
As the mapping $(y_1, \ldots, y_n) \longmapsto y_1.y_2 \ldots y_n$ from $\mathbb{R}^n$ to $\mathbb{R}$ is measurable, then it is clear that f is measurable if the mappings f_j are measurable. For proving the measurability of f_1 for example knowing that f is measurable, we choose $a_2, \ldots, a_n$ such that $f_j(a_j) \neq 0$ for any $j = 2, \ldots, n$. For $x \in X_1$ we have:

$$f_1(x) = \frac{f(x, a_2, \ldots, a_n)}{\prod_{j=2}^n (f_j(a_j))}$$

This proves that f_1 is measurable.
In particular a non empty rectangle $\prod_{j=1}^n A_j$ is measurable if and only if each A_j is.

Proposition 7.9
Let $(X, \mathscr{B})$ be a measurable space.

a) If f is measurable of $(X, \mathscr{B})$ with values in $\mathbb{R}$ or $\overline{\mathbb{R}}$, then $|f|$ is measurable.

b) If $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions of $(X, \mathscr{B})$ with values in $\mathbb{R}$ or in $\overline{\mathbb{R}}$, then the functions g,h,k defined by $g(x) = \operatorname{Sup}\limits_{n\in\mathbb{N}} f_n(x)$, $h(x) = \overline{\lim}_{n\to+\infty} f_n(x)$ and $k(x) = \underline{\lim}_{n\to+\infty} f_n(x)$ are measurable.

Proof .

a) If $a < 0$; $\{x \in X;\ |f(x)| > a\} = X$.
If $a \geq 0$; $\{x \in X;\ |f(x)| > a\} = \{x \in X;\ f(x) > a\} \cup \{x \in X;\ f(x) < -a\} = f^{-1}(]a, +\infty]) \cup f^{-1}([-\infty, -a[) \in \mathscr{B}$.

b) $\{x \in X;\ g(x) > a\} = \bigcup_{n\in\mathbb{N}} \{x \in X;\ f_n(x) > a\} \in \mathscr{B}$.
$h(x) = \inf_{n\in\mathbb{N}}(\operatorname{Sup}_{j\geq n} f_j(x))$

$$\{x \in X;\ h(x) > a\} = \bigcap_{n=1}^{+\infty} \bigcup_{j=n}^{\infty} \{x \in X;\ f_j(x) > a\} \in \mathscr{B}.$$

$k(x) = \operatorname{Sup}_{n\in\mathbb{N}}(\inf_{j\geq n} f_j(x))$

$$\{x \in X;\ k(x) > a\} = \bigcup_{n=1}^{+\infty} \bigcap_{j=n}^{\infty} \{x \in X;\ f_j(x) > a\} \in \mathscr{B}.$$

Remark .
It results from the previous proposition that if f is measurable then the functions $f^+ = \operatorname{Sup}(f, 0)$ and $f^- = \inf(f, 0)$ are measurable, and if $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions which converges point wise toward a function f on X, then f is measurable. □

Corollary 7.10
For any sequence $(f_n)_{n\in\mathbb{N}}$ of measurable functions with real values on a measurable space X, if $C = \{x \in X;\ \lim\limits_{n\to+\infty} f_n(x)$ exists in $\overline{\mathbb{R}}\}$. Then C is measurable.

Proof .
We put $D = C^c$, $D = \{x \in X;\ \underline{\lim}_{n\to+\infty} f_n(x) < \overline{\lim}_{n\to+\infty} f_n(x)\}$. If we put $g = \underline{\lim}_{n\to+\infty} f_n$ and $h = \overline{\lim}_{n\to+\infty} f_n$. For each rational r, let

$$D_r = \{x \in X;\ g(x) < r < h(x)\} = \{g(x) < r\} \cap \{h(x) > r\}$$

which is measurable. $D = \bigcup_{r\in\mathbb{Q}} D_r$ which proves the measurability of D. □

Theorem 7.11
Let $A \subset \mathbb{R}^m$ and $f\colon A \longrightarrow \mathbb{R}^n$ a mapping. Assume that for any point $a \in A$, there exists a neighborhood $V(a)$ such that

$$\mu_n^*(f(A \cap V(a))) = 0$$

Then $\mu_n^*(f(A)) = 0$.

Proof .
For any $a \in A$, there exists a ball $B \subset \mathbb{R}^m$ of center of rational coordinates such that $a \in B$ and $\mu_n^*(f(A \cap B)) = 0$. The family $\mathcal{B}$ of these balls is at least countable and cover A. It follows that $f(A)$ is covered by the sequence $f(A \cap B)$, $B \in \mathcal{B}$, and every one is of measure zero. It follows that $\mu_n^*(f(A)) = 0$.

Theorem 7.12
Let $A \subset \mathbb{R}^m$ and $f\colon \mathbb{R}^m \longrightarrow \mathbb{R}^n$ a mapping such that, there exists $s \geq m/n$ and

$$|f(x) - f(y)| \leq M^s |x - y|^s, \quad \forall x, y \in A.$$

Then

1. *If $s > m/n \Rightarrow \mu_n^*(f(A)) = 0$.*
2. *If $s = m/n \Rightarrow \mu_n^*(f(A)) \leq 2^n (M\sqrt{m})^m \mu_n^*(A)$.*

Proof .
We can suppose that $\mu_m^*(A) < \infty$, if not we take the sequence $A \cap [-p, p]$; $p \in \mathbb{N}$. We denote $||x||_\infty = \text{Sup}_{1 \leq j \leq k} |x_j|$ if $x \in \mathbb{R}^k$. We have $||x||_\infty \leq |x| \leq \sqrt{n}||x||_\infty$ on $\mathbb{R}^n$ and $||x||_\infty \leq |x| \leq \sqrt{m}||x||_\infty$ on $\mathbb{R}^m$. Thus

$$||f(x) - f(y)||_\infty \leq (M\sqrt{m})^s ||x - y||_\infty^s, \quad \forall\ x, y \in A$$

Let $0 < \varepsilon < 1$ and $P = P(b, r)$ a rectangle with $r < \varepsilon < 1$. Assume that $P \cap A \neq \emptyset$. Let $a, b \in A \cap P \Rightarrow ||x - b||_\infty \leq r/2$, $||a - b||_\infty \leq r/2$ and $||x - a||_\infty \leq r$. Then it follows that $||f(x) - f(a)||_\infty \leq (M\sqrt{m})^s r^s$ and

$$f(A \cap P) \subset P(f(a)), 2(M\sqrt{m})^s r^s \Rightarrow \mu_n^*(f(A \cap P)) \leq 2^n (M\sqrt{m})^{ns} r^m r^{ns-m}$$

If $(P_k)_k$ is a covering of A by of the rectangles of thisôtés $\leq \varepsilon$, then

$$\mu_n^*(f(A)) \leq 2^n (M\sqrt{m})^{ns} \varepsilon^{ns-m} \sum_k Vol(P_k)$$

Thus $\mu_n^*(f(A)) \leq 2^n (M\sqrt{m})^{ns} \varepsilon^{ns-m} \mu_m^*(A)$. □

Corollary 7.13

1. *Every null set in $\mathbb{R}^n$ is of measure zero in any system of coordinate in $\mathbb{R}^n$.*
2. *Every subspace of dimension $m < n$ is a null set in $\mathbb{R}^n$.*

zero.

Proof .

1. Every linear mapping $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ fulfills $\|f(x)\| \leq M\|x\|$. The result follows from the previous theorem with $m \leq n$ and $s = 1$.

2. If V is a subspace of dimension $m < n$, $V = f(\mathbb{R}^m)$ and we applied the first result of this corollary.

□

Corollary 7.14
Let $f\colon \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a mapping of class $\mathcal{C}^1$ in any point a of $A \subset \mathbb{R}^m$. If $m < n$ then $\mu_n^*(f(A)) = 0$.

Proof .
For any $a \in A$ there exists an open ball $B(a, r)$ such that

$$\|f(x) - f(y)\| \leq (1 + ||df(a)||)\|x - y\|$$

for any $x, y \in B(a, r)$, $df(a)$ is the differential of f in the point a. It follows that

$$\mu_n^*(f(A \cap B(a, r))) = 0 \Rightarrow \mu_n^*(f(A)) = 0$$

□

Corollary 7.15
Let $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a mapping of class $\mathcal{C}^1$ in any point a of $A \subset \mathbb{R}^n$. If $\mu_n^*(A) = 0$ then $\mu_n^*(f(A)) = 0$.

Exercise .
Let $f\colon \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a mapping of class $\mathcal{C}^p$ and let A a subset of $\mathbb{R}^m$. Assume that $p > m/n$, $D_j f = 0$ on A for any $0 \leq j \leq p - 1$. Show that $\mu_n^*(f(A)) = 0$. (ind: we can prove that $\|f(x) - f(y)\| \leq M\|x - y\|^p$ locally on A)

Exercise .
Let $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear mapping such that $f(e_j) = \lambda_j e_j$, $e_1, \ldots, e_n$ is a base of $\mathbb{R}^n$. Show that if A is a subset of $\mathbb{R}^n$

$$\mu_n^*(f(A)) \leq |\lambda_1 \ldots . \lambda_n| \mu_n^*(A)$$

(ind: if P is a rectangle of center a and of sides of lengths $s_1, , \ldots, s_n$, then $f(P)$ is a rectangle of center $f(a)$ and of sides of lengths $|\lambda_1|s_1, \ldots, |\lambda_n|s_n$. If any $|\lambda_j| = 0$ the result is trivial and if not we can applied the result to f^{-1}.

Theorem 7.16 *(Egoroff)*
Let $(X, \mathscr{B}), \mu)$ be a measure space. Assume that the measure μ is bounded. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of real or complex measurable functions on X which converges point wise on X to a function f. For any $\varepsilon > 0$ there exists a set $A_\varepsilon \in \mathscr{B}$, such that $\mu(A_\varepsilon) \leq \varepsilon$ and the restriction of the sequence (f_n) on the complementary of A_ε is uniformly convergent.

Proof .
The function f is measurable. For any integers (n,k), $k>0$, let

$$E_n^{(k)} = \bigcap_{p=n}^{+\infty}\{x; |f_p(x)-f(x)| \leq \frac{1}{k}.\}$$

This set is measurable. For a given k, the sequence $(E_n^{(k)})_{n\in\mathbb{N}}$ is increasing and $\lim_{n\to+\infty} E_n^{(k)} = X$. (Because the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f on X). As μ is bounded, $\lim_{n\to+\infty}\mu(E_n^{(k)})^c = 0$. Then there exists an integer $n(k)$ such that $\mu(E_{n(k)}^{(k)})^c \leq \varepsilon/2^k$. The set $A_\varepsilon = \bigcup_{k=1}^{+\infty}(E_{n(k)}^{(k)})^c$ is appropriate. In fact $\mu(A_\varepsilon) \leq \varepsilon$, and on the complementary of A_ε the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f.

Remark .
The requirement that μ is bounded is essential. For constructing a counterexample it suffices of take μ the Lebesgue measure on $\mathbb{R}$ and f_n the characteristic function of the range $[n,+\infty[$. (Assume the existence of an invariant measure by translation on $\mathbb{R}$, called Lebesgue measure.)

The classical Cantor ternary set .
Let $a<b$ two real numbers. We call **"tiers median"** of the interval $I\subset[a,b]$, the open interval of length $\frac{b-a}{3}$ and of the same center that $[a,b]$. $(I =]\frac{b-a}{3}, \frac{2(b-a)}{3}[)$.
Let $E_0 = [0,1]$. We remove the tiers-median of E_0, and we recall E_1 this which remains. $E_1 = [0,\frac{1}{3}]\cup[\frac{2}{3},1]$. We remove the tiers-median of these two intervals and we recall E_2 this which remains

$$E_2 = [0,\frac{1}{9}]\cup[\frac{2}{9},\frac{3}{9}]\cup[\frac{6}{9},\frac{7}{9}]\cup[\frac{8}{9},1].$$

By repeating this operation successively, we construct a sequence of decreasing sets $(E_n)_{n\in\mathbb{N}}$ such that each E_n is union of 2^n intervals each one is of length $\frac{1}{3^n}$. We denote $I_{n,k}$ $(k=1,\dots,2^n)$ the intervals of E_n. We call **triadic Cantor's set** the set

$$P = \bigcap_{n=1}^{\infty} E_n$$

$P\neq\emptyset$ because it is clear that 0 and 1 are in P. P is compact because P is closed and bounded. P does not contain any non empty open interval. In fact E_n can not contain intervals of length greater than $\frac{1}{3^n}$. If I is an interval in P, $I\subset P\subset E_n$, thus the length of I is small that $\frac{1}{3^n}$, this for any n, then I is of length zero, and thus P is of interior empty. From the construction if x is an endpoint of an interval $I_{n,k}$, then x remains an endpoint of an interval $I_{n+p,k(p)}$ for any $p\in\mathbb{N}$. Thus $x\in P$. It results that P is a perfect set; in fact for any $x\in P$ and for any $n\in\mathbb{N}$, there exists a_n and b_n in P such that $a_n\leq x\leq b_n$ and $\lim_{n\to+\infty}(b_n-a_n)=0$. It suffices to take a_n and b_n the endpoints of the intervals $I_{m,k}$. The sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are bounded, then we can extract a convergent sub-sequence. And as $b_n-a_n>0$ and $\lim_{n\to+\infty}(b_n-a_n)=0$, $x=\lim_{n\to+\infty}b_n=\lim_{n\to+\infty}a_n$ and it is an accumulation point.

It is easy to verify that the left endpoints of the intervals $I_{n,k}$ are of the form $\sum_{p=1}^{n} \frac{\alpha_p}{3^p}$ where $\alpha_p = 0$ or 2. There result that any point x of P is limit of a sequence of points of P which are of the endpoints space of intervals of the form $I_{n,k}$. Thus $x = \sum_{p=1}^{+\infty} \frac{\alpha_p}{3^p}$, with $\alpha_p = 0$ or 2. It result that P is in bijection with the sets of the mapping of $\mathbb{N} \longrightarrow \{0, 2\}$ which is not countable. We have P is in bijection with $[0, 1]$. Thus P is a compact of measure zero and in bijection with $[0, 1]$.