3. Find the rates of convergence of the following functions as $h \to 0$:

a. $\displaystyle\lim_{h\to 0} \frac{\sin h}{h} = 1$ **b.** $\displaystyle\lim_{h\to 0} \frac{1-\cos h}{h} = 0$

c. $\displaystyle\lim_{h\to 0} \frac{\sin h - h\cos h}{h} = 0$ **d.** $\displaystyle\lim_{h\to 0} \frac{1-e^h}{h} = -1$

Answer: Here, Maclaurin series are the easiest way to get a solution:

$$\frac{\sin h}{h} = \frac{h - \frac{h^3}{6} + \cdots}{h} = 1 - \frac{h^2}{6} + \cdots$$

and so $\frac{\sin h}{h} = 1 + O(h^2)$.

For **b**, we have

$$\frac{1-\cos h}{h} = \frac{1 - (1 - \frac{h^2}{2} + \cdots)}{h} = \frac{h}{2} + \cdots,$$

so $\frac{1-\cos h}{h} = O(h)$.

For **c**, we have

$$\frac{\sin h - h\cos h}{h} = \frac{(h - \frac{h^3}{6} + \cdots) - h(1 - \frac{h^2}{4} + \cdots)}{h} = \frac{-h^2}{6} + \frac{h^2}{4}$$

so $\frac{\sin h - h\cos h}{h} = O(h^2)$.

Finally, for **d**, we have

$$\frac{1-e^h}{h} = \frac{1 - (1 + h + \frac{h^2}{2} + \cdots)}{h} = -1 - \frac{h}{2} + \cdots,$$

so $\frac{1-e^h}{h} = -1 + O(h)$.