Integrals Involving Quadratic Expressions

Bander Almutairi

King Saud University

November 9, 2015

1. Integrals Involving Quadratic Expressions

Partial fraction decompositions may lead to integrands containIng an irreducible quadratic expression $ax^2 + bx + c$.

Partial fraction decompositions may lead to integrands contaInIng an irreducible quadratic expression $ax^2 + bx + c$. When $b \neq 0$, we may need to complete square:

Partial fraction decompositions may lead to integrands contaInIng an irreducible quadratic expression $ax^2 + bx + c$. When $b \neq 0$, we may need to complete square:

$$ax^2 + bx + c$$

Partial fraction decompositions may lead to integrands contaInIng an irreducible quadratic expression $ax^2 + bx + c$. When $b \neq 0$, we may need to complete square:

$$ax^2 + bx + c = a(x^2 + \frac{b}{a}x) + c$$

Partial fraction decompositions may lead to integrands contaInIng an irreducible quadratic expression $ax^2 + bx + c$. When $b \neq 0$, we may need to complete square:

$$
\begin{aligned}
ax^2 + bx + c &= a(x^2 + \frac{b}{a}x) + c \\
&= a(x^2 + \frac{b}{a}x + (\frac{b}{2a})^2) + c - a.(\frac{b}{2a})^2
\end{aligned}
$$

Partial fraction decompositions may lead to integrands contaInIng an irreducible quadratic expression $ax^2 + bx + c$. When $b \neq 0$, we may need to complete square:

$$\begin{aligned} ax^2 + bx + c &= a(x^2 + \frac{b}{a}x) + c \\ &= a(x^2 + \frac{b}{a}x + (\frac{b}{2a})^2) + c - a.(\frac{b}{2a})^2 \\ &= a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{2.a}) \end{aligned}$$

Partial fraction decompositions may lead to integrands contaInIng an irreducible quadratic expression $ax^2 + bx + c$. When $b \neq 0$, we may need to complete square:

$$\begin{aligned} ax^2 + bx + c &= a(x^2 + \frac{b}{a}x) + c \\ &= a(x^2 + \frac{b}{a}x + (\frac{b}{2a})^2) + c - a.(\frac{b}{2a})^2 \\ &= a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{2.a}) \end{aligned}$$

We could use the substitution $u = x + \frac{b}{2a}$, If necessary.

Partial fraction decompositions may lead to integrands contaInIng an irreducible quadratic expression $ax^2 + bx + c$. When $b \neq 0$, we may need to complete square:

$$\begin{aligned} ax^2 + bx + c &= a(x^2 + \frac{b}{a}x) + c \\ &= a(x^2 + \frac{b}{a}x + (\frac{b}{2a})^2) + c - a.(\frac{b}{2a})^2 \\ &= a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{2.a}) \end{aligned}$$

We could use the substitution $u = x + \frac{b}{2a}$, If necessary.

Examples:

Partial fraction decompositions may lead to integrands contaInIng an irreducible quadratic expression $ax^2 + bx + c$. When $b \neq 0$, we may need to complete square:

$$\begin{aligned} ax^2 + bx + c &= a(x^2 + \frac{b}{a}x) + c \\ &= a(x^2 + \frac{b}{a}x + (\frac{b}{2a})^2) + c - a.(\frac{b}{2a})^2 \\ &= a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{2.a}) \end{aligned}$$

We could use the substitution $u = x + \frac{b}{2a}$, If necessary.

Examples:

1. $x^2 - 6x + 13$

Partial fraction decompositions may lead to integrands contaInIng an irreducible quadratic expression $ax^2 + bx + c$. When $b \neq 0$, we may need to complete square:

$$\begin{aligned} ax^2 + bx + c &= a(x^2 + \frac{b}{a}x) + c \\ &= a(x^2 + \frac{b}{a}x + (\frac{b}{2a})^2) + c - a.(\frac{b}{2a})^2 \\ &= a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{2.a}) \end{aligned}$$

We could use the substitution $u = x + \frac{b}{2a}$, If necessary.

Examples:

1. $x^2 - 6x + 13 = (x - 3)^2 + 4.$

Partial fraction decompositions may lead to integrands contaInIng an irreducible quadratic expression $ax^2 + bx + c$. When $b \neq 0$, we may need to complete square:

$$\begin{aligned} ax^2 + bx + c &= a(x^2 + \frac{b}{a}x) + c \\ &= a(x^2 + \frac{b}{a}x + (\frac{b}{2a})^2) + c - a.(\frac{b}{2a})^2 \\ &= a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{2.a}) \end{aligned}$$

We could use the substitution $u = x + \frac{b}{2a}$, If necessary.

Examples:

1. $x^2 - 6x + 13 = (x-3)^2 + 4.$
2. $8 + 2x - x^2$

Partial fraction decompositions may lead to integrands contaInIng an irreducible quadratic expression $ax^2 + bx + c$. When $b \neq 0$, we may need to complete square:

$$\begin{aligned} ax^2 + bx + c &= a(x^2 + \frac{b}{a}x) + c \\ &= a(x^2 + \frac{b}{a}x + (\frac{b}{2a})^2) + c - a.(\frac{b}{2a})^2 \\ &= a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{2.a}) \end{aligned}$$

We could use the substitution $u = x + \frac{b}{2a}$, If necessary.

Examples:

1. $x^2 - 6x + 13 = (x-3)^2 + 4.$
2. $8 + 2x - x^2 = 8 - (x^2 - 2x)$

Partial fraction decompositions may lead to integrands contaInIng an irreducible quadratic expression $ax^2 + bx + c$. When $b \neq 0$, we may need to complete square:

$$\begin{aligned} ax^2 + bx + c &= a(x^2 + \frac{b}{a}x) + c \\ &= a(x^2 + \frac{b}{a}x + (\frac{b}{2a})^2) + c - a.(\frac{b}{2a})^2 \\ &= a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{2.a}) \end{aligned}$$

We could use the substitution $u = x + \frac{b}{2a}$, If necessary.

Examples:

1. $x^2 - 6x + 13 = (x-3)^2 + 4.$
2. $8 + 2x - x^2 = 8 - (x^2 - 2x) = 9 - (x-1)^2.$

Partial fraction decompositions may lead to integrands contaInIng an irreducible quadratic expression $ax^2 + bx + c$. When $b \neq 0$, we may need to complete square:

$$\begin{aligned} ax^2 + bx + c &= a(x^2 + \frac{b}{a}x) + c \\ &= a(x^2 + \frac{b}{a}x + (\frac{b}{2a})^2) + c - a.(\frac{b}{2a})^2 \\ &= a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{2.a}) \end{aligned}$$

We could use the substitution $u = x + \frac{b}{2a}$, If necessary.

Examples:

1. $x^2 - 6x + 13 = (x-3)^2 + 4.$
2. $8 + 2x - x^2 = 8 - (x^2 - 2x) = 9 - (x-1)^2.$
3. $x^2 + 8x + 25$

Partial fraction decompositions may lead to integrands contaInIng an irreducible quadratic expression $ax^2 + bx + c$. When $b \neq 0$, we may need to complete square:

$$\begin{aligned} ax^2 + bx + c &= a(x^2 + \frac{b}{a}x) + c \\ &= a(x^2 + \frac{b}{a}x + (\frac{b}{2a})^2) + c - a.(\frac{b}{2a})^2 \\ &= a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{2.a}) \end{aligned}$$

We could use the substitution $u = x + \frac{b}{2a}$, If necessary.

Examples:

1. $x^2 - 6x + 13 = (x - 3)^2 + 4.$
2. $8 + 2x - x^2 = 8 - (x^2 - 2x) = 9 - (x - 1)^2.$
3. $x^2 + 8x + 25 = (x^2 + 8x + 4^2) + 25 - 4^2$

Partial fraction decompositions may lead to integrands contaInIng an irreducible quadratic expression $ax^2 + bx + c$. When $b \neq 0$, we may need to complete square:

$$\begin{aligned} ax^2 + bx + c &= a(x^2 + \frac{b}{a}x) + c \\ &= a(x^2 + \frac{b}{a}x + (\frac{b}{2a})^2) + c - a.(\frac{b}{2a})^2 \\ &= a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{2.a}) \end{aligned}$$

We could use the substitution $u = x + \frac{b}{2a}$, If necessary.

Examples:

1. $x^2 - 6x + 13 = (x-3)^2 + 4.$
2. $8 + 2x - x^2 = 8 - (x^2 - 2x) = 9 - (x-1)^2.$
3. $x^2 + 8x + 25 = (x^2 + 8x + 4^2) + 25 - 4^2 = (x+4)^2 + 9.$

Example (1):

Example (1):

$$\int \frac{2x-1}{x^2-6x+13}dx$$

Example (1):

$$\int \frac{2x-1}{x^2-6x+13}dx \quad = \quad \int \frac{2x-1}{(x-3)^2+4}dx$$

Example (1):

$$\int \frac{2x-1}{x^2-6x+13}dx = \int \frac{2x-1}{(x-3)^2+4}dx$$

$$= \int \frac{2(u+3)-1}{u^2+4}du, \qquad \text{by puting } u = x-3$$

Example (1):

$$\begin{aligned}\int \frac{2x-1}{x^2-6x+13}dx &= \int \frac{2x-1}{(x-3)^2+4}dx \\ &= \int \frac{2(u+3)-1}{u^2+4}du, \qquad \text{by puting } u = x-3 \\ &= \int \frac{2u}{u^2+4}du + 5\int \frac{1}{u^2+4}du\end{aligned}$$

Example (1):

$$\begin{aligned}\int \frac{2x-1}{x^2-6x+13}dx &= \int \frac{2x-1}{(x-3)^2+4}dx \\ &= \int \frac{2(u+3)-1}{u^2+4}du, \qquad \text{by puting } u=x-3 \\ &= \int \frac{2u}{u^2+4}du + 5\int \frac{1}{u^2+4}du \\ &= \ln|u^2+4| + 5.\frac{1}{2}\tan^{-1}(\frac{u}{2}) + c\end{aligned}$$

Example (1):

$$\begin{aligned}\int \frac{2x-1}{x^2-6x+13}dx &= \int \frac{2x-1}{(x-3)^2+4}dx \\ &= \int \frac{2(u+3)-1}{u^2+4}du, \qquad \text{by puting } u = x-3 \\ &= \int \frac{2u}{u^2+4}du + 5\int \frac{1}{u^2+4}du \\ &= \ln|u^2+4| + 5.\frac{1}{2}\tan^{-1}(\frac{u}{2}) + c \\ &= \ln|x^2-6x+13| + \frac{5}{2}\tan^{-1}(\frac{x-3}{2}) + c.\end{aligned}$$

Example (1):

$$\begin{aligned}
\int \frac{2x-1}{x^2-6x+13}dx &= \int \frac{2x-1}{(x-3)^2+4}dx \\
&= \int \frac{2(u+3)-1}{u^2+4}du, \qquad \text{by puting } u = x-3 \\
&= \int \frac{2u}{u^2+4}du + 5\int \frac{1}{u^2+4}du \\
&= \ln|u^2+4| + 5.\frac{1}{2}\tan^{-1}(\frac{u}{2}) + c \\
&= \ln|x^2-6x+13| + \frac{5}{2}\tan^{-1}(\frac{x-3}{2}) + c.
\end{aligned}$$

Example (2):

Example (2):

$$\int \frac{1}{\sqrt{8+2x-x^2}}dx$$

Example (2):

$$\int \frac{1}{\sqrt{8+2x-x^2}}dx = \int \frac{1}{\sqrt{9-(x-1)^2}}dx$$

Example (2):

$$\int \frac{1}{\sqrt{8+2x-x^2}}dx = \int \frac{1}{\sqrt{9-(x-1)^2}}dx$$

$$= \int \frac{1}{\sqrt{3^2-u^2}}du, \qquad \text{by puting } u = x-1$$

Example (2):

$$\int \frac{1}{\sqrt{8+2x-x^2}}dx = \int \frac{1}{\sqrt{9-(x-1)^2}}dx$$
$$= \int \frac{1}{\sqrt{3^2-u^2}}du, \qquad \text{by puting } u = x-1$$
$$= \sin^{-1}(\frac{x-1}{3}) + c.$$

Example (2):

$$\int \frac{1}{\sqrt{8+2x-x^2}}dx = \int \frac{1}{\sqrt{9-(x-1)^2}}dx$$

$$= \int \frac{1}{\sqrt{3^2-u^2}}du, \qquad \text{by puting } u = x-1$$

$$= \sin^{-1}(\frac{x-1}{3}) + c.$$

Example (3):

Example (3):

$$\int \frac{1}{\sqrt{x^2 + 8x + 25}} dx$$

Example (3):

$$\int \frac{1}{\sqrt{x^2+8x+25}}dx \quad = \quad \int \frac{1}{\sqrt{(x+4)^2+9}}dx$$

Example (3):

$$\int \frac{1}{\sqrt{x^2+8x+25}}dx = \int \frac{1}{\sqrt{(x+4)^2+9}}dx$$

$$= \int \frac{3\sec(\theta)}{\sqrt{(3\tan(\theta))^2+9}}d\theta, \text{ puting } x+4 = 3\tan(\theta)$$

Example (3):

$$\int \frac{1}{\sqrt{x^2+8x+25}}dx = \int \frac{1}{\sqrt{(x+4)^2+9}}dx$$

$$= \int \frac{3\sec(\theta)}{\sqrt{(3\tan(\theta))^2+9}}d\theta, \text{ puting } x+4 = 3\tan(\theta)$$

$$= \int \frac{3.\sec^2(\theta)}{3\sec(\theta)}d\theta$$

Example (3):

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2+8x+25}}dx &= \int \frac{1}{\sqrt{(x+4)^2+9}}dx \\
&= \int \frac{3\sec(\theta)}{\sqrt{(3\tan(\theta))^2+9}}d\theta, \text{ puting } x+4 = 3\tan(\theta) \\
&= \int \frac{3.\sec^2(\theta)}{3\sec(\theta)}d\theta \\
&= \ln\left|\sec(\theta)+\tan(\theta)\right| + c
\end{aligned}$$

Example (3):

$$
\begin{aligned}
\int \frac{1}{\sqrt{x^2+8x+25}}dx &= \int \frac{1}{\sqrt{(x+4)^2+9}}dx \\
&= \int \frac{3\sec(\theta)}{\sqrt{(3\tan(\theta))^2+9}}d\theta, \text{ puting } x+4 = 3\tan(\theta) \\
&= \int \frac{3.\sec^2(\theta)}{3\sec(\theta)}d\theta \\
&= \ln|\sec(\theta)+\tan(\theta)| + c \\
&= \ln\left|\frac{\sqrt{x^2+8x+25}}{3} + \frac{x+4}{3}\right| + c
\end{aligned}
$$

Example (3):

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2+8x+25}}dx &= \int \frac{1}{\sqrt{(x+4)^2+9}}dx \\
&= \int \frac{3\sec(\theta)}{\sqrt{(3\tan(\theta))^2+9}}d\theta, \text{ puting } x+4=3\tan(\theta) \\
&= \int \frac{3.\sec^2(\theta)}{3\sec(\theta)}d\theta \\
&= \ln|\sec(\theta)+\tan(\theta)|+c \\
&= \ln\left|\frac{\sqrt{x^2+8x+25}}{3}+\frac{x+4}{3}\right|+c \\
&= \ln\left|\frac{\sqrt{x^2+8x+25}+x+4}{3}\right|+c
\end{aligned}$$

Example (3):

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2+8x+25}}dx &= \int \frac{1}{\sqrt{(x+4)^2+9}}dx \\
&= \int \frac{3\sec(\theta)}{\sqrt{(3\tan(\theta))^2+9}}d\theta, \text{ puting } x+4 = 3\tan(\theta) \\
&= \int \frac{3.\sec^2(\theta)}{3\sec(\theta)}d\theta \\
&= \ln|\sec(\theta)+\tan(\theta)| + c \\
&= \ln\left|\frac{\sqrt{x^2+8x+25}}{3} + \frac{x+4}{3}\right| + c \\
&= \ln\left|\frac{\sqrt{x^2+8x+25}+x+4}{3}\right| + c \\
&= \ln\left|\sqrt{x^2+8x+25}+x+4\right| - \ln(3) + c
\end{aligned}$$

Exercises: Evaluate the following integrals:

1. $\int \frac{1}{\sqrt{4x-x^2}}dx.$
2. $\int \frac{1}{\sqrt{7+6x-x^2}}dx.$
3. $\int \frac{x+5}{9x^2+6x+17}dx.$
4. $\int \frac{1}{\sqrt{x(6-x)}}dx$