1 A general single period market model

We will now consider a general single period market model, in which the agent is allowed to invest in a money market account (riskless asset or saving account) and a finite number of stocks (risky assets) described by their prices S^1, ..., S^d. The price of the i^{th} stock is denoted at time $t = 0$ by S_0^i and $t = 1$ by S_1^i. The riskless asset is modeled $S_0^0 = 1$ and $S_1^0 = 1 + r$. The prices of the stocks at time $t = 0$ are known (can be observed), but the prices the stocks will have at time $t = 1$ are not known at time $t = 0$ and are considered to be random.

We assume that the state of the world at time $t = 1$ can be one of the k states of the world ω_1, ω_2, ..., ω_k which we all put together into a set Ω, i.e. $\Omega := \{\omega_1, \omega_2, \ldots, \omega_k\}$.

We assume that on Ω is defined a probability measure P which tells us about the likelihood $P(\omega_i) > 0$ for all $i = \{1, 2, \ldots, k\}$. The stock prices S_1^i can therefore be considered as random variables

$$\begin{aligned} S_1^i &: \Omega \longrightarrow \mathbb{R} \\ &: \omega \longmapsto S_1^i(\omega) \end{aligned}$$

hence $S_1^i(\omega)$ denotes the price of the i^{th} stock at time $t = 1$ if the world is in state $\omega \in \Omega$ at time $t = 1$.

Let us now formally define the trading strategies which agents are going to use for this model.

Definition 1.1 *A trading strategy for an agent in our general single period market model (GSPMM) is a pair (x, Δ), where $\Delta = (\Delta^1, \ldots, \Delta^d) \in \mathbb{R}^d$ is an d–dimensional vector, specifying the initial total investment x at time $t = 0$ and Δ^i the number of shares i hold ought from the i^{th} stock.*

Given a trading strategy (x, Δ) as above, we always assume that the rest money $x - \sum_{i=1}^d \Delta^i S_0^i$ is invested in the money market account.

We define the corresponding value process to a trading strategy.

Definition 1.2 *The value process of the trading strategy (x, Δ) in our (GSPMM) is given by $(V_0^{(x,\Delta)}, V_1^{(x,\Delta)})$ or simply (V_0, V_1) where $V_0 = x$ and V_1 is the random variable*

$$\begin{aligned} V_1 &= \left(x - \sum_{i=1}^d \Delta^i S_0^i \right) (1 + r) + \sum_{i=1}^d \Delta^i S_1^i \\ &= x\,(1 + r) + \sum_{i=1}^d \Delta^i \left(S_1^i - S_0^i\,(1 + r) \right). \end{aligned}$$

1.1 Characterization of absence of arbitrages

Often it is very useful to compare and study the prices of the stocks in relation to the saving account. For this reason we introduce the discounted stock prices $\widetilde{S}_1^i$ defined as follows :

$$\widetilde{S}_1^i = \frac{S_1^i}{1 + r}, \text{ for } i = 1, \ldots, d.$$

We define also the discounted value process corresponding to the trading strategy (x, Δ) by

$$\widetilde{V}_1 = \frac{V_1}{1 + r} = x + \sum_{i=1}^d \Delta^i \left(\widetilde{S}_1^i - S_0^i \right). \tag{1.1}$$

1.1.1 Risk neutral probability measure

Definition 1.3 *A measure Q on Ω is called a risk neutral probability measure (RNPM) for our general single period market model (GSPMM) if:*

1. *$Q(\{\omega\}) > 0$ for all $\omega \in \Omega$.*
2. *$E_Q[\widetilde{S}_1^i] = E_Q[\frac{S_1^i}{1+r}] = S_0^i$ for $i = 1, \ldots, d$ or equivalently $E_Q[\frac{S_1^i}{1+r} - S_0^i] = 0$ for $i = 1, \ldots, d$.*

The following Theorem is one of the cornerstones of Financial Mathematics which says that risk neutral measures are closely connected to the question whether there is arbitrage in the model.

Theorem 1.1 ***(Fundamental Theorem of Asset Pricing)*** *In the general single period market model (GSPMM), there are no arbitrages if and only if there exist a risk neutral measure for the market model.*

Example 1.1 *We consider the following model featuring two stocks S^1 and S^2 as well as states $\Omega = \{\omega_1, \omega_2, \omega_3\}$. The prices of the stocks are given by the following table*

t	S_t^0	S_t^1			S_t^2		
		ω_1	ω_2	ω_3	ω_1	ω_2	ω_3
0	1	5	5	5	10	10	10
1	$\frac{10}{9}$	$\frac{60}{9}$	$\frac{60}{9}$	$\frac{40}{9}$	$\frac{40}{3}$	$\frac{80}{9}$	$\frac{80}{9}$

Question: *Is this model arbitrage free ?*
Answer: *Assume that there exists a probability measure Q such that*

$$E_Q\left[\frac{S_1^1}{1+r}\right] = S_0^1 \quad \text{and} \quad E_Q\left[\frac{S_1^2}{1+r}\right] = S_0^2.$$

This leads to the following system

$$\begin{cases} 5 = 6(q_1 + q_2) + 4q_3 \\ 10 = 12q_1 + 8(q_2 + q_3) \end{cases} \iff \begin{cases} 5 = 6(1 - q_3) + 4q_3 \\ 10 = 12q_1 + 8(1 - q_1) \end{cases}$$

Hence $Q = (q_1, q_2, q_3) = (\frac{1}{2}, 0, \frac{1}{2})$. Then this model is not arbitrage free.

Example 1.2 *Consider the following model $k = 3$, $d = 1$, $r = \frac{1}{9}$*

n	S_n^0	S_n^1		
		ω_1	ω_2	ω_3
0	1	5	5	5
1	$\frac{10}{9}$	$\frac{20}{3}$	$\frac{40}{9}$	$\frac{30}{9}$

Question: *Is this model arbitrage free ?*
Answer: *Assume that there exists a probability measure Q such that*

$$E_Q\left[\frac{S_1^1}{1+r}\right] = S_0^1 \iff 5 = 6q_1 + 4q_2 + 3\left(1 - q_1 - q_2\right).$$

which is equivalent $3q_1 + q_2 = 2$, hence $q_2 = 2 - 3q_1$ and $q_3 = 1 - q_1 - 2 + 3q_1 = 2q_1 - 1$. Hence $Q = (q, 2-3q, 2q-1)$ such that $\frac{1}{2} < q < \frac{2}{3}$. Therefore there infinite many solutions hence infinite RNPM.

Example 1.3 *Consider now, the following model: given by $k=3$, $d=2$, $r=\frac{1}{9}$ and the discounted price*

n	S_n^0	$\widetilde{S}_n^1$			$\widetilde{S}_n^2$		
		ω_1	ω_2	ω_3	ω_1	ω_2	ω_3
0	1	5	5	5	10	10	10
1	$\frac{10}{9}$	6	6	3	12	8	8

Question: *Is this model arbitrage free ?*
Answer: *Assume that there exists a probability measure Q such that*

$$E_Q\left[\frac{S_1^1}{1+r}\right] = S_0^1 \quad \text{and} \quad E_Q\left[\frac{S_1^2}{1+r}\right] = S_0^2.$$

This implies that

$$\left\{\begin{array}{l} 5 = 6(q_1+q_2)+3q_3 \\ 10 = 12q_1+8(q_2+q_3) \end{array}\right. \iff \left\{\begin{array}{l} 5 = 6(1-q_3)+3q_3 \\ 10 = 12q_1+8(1-q_1) \end{array}\right.$$

the solution of the system is given by $Q=(\frac{1}{2},\frac{1}{6},\frac{1}{3})$. Hence this model is arbitrage free.

1.2 Completeness: Attainable contingent claims

We now come back to the question "What should the price of an option in our model be ?" In one single period binomial model we considered options of the type $f(S_1)$where f is a payoff function. In the model considered in this section we now have more than one stock and the payoff may look more complicated. For this reason we generalize our definition of an option. We call this more general product contingent claim.

Definition 1.4 ***A contingent claim*** *in a market model is a random variable F on Ω representing a payoff at time $t=1$.*

To price a contingent claim, we may follow the same approach as in the binomial model and apply the replication principle.

Proposition 1.1 *Let F be a contingent claim in GSPMM and let (x,Δ) be a hedging strategy for F, that is a trading strategy which satisfies $V_1^{(x,\Delta)}=F$, then the only price of F which complies with the no arbitrage principle is $x=V_0^{(x,\Delta)}$.*

A crucial difference to the elementary single period model as discussed in the binomial model is however, that in the general single period market model, a replicating strategy might not exist. This can happen, when there are more effective sources of randomness, than there are stocks to invest in. Let us consider the following example, which represents an elementary version of a so called **stochastic volatility model**.

Example 1.4 *We consider the following market model. It consists of two tradeable assets, one money market account S_t^0 and one stock S_t $(t=0,1)$, as well as third object which we call the volatility σ. The volatility determines whether the stock price can make big jumps or small jumps. In this model the volatility is assumed to be random, or in other words stochastic. Such models are called stochastic volatility models. To be a bit more precise, we assume that our state space consists of four states: $\Omega := \{\omega_1,\omega_2,\omega_3,\omega_4\}$ and that the volatility is given by*

$$\sigma(\omega) = \left\{\begin{array}{ll} h & \text{if } \omega \in \{\omega_1,\omega_2\} \\ l & \text{if } \omega \in \{\omega_3,\omega_4\} \end{array}\right.$$

where $0 < l < h < 1$ and l stands for low volatility whereas h stands for high volatility. The stock price S_1 is then modeled by:

$$S_1(\omega) = \begin{cases} S_0\,(1+\sigma) & \text{if } \omega \in \{\omega_1, \omega_3\} \\ S_0\,(1-\sigma) & \text{if } \omega \in \{\omega_2, \omega_4\} \end{cases}$$

where S_0 denotes the initial stock price. The stock price can therefore jump up or jump down, as in the binomial model. The difference to this model is that the amount by which it jumps is itself random, determined by the volatility. Finally the saving account is modeled by

$$S_0^0 = 1 \quad \text{and} \quad S_1^0 = 1 + r$$

Let us now consider a digital call in this model, i.e.

$$F = \mathbf{1}_{]K,+\infty[}(S_1) = \begin{cases} 1 & \text{on the set } \{S_1 > K\} \\ 0 & \text{on the set } \{S_1 \leq K\} \end{cases}$$

Let us assume that the strike price K satisfies

$$S_0\,(1+l) < K < S_0\,(1+h)$$

Then a nonzero payoff is only possible if the volatility is high, and the stock jumps up. This is the case if and only if the state of the world at time $t = 1$ is given by $\omega = \omega_1$. The contingent claim F can therefore alternatively be written as

$$F(\omega) = \begin{cases} 1 & \text{if } \omega = \omega_1 \\ 0 & \text{if } \omega \in \{\omega_2, \omega_3, \omega_4\}. \end{cases}$$

Let us now see whether there exists a replicating strategy for this contingent claim, i.e. a trading strategy (x, Δ) satisfying

$$V_1^{(x,\Delta)} = F \iff (x - \Delta S_0)(1+r) + \Delta S_1 = F$$

which leads to the system of equations

$$\begin{aligned} (x - \Delta S_0)(1+r) + \Delta S_0\,(1+h) &= 1 \\ (x - \Delta S_0)(1+r) + \Delta S_0\,(1+l) &= 0 \\ (x - \Delta S_0)(1+r) + \Delta S_0\,(1-l) &= 0 \\ (x - \Delta S_0)(1+r) + \Delta S_0\,(1-h) &= 0 \end{aligned}$$

which is equivalent to

$$\begin{aligned} x(1+r) &= 1 - \Delta S_0\,(r-h) \\ x(1+r) &= \Delta S_0\,(r-l) \\ x(1+r) &= \Delta S_0\,(r+l) \\ x(1+r) &= \Delta S_0\,(r+h) \end{aligned}$$

Solving for x in the second respectively fourth component yields,

$$x = \frac{\Delta S_0(h+r)}{1+r} \quad \text{and } x = \frac{\Delta S_0(l+r)}{1+r}$$

This is not possible unless $\Delta = 0$ since $S_0(h+r) \neq S_0(l+r)$. The conclusion is, that there is no trading strategy (x, Δ) which replicates F. The heuristic explanation is, that there is a source of randomness in the volatility, which cannot be hedged, since the volatility is not tradeable. The mathematical explanation is just, that the system of linear equations above has no solution. To take account of this difficulty we introduce the following Definition.

Definition 1.5 *A contingent claim F is said to be attainable, if there exists a trading strategy (self-financing portfolio) (x,Δ) which replicates F, i.e. satisfies $V_1^{(x,\Delta)} = F$.*

For attainable contingent claims the replication principle applies and it is clear how to price them, namely by the total initial investment needed for a replicating strategy. There might be more than one replicating strategy, but it follows again from the no arbitrage principle, that the total initial investment for replicating strategies is unique.

Proposition 1.2 *Let F be an attainable contingent claim and Q be an arbitrary risk neutral probability measure (RNPM). Then the price x of F at time $t = 0$ defined via a replicating strategy can be computed by the formula*

$$x = E_Q\left[\frac{F}{1+r}\right].$$

Proof. Let (x,Δ) be a replicating strategy of F, that is $V_1^{(x,\Delta)} = F$. It follow the (1.1) that

$$\begin{aligned} E_Q\left[\frac{F}{1+r}\right] &= E_Q\left[\frac{V_1^{(x,\Delta)}}{1+r}\right] \\ &= E_Q\left[x + \sum_{i=1}^{d} \Delta^i \left(\frac{S_1^i}{1+r} - S_0^i\right)\right] \\ &= x + \sum_{i=1}^{d} \Delta^i E_Q\left[\widetilde{S}_1^i - \widetilde{S}_0^i\right] = x \end{aligned}$$

since $\widetilde{S}_0^i = S_0^i$ and by the definition of Q

$$E_Q\left[\frac{S_1^i}{1+r} - S_0^i\right] = 0 \ \text{ for all } i = \{1,\ldots,d\}.$$

Remark that in the following model $k = 3$, $d = 1$, $r = \frac{1}{9}$

n	S_n^0	S_n^1		
		ω_1	ω_2	ω_3
0	1	5	5	5
1	$\frac{10}{9}$	$\frac{20}{3}$	$\frac{40}{9}$	$\frac{30}{9}$

the RNPM exists by is not unique, hence the prices of contingent claims are not unique.

Now let us characterize the models, in which the problem of non uniqueness of prices does not occur

Definition 1.6 *A financial market model is said to be* ***complete****, if for any contingent claim F there exists a replicating strategy (x,Δ). A model which is not complete is called* ***incomplete****.*

The following proposition gives us a criterion for completeness.

Proposition 1.3 *Assume a general single period market model consisting of d stocks S^1, $\ldots$, S^d and a saving account S^0 modeled on the state space $\Omega = \{\omega_1,\omega_3,\ldots,\omega_k\}$ is arbitrage free. Then this model is complete if and only if the $k\times(d+1)$ matrix A given*

$$A := \begin{pmatrix} 1+r & S_1^1(\omega_1) & S_1^2(\omega_1) & \cdots & \cdots & \cdots & S_1^d(\omega_1) \\ 1+r & S_1^1(\omega_2) & S_1^2(\omega_2) & \cdots & \cdots & \cdots & S_1^d(\omega_2) \\ \vdots & & & & & & \\ \vdots & & & & & & \\ \vdots & & & & & & \\ 1+r & S_1^1(\omega_{k-1}) & S_1^2(\omega_{k-1}) & \cdots & \cdots & \cdots & S_1^d(\omega_{k-1}) \\ 1+r & S_1^1(\omega_k) & S_1^2(\omega_k) & \cdots & \cdots & \cdots & S_1^d(\omega_k) \end{pmatrix}$$

has full rank, i.e. rank$(A)=k$.

Proof. The model is complete if there exist a replicating strategy (x,Δ) for a given contingent claim F. Then

$$V_1^{(x,\Delta)}=\left(x-\sum_{i=1}^{d}\Delta^i S_0^i\right)(1+r)+\sum_{i=1}^{d}\Delta^i S_1^i=F. \tag{1.2}$$

If we set $\alpha=x-\sum_{i=1}^{d}\Delta^i S_0^i$, the equation (1.2) can be written as $A\binom{\alpha}{\Delta}=F$. Therefore (α,Δ) is unique if the rank$(A)=k$. Hence (x,Δ) is unique.

Example 1.5 *The volatility model defined on* $\Omega:=\{\omega_1,\omega_2,\omega_3,\omega_4\}$ *by*

$$S_1(\omega)=\begin{cases} S_0(1+\sigma) & \text{if } \omega\in\{\omega_1,\omega_3\}\\ S_0(1-\sigma) & \text{if } \omega\in\{\omega_2,\omega_4\}\end{cases}$$

where

$$\sigma(\omega)=\begin{cases} h & \text{if } \omega\in\{\omega_1,\omega_2\}\\ l & \text{if } \omega\in\{\omega_3,\omega_4\}.\end{cases} \quad \text{such that } 0<l<h<1$$

is not complete since the rank of the matrix

$$A=\begin{pmatrix} 1+r & S_0(1+h)\\ 1+r & S_0(1-h)\\ 1+r & S_0(1+l)\\ 1+r & S_0(1-l)\end{pmatrix}$$

is equal to $2<4$.

The previous proposition presents a method how to determine whether a model is complete, without computing replicating strategies. Now, if the model is not complete, is there a method how to determine whether a specific contingent claim is attainable, without trying to compute the replicating strategy ? Yes there is. The following proposition shows how.

Proposition 1.4 *The contingent claim* F *is attainable, if and only if* $E_Q[\frac{F}{1+r}]$ *takes the same value for all RNPM* Q.

The following model

n	S_n^0	S_n^1		
		ω_1	ω_2	ω_3
0	1	5	5	5
1	$\frac{10}{9}$	$\frac{20}{3}$	$\frac{40}{9}$	$\frac{30}{9}$

is arbitrage free and the RNPM are given by $Q=(q,2-3q,2q-1)$ for $\frac{1}{2}<q<\frac{2}{3}$.
If $F(\omega_1)=f_1$, $F(\omega_2)=f_2$ and $F(\omega_3)=f_3$ then

$$\begin{aligned} E_Q\left[\frac{F}{1+\frac{1}{9}}\right] &= \frac{9}{10}(qf_1+(2-3q)f_2+(2q-1)f_3)\\ &= \frac{9}{10}q(f_1-3f_2+2f_3)+\frac{9}{10}(2f_2-f_3). \end{aligned}$$

Therefore if $f_1-3f_2+2f_3=0$ then

$$E_Q\left[\frac{F}{1+\frac{1}{9}}\right]=\frac{9}{10}(2f_2-f_3).$$

Hence the contingent claim $F=(f_1,f_2,f_3)$ is attainable if and only if $f_1-3f_2+2f_3$. (for example $F=(4,1,0)$ is not attainable.

An important consequence of this proposition is the following theorem:

Theorem 1.2 *A market model is complete if and only if there exists only one risk neutral probability measure (RNPM).*

Example 1.6 *a. Consider a one–period (annual) market model consisting in a non-risky asset (paying a risk-free rate of 5% per year) and in two stocks with prices S^1 and S^2:*

n	S_n^0	S_n^1			S_n^2		
		ω_1	ω_2	ω_3	ω_1	ω_2	ω_3
0	1	10	10	10	5	5	5
1	$1+0.05$	12	8	6	10	4	5

Is this model arbitrage free and/or complete ?
a) If a RNPM $Q=(q_1,q_2,q_3)$ exists it should satisfy $1=q_1+q_2+q_3$ and

$$\begin{cases} 10 = \dfrac{12}{1.05}q_1 + \dfrac{8}{1.05}q_2 + \dfrac{6}{1.05}(1-q_1-q_2) \\ 5 = \dfrac{10}{1.05}q_1 + \dfrac{4}{1.05}q_2 + \dfrac{5}{1.05}(1-q_1-q_2) \end{cases}$$

$$\begin{cases} 10.5 = 12q_1+8q_2+6q_3 \\ 5.25 = 10q_1+4q_2+5q_3 \end{cases} \quad \begin{cases} 10.5 = 12q_1+8q_2+6q_3 \\ 10.5 = 20q_1+8q_2+10q_3 \end{cases}$$

which leads to $q_1=q_3=0$ and $q_2=1=\frac{5.25}{4}$, hence there is no RNPM for this model and the model has arbitrage opportunities. An example of arbitrage opportunity is given by the strategy $\phi=(0,-1,2)$ consisting in zero positions in the bond, a short position in stock S^1 and two long positions in stock S^2. Indeed we $V_0^\phi = 0\times 1 + (-1)\times 10 + 2\times 5 = 0$ while

$$V_1^\phi = 0\times(1+0.05) + (-1)S_1^1 + 2S_1^2 = \begin{cases} 8 & \text{if } \omega=\omega_1 \\ 0 & \text{if } \omega=\omega_2 \\ 4 & \text{if } \omega=\omega_3 \end{cases}$$

and $V_1^\phi \geq 0$ and $P(V_1^\phi>0)=\frac{2}{3}$ if ω_1, ω_2 and ω_3 are equally likely to appear. We can also remark that $E[V_1^\phi] = 8P(V_1^\phi=8)+4P(V_1^\phi=4)>0$.
b) Consider now the following model

n	S_n^0	S_n^1			S_n^2		
		ω_1	ω_2	ω_3	ω_1	ω_2	ω_3
0	1	10	10	10	10	10	10
1	$1+0.05$	12	10	6	15	8	8

Is this model arbitrage free and/or complete ?
a) If a RNPM Q exists it should satisfy

$$\begin{cases} 10.5 = 12q_1+10q_2+6(1-q_1-q_2) \\ 10.5 = 15q_1+8(1-q_1) \end{cases} \quad \begin{cases} q_1 = \frac{5}{14}. \\ 4.5 = 6\frac{5}{14}+4q_2 \end{cases}$$

which gives a unique solution $Q=(\frac{5}{14},\frac{33}{56},\frac{3}{56})$, hence the model is arbitrage free and complete

Example 1.7 *1. The volatility model defined on $\Omega := \{\omega_1,\omega_2,\omega_3,\omega_4\}$ by*

$$S_1(\omega) = \begin{cases} S_0(1+\sigma) & \text{if } \omega\in\{\omega_1,\omega_3\} \\ S_0(1-\sigma) & \text{if } \omega\in\{\omega_2,\omega_4\} \end{cases}$$

where

$$\sigma(\omega) = \begin{cases} h & \text{if } \omega\in\{\omega_1,\omega_2\} \\ l & \text{if } \omega\in\{\omega_3,\omega_4\}. \end{cases} \quad \text{such that } 0<l<h<1$$

Is this model arbitrage free and /or complete ?