قاعدة السلسلة

لنفرض أن

$$\begin{cases} f(x,y) = xy^2 + x^2 + y^2 \\ x = t+1 \, ; \; y = t^2 \end{cases}$$

إذن:

$$f(x,y) = xy^2 + x^2 + y^2 = (t+1)(t^2)^2 + (t+1)^2 + (t^2)^2$$

$$= t^4(t+1) + t^2 + 2t + 1 + t^4$$

$$= t^5 + 2t^4 + t^2 + 2t + 1 = u(t)$$

لدينا:

$$u'(t) = \frac{du}{dt}(t) = 5t^4 + 8t^3 + 2t + 2$$

ولدينا:

$$\frac{\partial f}{\partial x}(x,y) \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y}(x,y) \cdot \frac{dy}{dt} = (y^2 + 2x) \cdot 1 + (2xy + 2y)(2t)$$

$$= \left((t^2)^2 + 2(t+1)\right) + \left(2(t+1)t^2 + 2t^2\right)2t$$

$$= t^4 + 2t + 2 + (2t^3 + 2t^2 + 2t^2)2t$$

$$= \underline{t^4} + 2t + 2 + \underline{4t^4} + 4t^3 + 4t^3$$

$$= 5t^4 + 8t^3 + 2t + 2$$

مبرهنة: قاعدة السلسلة

(1) لتكن f دالة في متغيرين x, y حيث $x=x(t)$, $y=y(t)$
إذا كان f لها مشتقات جزئية أولى متصلة عند كل نقطة، و $x(t)$, $y(t)$
كل منها لها مشتقات متصلة فإن $f(x,y)=w(t)$ لها مشتقات
حيث:

$$\boxed{\frac{dw}{dt}(t) = w'(t) = \frac{\partial f}{\partial x}(x(t),y(t))\cdot x'(t) + \frac{\partial f}{\partial y}(x(t),y(t))\cdot y'(t)}$$

(2) لتكن f دالة في متغيرين x, y حيث $f(x,y)=w(u,v)$
و $x=g(u,v)$ و $y=h(u,v)$، وكلها لها مشتقات
جزئية من الرتبة الثانية متصلة فإن:

$$\boxed{\begin{aligned} \frac{\partial w}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \\ \frac{\partial w}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v} \end{aligned}}$$

مثال 1 ص 94

إذا كانت $w = x^2 + 2xy + y^2$ ، وكانت $x = t\cos t$, $y = t\sin t$

أوجد $\frac{dw}{dt}$

الحل: طريقة أولى:

$$w = x^2 + 2xy + y^2 = (t\cos t)^2 + 2\, t\cos t \cdot t\sin t + (t\sin t)^2$$

$$w = t^2\left(\cos^2 t + 2\cos t\sin t + \sin^2 t\right) = t^2(1 + 2\cos t\sin t)$$

$$\frac{dw}{dt} = 2t(1 + 2\cos t\sin t) + t^2(2\cos t\cos t - 2\sin t\sin t)$$

$$\boxed{\frac{dw}{dt} = 2t\left(1 + 2\cos t\sin t + t\cos^2 t - t\sin^2 t\right)}$$

الطريقة الثانية باستعمال قاعدة السلسلة

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} = (2x + 2y)(\cos t - t\sin t) + (2x + 2y)(\sin t + t\cos t)$$

$$= 2(t\cos t + t\sin t)(\cos t - t\sin t) + 2(t\cos t + t\sin t)(\sin t + t\cos t)$$

$$= 2(t\cos t + t\sin t)(\cos t - t\sin t + \sin t + t\cos t)$$

$$= 2t(\cos t + \sin t)(\cos t + \sin t + t(\cos t - \sin t))$$

$$= 2t\left[(\cos t + \sin t)^2 + t(\cos t + \sin t)(\cos t - \sin t)\right]$$

$$= 2t\left(\cos^2 t + 2\cos t\sin t + \sin^2 t + t(\cos^2 t - \sin^2 t)\right)$$

$$\boxed{\frac{dw}{dt} = 2t\left(1 + 2\cos t\sin t + t\cos^2 t - t\sin^2 t\right)}$$

مثال 2 ص 95

إذا كانت $w = f(x,y) = \ln\left(\sqrt{x^2+y^2}\right)$, كانت $x = re^{s}$, $y = re^{-s}$

احسب كلاً من $\frac{\partial w}{\partial r}$, $\frac{\partial w}{\partial s}$.

$\ln u^r = r\ln u$

$\ln a.b = \ln a + \ln b$

الحل: طريقة أولى

$$w = \frac{1}{2}\ln(x^2+y^2) = \frac{1}{2}\ln\left((re^{s})^2 + (re^{-s})^2\right)$$

$$w = \frac{1}{2}\ln\left(r^2(e^{2s}+e^{-2s})\right) = \frac{1}{2}\ln r^2 + \frac{1}{2}\ln(e^{2s}+e^{-2s}) = \ln r + \frac{1}{2}\ln(e^{2s}+e^{-2s})$$

$$\boxed{\frac{\partial w}{\partial r} = \frac{1}{r} \quad , \quad \frac{\partial w}{\partial s} = \frac{1}{2}\,\frac{2e^{2s}-2e^{-2s}}{e^{2s}+e^{-2s}} = \frac{e^{2s}-e^{-2s}}{e^{2s}+e^{-2s}}}$$

الطريقة الثانية باستعمال قاعدة السلسلة :

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{x}{x^2+y^2}\cdot e^{s} + \frac{y}{x^2+y^2}\cdot e^{-s} = \frac{re^{s}\cdot e^{s} + re^{-s}e^{-s}}{(re^{s})^2+(re^{-s})^2}$$

$$\frac{\partial w}{\partial r} = \frac{r(e^{2s}+e^{-2s})}{r^2(e^{2s}+e^{-2s})} = \frac{1}{r} \qquad \boxed{\frac{\partial w}{\partial r} = \frac{1}{r}}$$

بأن

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} = \frac{x}{x^2+y^2}\,r.e^{s} - \frac{y}{x^2+y^2}\cdot re^{-s} = \frac{re^{s}.re^{s} - re^{-s}.re^{-s}}{r^2(e^{2s}+e^{-2s})}$$

$$\frac{\partial w}{\partial s} = \frac{r^2(e^{2s}-e^{-2s})}{r^2(e^{2s}+e^{-2s})} \Rightarrow \boxed{\frac{\partial w}{\partial s} = \frac{e^{2s}-e^{-2s}}{e^{2s}+e^{-2s}}}$$

مثال 4 ص 96

برهن أن الدالة $z = f\left(\frac{1}{2}bx^2 - \frac{1}{3}ay^3\right)$ تحقق المعادلة:

$$ay^2\frac{\partial z}{\partial x} + bx\frac{\partial z}{\partial y} = 0$$

الحل:

نفرض أن $u = \frac{1}{2}bx^2 - \frac{1}{3}ay^3$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} = bx\frac{df}{du}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} = -ay^2\frac{df}{du}$$

وبالتالي

$$ay^2\frac{\partial z}{\partial x} + bx\frac{\partial z}{\partial y} = ay^2\left(bx\frac{df}{du}\right) + bx\left(-ay^2\frac{df}{du}\right)$$

$$= \left(\underbrace{abxy^2 - abxy^2}_{0}\right)\frac{df}{du} = 0$$

إذن

$$\boxed{ay^2\frac{\partial z}{\partial x} + bx\frac{\partial z}{\partial y} = 0}$$

مثال 2 ص 46

برهن أن الدالة $z = f(s^2-t^2, t^2-s^2)$

تحقق العلاقة: $t\frac{\partial z}{\partial s} + s\frac{\partial z}{\partial t} = 0$

الحل: لدينا $x = s^2-t^2$, $y = t^2-s^2$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} = \frac{\partial f}{\partial s}2s + \frac{\partial f}{\partial y}(-2s) = 2s\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}2t = 2t\left(-\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right)$$

فإن:

$$t\frac{\partial z}{\partial s} + s\frac{\partial z}{\partial t} = t\left(2s\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right)\right) + s\left(2t\left(-\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right)\right)$$

$$= 2ts\underbrace{\left[\frac{\partial f}{\partial x} - \cancel{\frac{\partial f}{\partial y}} - \frac{\partial f}{\partial x} + \cancel{\frac{\partial f}{\partial y}}\right]}_{0} = 0$$

وبالتالي:

$$\boxed{t\frac{\partial z}{\partial s} + s\frac{\partial z}{\partial t} = 0}$$

مثال 6 ص 97

نفرض أن f دالة في متغيرين، وأن مشتقاتها الجزئية من الرتبة الثانية متصلة. إذا كانت: $w=f(x,y)$ حيث $x=u+v$ ، $y=u-v$

برهن أن $\frac{\partial^2 w}{\partial u \partial v}=\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w}{\partial y^2}$

الحل:

لدينا: $\frac{\partial w}{\partial u}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}$

فإن: $\frac{\partial^2 w}{\partial u \partial v}=\frac{\partial}{\partial v}\left(\frac{\partial w}{\partial u}\right)=\frac{\partial}{\partial v}\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\right)\frac{\partial x}{\partial v}+\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\right)\frac{\partial y}{\partial v}$

$$=\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y \partial x}\right)(1)+\left(\frac{\partial^2 f}{\partial x \partial y}+\frac{\partial^2 f}{\partial y^2}\right)(-1)$$

$$=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y \partial x}-\frac{\partial^2 f}{\partial x \partial y}-\frac{\partial^2 f}{\partial y^2}$$

وبما أن f لها مشتقات جزئية من الرتبة الثانية متصلة فإن $\frac{\partial^2 f}{\partial x \partial y}=\frac{\partial^2 f}{\partial y \partial x}$

فإن

$$\boxed{\frac{\partial^2 w}{\partial u \partial v}=\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 f}{\partial y^2}}$$

مثال 7 ص 98

u دالة في متغيرين x و y وكانت $x=e^s$, $y=e^t$

برهن أن: $x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$

يمكن كتابتها بالصيغة: $\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}=0$

الحل:

$$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}=\frac{\partial u}{\partial x}e^s+\frac{\partial u}{\partial y}\cdot 0=e^s\cdot\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 u}{\partial s^2}=\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial s}\right)=\frac{\partial}{\partial s}\left(e^s\cdot\frac{\partial u}{\partial x}\right)=e^s\cdot\frac{\partial u}{\partial x}+e^s\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial^2 u}{\partial s^2}=x\cdot\frac{\partial u}{\partial x}+x\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\cdot\frac{\partial x}{\partial s}+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)\cdot\frac{\partial y}{\partial s}\right)$$

$$=x\frac{\partial u}{\partial x}+x\left(\frac{\partial^2 u}{\partial x^2}e^s+\frac{\partial^2 u}{\partial x\partial y}\cdot 0\right)\Rightarrow\boxed{\frac{\partial^2 u}{\partial s^2}=x\frac{\partial u}{\partial x}+x^2\frac{\partial^2 u}{\partial x^2}}\ (1)$$

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t}=\frac{\partial u}{\partial x}\cdot 0+\frac{\partial u}{\partial y}\cdot e^t=y\cdot\frac{\partial u}{\partial y}=e^t\frac{\partial u}{\partial y}$$

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)=\frac{\partial}{\partial t}\left(e^t\frac{\partial u}{\partial y}\right)=e^t\frac{\partial u}{\partial y}+e^t\cdot\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)$$

$$\frac{\partial^2 u}{\partial t^2}=y\frac{\partial u}{\partial y}+y\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)\cdot\frac{\partial x}{\partial t}+\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\cdot\frac{\partial y}{\partial t}\right)=y\frac{\partial u}{\partial y}+y\left(\frac{\partial^2 u}{\partial y^2}\cdot e^t\right)=y\frac{\partial u}{\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$$

$$\boxed{\frac{\partial^2 u}{\partial t^2}=y\frac{\partial u}{\partial y}+y^2\frac{\partial^2 u}{\partial y^2}}\ (2)$$

حسب (1) , (2) لدينا:

$$\boxed{\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}=x\frac{\partial u}{\partial x}+x^2\frac{\partial^2 u}{\partial x^2}+y\frac{\partial u}{\partial y}+y^2\frac{\partial^2 u}{\partial y^2}}$$

وبالتالي العبارة:

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$$

يمكن كتابتها:

$$\frac{\partial^2 u}{\partial s^2}+\frac{\partial^2 u}{\partial t^2}=0$$

مثال 8 ص 99

f مشتقاتها من الرتبة الأولى والثانية متصلة

حيث $x = e^s \cos t$, $y = e^s \sin t$

برهن أن: الدالة $w = f(x,y)$ تحقق العلاقة:

$$e^{-2s}\left(\frac{\partial^2 w}{\partial s^2} + \frac{\partial^2 w}{\partial t^2}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

الحل:

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}\cdot e^s\cos t + \frac{\partial f}{\partial y}\cdot e^s\sin t$$

$$\frac{\partial^2 w}{\partial s^2} = \frac{\partial}{\partial s}\left(\frac{\partial w}{\partial s}\right) = \frac{\partial}{\partial s}\left(e^s\cos t\cdot\frac{\partial f}{\partial x} + e^s\sin t\,\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial s}\left(e^s\cos t\,\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial s}\left(e^s\sin t\,\frac{\partial f}{\partial y}\right)$$

$$= \cos t\left(e^s\frac{\partial f}{\partial x} + e^s\cdot\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial x}\right)\right) + \sin t\left(e^s\cdot\frac{\partial f}{\partial y} + e^s\cdot\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial y}\right)\right)$$

$$= e^s\cos t\left(\frac{\partial f}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\cdot\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\frac{\partial y}{\partial s}\right) + e^s\sin t\left(\frac{\partial f}{\partial y} + \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\cdot\frac{\partial x}{\partial s} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right.$$

$$\frac{\partial^2 w}{\partial s^2} = e^s\cos t\left(\frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial x^2}e^s\cos t + \frac{\partial^2 f}{\partial x\partial y}\cdot e^s\sin t\right) + e^s\sin t\left(\frac{\partial f}{\partial y} + \frac{\partial^2 f}{\partial y\partial x}e^s\cos t + \frac{\partial^2 f}{\partial y^2}\cdot e^s\sin t\right.$$

$$\boxed{\frac{\partial^2 w}{\partial s^2} = e^s\cos t\,\frac{\partial f}{\partial x} + e^s\sin t\,\frac{\partial f}{\partial y} + e^{2s}\cos^2 t\,\frac{\partial^2 f}{\partial x^2} + 2e^{2s}\cos t\sin t\,\frac{\partial^2 f}{\partial x\partial y} + e^{2s}\sin^2 t\,\frac{\partial^2 f}{\partial y^2}}$$

$$\frac{\partial^2 w}{\partial t^2} = ?$$

الدوال المتجانسة

تعريف: ليكن f دالة في متغيرين x و y.

f دالة متجانسة من درجة k $(k \in \mathbb{R})$ إذا كان:

(1) لكل $(x,y) \in D_f$ ولكل $t > 0$ ، $(tx,ty) \in D_f$

(2) لكل $(x,y) \in D_f$ ولكل $t > 0$ ، $f(tx,ty) = t^k f(x,y)$

مثال: (1) $f(x,y)=\dfrac{xy}{x^2+y^2}$

$D_f=\mathbb{R}^2\setminus\{(0,0)\}$, لكل $(x,y)\in D_f$ فإن $(x,y)\neq(0,0)$, لكل $t>0$

، فإن $(tx,ty)=(0,0)$ $\big(=t(x,y)\big)$ فإن $(tx,ty)\in D_f$

، لكل $(x,y)\in D_f$, لكل $t>0$.

$$f(tx,ty)=\frac{tx\cdot ty}{(tx)^2+(ty)^2}=\frac{t^2xy}{t^2(x^2+y^2)}=\frac{xy}{x^2+y^2}=f(x,y)\ \big(=t^0f(x,y)\big)$$

فإن f دالة متجانسة من درجة 0.

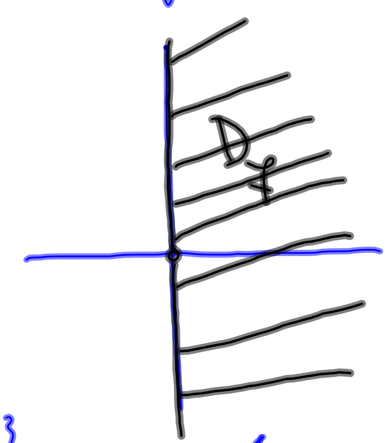
(2) $f(x,y)=\dfrac{\sqrt{x}\cdot y}{x^2+y^2}$, $D_f=\{(x,y): x\geq 0,\ y\in\mathbb{R}\}\setminus\{(0,0)\}$

لكل $(x,y)\in D_f$, $t>0$

فإن $(tx,ty)\neq(0,0)$, $tx\geq 0$

فإن $(tx,ty)\in D_f$

. لكل $(x,y)\in D_f$, $t>0$



$$f(tx,ty)=\frac{\sqrt{tx}\cdot ty}{(tx)^2+(ty)^2}=\frac{t\sqrt{t}\sqrt{x}\,y}{t^2(x^2+y^2)}=\frac{t^{\frac{3}{2}}\sqrt{x}\,y}{t^2(x^2+y^2)}=t^{-\frac{1}{2}}f(x,y)$$

فإن f دالة متجانسة من درجة $\left(-\frac{1}{2}\right)$

(٣) $f(x,y)=x^2+xy^2$, $D_f=\mathbb{R}^2$

• لكل $(x,y)\in D_f$, $t>0$ فإن $(tx,ty)\in D_f$

• لكل $(x,y)\in D_f$, لكل $t>0$:

$$f(tx,ty)=(tx)^2+(tx)(ty)^2=t^2x^2+t^3xy^2=t^2(x^2+txy^2)$$

ولدينا: $t=2$; $f(x,y)=x^2+xy^2$

$$f(2x,2y)=2^2(x^2+2xy^2)\neq 2^2(x^2+xy^2)=2^2f(x,y)$$

وبالتالي f ليست دالة متجانسة.

خصائص الدالة المتجانسة

(1) إذا كان f دالة متجانسة من درجة k ولها مشتقات أولية متصلة

فإن $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ كل منها دالة متجانسة من درجة $(k-1)$

(2) إذا كان f دالة متجانسة من درجة k ومشتقاتها الجزئية الأولى متصلة

فإن لكل $(x,y)\in D_f$:

$$\boxed{x\frac{\partial f}{\partial x}(x,y)+y\frac{\partial f}{\partial y}(x,y)=k\cdot f(x,y)}$$

البرهان: (1) f دالة متجانسة من درجة k ومشتقاتها الجزئية الأولى متصلة

إذن $\left\{\begin{array}{l} \text{لكل } (x,y)\in D_f \text{ ، } t>0 \text{ فإن } (tx,ty)\in D_f \\ \text{ولكل } (x,y)\in D_f \text{ ، } t>0 \text{ فإن } f(tx,ty)=t^k f(x,y) \end{array}\right.$

إذن $$\frac{\partial f}{\partial x}(tx,ty)\cdot\frac{\partial(tx)}{\partial x}=t^k\cdot\frac{\partial f}{\partial x}(x,y)$$

إذن $$\frac{\partial f}{\partial x}(tx,ty)\cdot t=t^k\frac{\partial f}{\partial x}(x,y)$$

إذن $$\boxed{\frac{\partial f}{\partial x}(tx,ty)=t^{k-1}\frac{\partial f}{\partial x}(x,y)}$$

(2) f دالة متجانسة من درجة k ومشتقاتها الجزئية الأولى متصلة.

إذن $\left\{\begin{array}{l} \text{إذا كان } t>0 \text{ ، } (x,y)\in D_f \text{ فإن } (tx,ty)\in D_f \\ \text{ولكل } t>0 \text{ ، } (x,y)\in D_f \text{ فإن } f(tx,ty)=t^k f(x,y) \end{array}\right.$

نشتق المعادلة السابقة حسب t ونجد باستعمال قاعدة السلسلة:

$$\frac{\partial f}{\partial x}(tx,ty)\cdot\frac{\partial(tx)}{\partial t}+\frac{\partial f}{\partial y}(tx,ty)\cdot\frac{\partial(ty)}{\partial t}=k\,t^{k-1}f(x,y)$$

إذن: $$x\cdot\frac{\partial f}{\partial x}(tx,ty)+y\frac{\partial f}{\partial y}(tx,ty)=k\,t^{k-1}f(x,y)$$

نأخذ $t=1$ إذن

$$\boxed{x\frac{\partial f}{\partial x}(x,y)+y\frac{\partial f}{\partial y}(x,y)=k f(x,y)}$$

ملاحظة: $t=2$:

$$x\frac{\partial f}{\partial x}(2x,2y)+y\frac{\partial f}{\partial y}(2x,2y)=k\,2^{k-1}f(x,y)$$