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Stochastic Aspects of Proportional Vitalities Model

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Abstract: In this paper, a family of models requiring proportional mean life vitalities is considered. The problem of estimation of the parameter(s) of the model is studied in two cases of known and unknown baselines along with some simulation studies to detect the adequacy of fitting. Closure and preservation properties of some ageing classes and stochastic orders are derived.

Keywords: vitality function; smoothing; empirical distribution; triangular distribution; hazard rate order; mean residual order; preservation; simulation

1. Introduction

A large segment of statistical methodology deals with the modeling and analysis of data representing the time until the occurrence of an event (cf. Nanda et al. [1]). In industrial and biomedical studies, these events represent the time to failure of a machine, an organ, an individual, or the completion of a certain task. These times are referred to as survival times in biomedical studies and as lifetimes or failure times in actuarial and engineering studies (cf. Henley and Kumamoto [2], Fleming [3] and Miller [4]). Reliability measures are ordinarily considered in lifetime sciences as characteristic quantities to quantify the amount of uncertainty during the lifetime process of an item to presignify its life span. Endless numbers of models are being introduced in the literature to model failure time data. The proportional hazard rates (PHR) model, used to model left-truncated and right-censored failure times, is a reputable model that has been greatly developed in the literature (see, e.g., Cox [5]). To concentrate on the average amounts in place of probability values Oakes and Dasu [6] introduced the proportional mean residual life (PMRL) model for the analysis of reliability and survival data in the situations where left-truncated or right-censored observations are available. Nanda et al. [7] studied some reliability aspects and further stochastic properties including a number of stochastic comparisons and ageing properties in the PMRL model. In contrast to the PHR and the PMRL models, there have been proposed specific models to account for right-truncated and left-censored observations. The proportional reversed hazard rates (PRHR) model has been introduced as an alternative model for the PHR model by Gupta et al. [8] and some stochastic aspects along with further inferential procedures about it have been studied by Gupta and Gupta [9]. The proportional mean past lifetime (PMPL) model that focuses on the mean inactivity time of items has been considered by Asadi and Berred [10] and several stochastic properties of it have been developed by Rezaei [11].

The advantage of the current investigation is to establish another model based on the vitality function of a distribution which is a common measure of life expectancy among demographers. The vitality function of a distribution is the conditional mean of the random variable on the right tail of distribution (see, e.g., Navarro et al. [12], Sunoj et al. [13] and Abdul-Sathar et al. [14]). The proposed model can be used for modelling right-censored and left-truncated distributions.

Let X be a non-negative random variable (RV) having absolutely continuous cumulative distribution function (CDF) F and probability density function (PDF) f. The vitality function of X is

$$v_X(t) = E(X \mid X > t) = \frac{\int_t^\infty xf(x)dx}{\bar{F}(t)}$$

and the hazard rate (HR) function of X is

$$h_X(t) = \frac{f(t)}{\bar{F}(t)}.$$

The mean residual life (MRL) of X after time point t is

$$m_X(t) = \int_t^\infty \frac{\bar{F}(x)}{\bar{F}(t)}dx.$$

If the support of X is (l_X, u_X), then $\bar{F}(t) = 0$ for all $t > u_X$. In such a case, $m_X(t)$, $v_X(t)$ and $h_X(t)$ are not defined for $t > u_X$. But it is customary, in such cases, to define $m_X(t) = v_X(t) = h_X(t) = 0$. The vitality function has a relationship with the HR and the MRL functions as follows:

$$h_X(t) = \frac{v'_X(t)}{v_X(t) - t}, \tag{1}$$

where $v'_X(t) = \frac{d}{dt}v_X(t)$ and

$$m_X(t) = v_X(t) - t. \tag{2}$$

For the random variable X with a finite mean such that $F(0^-) = 0$, the survival function (SF) of X ($\bar{F} = 1 - F$) can be retrieved from vitality function v_X by the inversion formula

$$\bar{F}(t) = \exp\left(-\int_0^t \frac{v'_X(x)dx}{v_X(x) - x}\right). \tag{3}$$

Demographers have used the vitality function or life expectancy or expectation of life function for centuries in studies of human populations (see, e.g, Vaupel et al. [15], Yashin et al. [16] and Sharrow et al. [17]). The vitality function is monotonically increasing over $[0, \infty)$. With this in mind, it is a valid conjecture that the curve of the ratio of the parametric or non-parametric estimated vitality functions based on two sets of data will indicate significant departures from a horizontal line with less probability than the case where hazard rate or mean residual life functions are utilized.

The regression models proposed for lifetime data generally consider the assumption of the PH model or the assumption of PMRL. For instance, suppose that we use the PMRL model, i.e., $m(t) = \theta m_0(t)$, for all $t \geq 0$ and for some unknown $\theta > 0$, where m_0 and m are, respectively, the baseline and the response mean residual life functions assumed to be unknown. In the cases that two samples from the underlying distributions of m and m_0 is available, there is a principle regarding the shape of the two MRL functions. The MRL functions may exhibit quite different behaviours, such as monotonically increasing, monotonically decreasing, bathtub shaped, upside-down bathtub shaped and roller-coaster shaped depending on the fluctuations of data. The data analyst observes that a strong assumption in the PMRL model exists as the behaviour of the MRL curves must be the same under the setup of the model. For example, consider that according to data in a two sample problem the shape of the estimated mean residual lives are a lot different, thus, it is unlikely to find an appropriate value for θ to incorporate that possibility and this leads to an inadequacy for fitting the model to data.

The survival models have developed rapidly in the literature to model time to event data (cf. Hosmer Jr and Lemeshow [18] and Liu [19]). When a semi-parametric model is proposed, there may exist some challenging issues regarding that model. For instance, one may question the suitability of the model to model data in many practical situations and that how well does it fit the data. The main

goal of this paper is to introduce a semi-parametric model based on the concept of vitality function, called proportional vitalities (PVIT) model. In the context of the PVIT, the involved age-specific vitality function is inherently monotonically increasing. The credibility of modelling lifetime data using the PVIT model may be worthy.

The entire structure of this paper is organized as follows. In Section 2, the model is proposed. In Section 3, making inferences about the parameter(s) of the model is discussed. In Section 4, we lay the foundations of several stochastic orderings and aging properties. In Section 5, a brief conclusion of the paper is emphasized and some generalizations for future work are outlined.

2. The PVIT Model

First of all, in what follows in the paper, we refer the readers for the definitions of the utilized stochastic orders to Shaked and Shanthikumar [20] and for the concepts of aging notions that will be used in the sequel to Barlow and Proschan [21]. The property of Totally positive of order 2 (TP_2) for bivariate functions is also adopted from Karlin [22]. To introduce the model, some motivations shall be provided. An examples of parametric distributions is brought to clarify the structure of the model. The following definition is stated.

Definition 1. *X and Y with respective vitality functions v_X and v_Y are said to satisfy the PVIT model, if for all $t \geq 0$,*

$$v_Y(t) = \xi v_X(t), \tag{4}$$

where ξ is a positive constant that we call it vitality growth parameter.

We can infer that $v_X(t) = E(X \mid X > t)$ is the expected life-length of an individual randomly drawn from the individuals grouped at the age t. The PVIT model induces that the ratio $v_Y(t)/v_X(t)$ is t-free which means that the vitality functions of X and Y are relatively free of their underlying aging processes. From a mathematical point of view,

$$\frac{E(Y \mid Y > t_1)}{E(X \mid X > t_1)} = \frac{E(Y \mid Y > t_2)}{E(X \mid X > t_2)}, \text{ for all } t_1, t_2 \in [0, \infty), \tag{5}$$

purporting that relative mean life-lengths of survivor individuals in both populations remain unchanged when the age of survival is variable. In particular, for almost surely positive random variables,

$$\frac{E(Y \mid Y > t)}{E(X \mid X > t)} = \frac{E(Y)}{E(X)} = \xi, \text{ for all } t > 0. \tag{6}$$

In view of (3), if X and Y satisfy the PVIT model as described in Definition 1, then

$$\bar{G}(t) = \exp\left(-\int_0^t \frac{\xi v_X'(x)}{\xi v_X(x) - x} dx\right), \ t \geq 0. \tag{7}$$

The identity (7) may not give a closed form for the SF of the random variable Y in terms of SF of X, whereas in the PMRL model the inversion formula provides an explicit expression for the SF of the response variable in terms of a combination of the SF of the underlying distribution function with that of its equilibrium distributions (see, e.g., Nanda et al. [23]). On that account, we are capable of proving that (7) presents a proper SF if the following conditions hold:

(i) For all $x \geq 0$, $\xi \geq x/v_X(x)$.

(ii) $\int_0^t \frac{\xi v_X'(x)}{\xi v_X(x) - x} dx < \infty$, for all $0 \leq t < \infty$.

(iii) $\int_0^\infty \frac{\xi v_X'(x)}{\xi v_X(x) - x} dx = \infty$.

In what follows, the PVIT model is characterized in terms of the relationship between the ratios of survival (or hazard rate) functions of the length-biased distributions and the ratio of those of the

underlying distributions. Recall that $\hat{X}$ and $\hat{Y}$ with PDF's $\hat{f}(t) = tf(t)/E(X)$, $t \geq 0$ and $\hat{g}(t) = tg(t)/E(Y)$, $t \geq 0$ (provided that the expectations exist and are finite) are said to follow length-biased distributions associated with f and g, respectively (cf. Gupta and Keating [24]). Length-biased sampling arises when a component already in use is sampled at a fixed time and then allowed to fail (cf. Scheaffer [25]). The concept of length-biased sampling and the arising hazard rates, as well as cumulative distribution functions and the relations with other reliability measures, are appreciated in the literature (cf. Gupta and Keating [24]).

Proposition 1. *The non-negative random variables X and Y satisfy the PVIT model, if and only if one of the following assertions holds;*

(i) $\frac{h_{\hat{X}}(t)}{h_{\hat{Y}}(t)} = \xi \frac{h_X(t)}{h_Y(t)}$, *for all* $t \geq 0$.

(ii) $\frac{1-\hat{F}(t)}{1-\hat{G}(t)} = \frac{\bar{F}(t)}{\bar{G}(t)}$ *for all* $t \geq 0$.

Proof of Proposition 1. By Proposition 1 in Izadkhah et al. [26], for all $t \geq 0$, we have

$$h_{\hat{X}}(t) = \frac{th_X(t)}{v_X(t)},$$

and

$$h_{\hat{Y}}(t) = \frac{th_Y(t)}{v_Y(t)}.$$

Hence, $v_Y(t) = \xi v_X(t)$, for all $t \geq 0$, if, and only if, $\frac{h_{\hat{X}}(t)}{h_{\hat{Y}}(t)} = \xi \frac{h_X(t)}{h_Y(t)}$, for all $t \geq 0$ which is equivalent to (i). By a further application of Proposition 1 of Izadkhah et al. [26], one has, for all $t \geq 0$,

$$1 - \hat{F}(t) = \frac{v_X(t)}{v_X(0)} \bar{F}(t),$$

and

$$1 - \hat{G}(t) = \frac{v_Y(t)}{v_Y(0)} \bar{G}(t).$$

It is perceptible that under the setup of the PVIT model, $v_Y(t)/v_X(t)$ is free of $t \geq 0$. Thus, one can deduce that $v_Y(t)/v_X(t) = v_Y(0)/v_X(0) = \xi$, for all $t \geq 0$. This, together with the recent identities above, establishes the equivalence of (ii) and the PVIT model. □

For studies in reliability, biometry and survival analysis, the length-biased distributions have been frequently appropriate for quite a number of sampling plans. Suppose that two random samples on X and Y cannot be reached due to an ungovernable biased sampling procedure. In spite of that, imagine that the available data come from the associated length-biased distributions. Proposition 1 states that in the context of the PVIT model, the ratio of survival functions (or the hazard rates) of the original distributions could be estimated under length-biased sampling as effectively as acting under random (unbiased) sampling. The following example indicates that by adding the vitality parameter ξ to the exponential distribution, the well-known Hall–Wellner family of distributions is generated.

Example 1. *Suppose that X is distributed with mean* $1/\lambda$ *in accordance with exponential distribution. In an easy way, we get* $v_X(x) = x + 1/\lambda, x \geq 0$. *Let Y be a random variable with life vitality* $v_Y(t) = \xi v_X(t)$, *for* $t \geq 0$ *where* $\xi > 0$. *Then, Y has mean residual life function* $m_Y(t) = (\xi - 1)t + (\xi/\lambda)$, *which is the characteristic of the Hall–Wellner family of distributions. To be clear, recall from Hall and Wellner [27], their family of survival functions* $\bar{G}(t) = [B/(At + B)]_+^{\frac{1}{A}+1}$ *which is valid for* $A > -1$ *and* $B > 0$. *When* $A > 0, A = 0$ *and* $-1 < A < 0$, *it gives respectively a Pareto, an exponential (at limit) and a rescaled beta distribution. Here, by choosing* $A = \xi - 1$ *and* $B = \xi/\lambda$, *it is clarified that the distribution of Y is in the Hall–Wellner family.*

Now, we discuss a characterization result within the class of distributions satisfying the PVIT model.

Theorem 1. *Let X and Y be two non-negative random variables with finite means and hazard rate functions h_X and h_Y, respectively, such that $\lim_{t\to\infty} th_X(t) = \lim_{t\to\infty} th_Y(t) = \infty$. Then, X and Y satisfy the PVIT model if and only if they are equal in distribution.*

Proof of Theorem 1. By the construction of the model, $\xi = v_Y(t)/v_X(t)$, for all $t \geq 0$, which is free of t. Therefore,

$$\begin{aligned}\xi &= \lim_{t\to\infty} \frac{v_Y(t)}{v_X(t)} \\ &= \frac{1+\lim_{t\to\infty}\{m_Y(t)/t\}}{1+\lim_{t\to\infty}\{m_X(t)/t\}}.\end{aligned}$$

Remark that since $E(X) < \infty$, thus, when $t \to \infty$, $t\bar{F}(t) \to 0$ and $\int_t^\infty \bar{F}(x)dx \to 0$. As a result, using l'Hopital's rule,

$$\begin{aligned}\lim_{t\to\infty} \frac{m_X(t)}{t} &= \lim_{t\to\infty} \frac{\int_t^\infty \bar{F}(x)dx}{t\bar{F}(t)} \\ &= \lim_{t\to\infty} \frac{\bar{F}(t)}{tf(t)-\bar{F}(t)} \\ &= \frac{1}{\lim_{t\to\infty} th_X(t) - 1} = 0.\end{aligned}$$

In a similar manner, $\lim_{t\to\infty} m_Y(t)/t = 0$. It thus concludes that $\xi = 1$, i.e., $v_Y(t) = v_X(t)$, for all $t \geq 0$, which means that X and Y have equal distributions. □

3. Estimation of the Vitality Growth Parameter

In this section, statistical inferences about the unknown parameter(s) of the model are made. In the PVIT model, as introduced in Definition 1, the baseline vitality function v_X may have either an accurate form (known) or an unknown feature. In this setting, we shall first consider the case when X has the Parametric distribution so that v_X is fully known. Then, the case where v_X is unknown but a random sample on X and also a random sample on Y are available will be considered.

3.1. One-Sample Case with Specified v_X

Here, by simulating a data set from Pareto distribution we initiate a process for estimation of the growth parameter ξ in the PVIT model. We generate a sample of size 50 from the Pareto distribution having survival function $\bar{F}_X(t) = 1/(1+t)^3$, $t \geq 0$ and denote them by $t_1, \ldots, t_{50}$, as presented in the Appendix. The vitality function is specified as $v_X(t) = (3t+1)/2$, $t \geq 0$. Now, assume that Y as another random variable has vitality function $v_Y(t) = \xi v_X(t)$, in which $\xi = 3$. To perform a linear regression analysis, set $z_i = \xi v_X(t_i) + \epsilon_i$, where ϵ_i's represent a generated random sample from $N(0, 0.01)$. The method of least square provides that

$$\hat{\xi} = \frac{\sum_{i=1}^{50} z_i v_X(t_i)}{\sum_{i=1}^{50} [v_X(t_i)]^2} = 2.999923.$$

The data and the calculated values are given in Appendix A.

3.2. Two-Sample Case when v_X is Unspecified

In this sub-section, we address the problem of estimating the parameter $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_p)$ in model (4) when, $X_1, X_2, \ldots, X_n$ is an independent random sample from F (the baseline) and $Y_1, Y_2, \ldots, Y_N$ drawn independently from the populations with distribution functions $G_1, G_2, \ldots, G_p$, (the output) respectively, so that $Y_1, \ldots, Y_{N_1}$ are adopted independently from G_1, and $Y_{N_1+1}, \ldots, Y_{N_1+N_2}$ are taken independently from $G_2, \ldots$ and $Y_{\sum_{i=1}^{p-1} N_i+1}, \ldots, Y_{N_p}$ taken independently from G_p. To this end, several steps will be taken to develop the procedure. Take into consideration that F_n is the empirical distribution function of X, defined as

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} I(x_{(i)} \leq x), \tag{8}$$

where I is the indicator (heavy-side) function and $x_{(i)}$ denotes the observed the ith order statistic in the sample drawn from F, and accordingly $\bar{F}_n = 1 - F_n$ represents the empirical SF. Suppose v_{F_n} stands for the vitality function of the distribution (8). It can be acquired from (2) that

$$v_{F_n}(t) = \int_t^{\infty} \frac{\bar{F}_n(x)}{\bar{F}_n(t)}\, dx + t = \frac{\sum_{i=1}^{n} x_{(i)} I(x_{(i)} > t)}{\sum_{i=1}^{n} I(x_{(i)} > t)}. \tag{9}$$

One disadvantage with the empirical distribution is that it assigns positive probability mass only to a finite number of points $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$, say, which may be undesirable when modeling continuous variables. Given the observed values $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ of the sample $X_1, X_2, \ldots, X_n$ from F, the smoothed empirical distribution is defined as a mixture distribution with the following SF

$$\tilde{\bar{F}}_n(x) = \frac{1}{n} \sum_{i=1}^{n} \bar{F}_{\delta_i}(x) \tag{10}$$

where $\bar{F}_{\delta_i}$ is a SF of a random variable distributed at a two-sided neighborhood of $x_{(i)}$. Note that if $\bar{F}_{\delta_i}$ is the SF of a degenerate random variable at $x_{(i)}$, then (10) corresponds to (8). We will take $\widetilde{F}_n$ and $\widetilde{f}_n$ as the CDF and the PDF associated with the SF (10). To show initiative in the case of our study, we are inclined to use triangular distribution. It is said that X^* follows the triangular distribution with the vector of parameters $\boldsymbol{\delta} = (a, c, b)$ in which $a < c < d \in \mathbb{R}$ (denoted by $X^* TA(\boldsymbol{\delta})$) when it has density

$$f_{\delta}(x) = \begin{cases} 0, & x < a \\ \frac{2(x-a)}{(b-a)(c-a)}, & a \leq x < c \\ \frac{2(b-x)}{(b-a)(b-c)}, & c \leq x < b \\ 0, & x > b. \end{cases} \tag{11}$$

The SF associated with (11) is derived as

$$\bar{F}_{\delta}(x) = \begin{cases} 1, & x \leq a \\ 1 - \frac{(x-a)^2}{(b-a)(c-a)}, & a < x \leq c \\ \frac{(b-x)^2}{(b-a)(b-c)}, & c < x < b \\ 0, & x \geq b. \end{cases} \tag{12}$$

In the case of (10), suppose that $\bar{F}_{\delta_i}$, $i = 2, \ldots, n-1$ is the SF of the triangular distribution with parameter $\delta_i = (x_{(i-1)}, x_{(i)}, x_{(i+1)})$. In parallel, assume that $\bar{F}_{\delta_1}$ is the survival functions associated with triangular distribution with parameter $\delta_1 = (0, x_{(1)}, x_{(2)})$ and that $\bar{F}_{\delta_n}$ is the SF of a symmetric triangular distribution with $\delta_n = (x_{(n-1)}, x_{(n)}, 2x_{(n)} - x_{(n-1)})$. We now derive the vitality function

of a random variable with SF (10) as it has already been fully characterized. Therefore, a smoothed estimation of vitality function is derived as

$$\begin{aligned} v_{\widetilde{F}_n}(t) &= t + \frac{\int_t^{\infty} \bar{\widetilde{F}}_n(x)dx}{\bar{\widetilde{F}}_n(t)} \\ &= t + \frac{\int_t^{x_{(2)}} \bar{F}_{\delta_1}(x)dx + \sum_{i=2}^{n-1} \int_t^{x_{(i+1)}} \bar{F}_{\delta_i}(x)dx + \int_t^{2x_{(n)}-x_{(n-1)}} \bar{F}_{\delta_n}(x)dx}{\bar{F}_{\delta_1}(t) + \sum_{i=2}^{n-1} \bar{F}_{\delta_i}(t) + \bar{F}_{\delta_n}(t)}. \end{aligned} \tag{13}$$

Computation of the integrals in (13) is not difficult as it can be easily accomplished by replacing the specified amounts of parameters of the survival functions, i.e., δ_i as in (12). For, $i = 2, \ldots, n-1$ when $t < x_{(i-1)}$ we have

$$\int_t^{x_{(i+1)}} \bar{F}_{\delta_i}(x)dx = \frac{x_{(i-1)} + x_{(i)} + x_{(i+1)}}{3} - t,$$

for $t \in [x_{(i-1)}, x_{(i)})$:

$$\int_t^{x_{(i+1)}} \bar{F}_{\delta_i}(x)dx = x_{(i)} + \frac{(x_{(i+1)} - x_{(i)})^2 - (x_{(i)} - x_{(i-1)})^2}{3(x_{(i+1)} - x_{(i-1)})} + \frac{(t - x_{(i-1)})^3}{3(x_{(i+1)} - x_{(i-1)})(x_{(i)} - x_{(i-1)})} - t,$$

for $t \in [x_{(i)}, x_{(i+1)})$ one can derive

$$\int_t^{x_{(i+1)}} \bar{F}_{\delta_i}(x)dx = \frac{(x_{(i+1)} - t)^3}{3(x_{(i+1)} - x_{(i-1)})(x_{(i+1)} - x_{(i)})},$$

and eventually $\int_t^{x_{(i+1)}} \bar{F}_{\delta_i}(x)dx = 0$, for $t \geq x_{(i+1)}$. The cases when $i = 1, n$ readily follow. Consider a setting by which Y_i is a random variable with vitality function $v_{Y_i}(t) = \xi_i v_X(t)$, for $i = 1, 2, \ldots, p$. In situations where v_X is unspecified, we are able to estimate it with the estimator (13) by applying data on X. In accordance with (7), we take $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_N)^T$ as a sample of independent random variables as described before. Consider $\boldsymbol{Y}_j = (Y_{1+S_{j-1}}, \ldots, Y_{S_j})^T$, which has likelihood function

$$\begin{aligned} L_j(\xi_j) &= \prod_{i=1+S_{j-1}}^{S_j} f_i(y_i \mid \xi_j) \\ &= \prod_{i=1+S_{j-1}}^{S_j} \frac{\xi_j v'_{\widetilde{F}_n}(y_i)}{\xi_j v_{\widetilde{F}_n}(y_i) - y_i} e^{-\int_0^{y_i} \frac{\xi_j v'_{\widetilde{F}_n}(t)}{\xi_j v_{\widetilde{F}_n}(t) - t} dt}, \end{aligned}$$

where $S_j = \sum_{k=0}^{j} N_j$, for $j = 1, 2, \ldots p$ with the convention that $S_0 = N_0 = 0$. Consider the PVIT model $v_{Y_i}(y_i \mid \xi_j) = \xi_j v_X(y_i)$, for $i = 1 + S_{j-1}, \ldots, S_j$. If $v_X(y_i)$ is estimated as in (13), then the resulting model is completely parametric. The method of maximum likelihood can be used to estimate $\boldsymbol{\xi}$. The full likelihood function obtains:

$$\begin{aligned} L(\boldsymbol{\xi}) &= \prod_{j=1}^{p} L_j(\xi_j) \\ &= \prod_{j=1}^{p} \prod_{i=1+S_{j-1}}^{S_j} \frac{\xi_j v'_{\widetilde{F}_n}(y_i)}{\xi_j v_{\widetilde{F}_n}(y_i) - y_i} e^{-\int_0^{y_i} \frac{\xi_j v'_{\widetilde{F}_n}(t)}{\xi_j v_{\widetilde{F}_n}(t) - t} dt}, \end{aligned}$$

from which the likelihood equations becomes:

$$\frac{N_k}{\xi_k} - \sum_{i=1+S_{k-1}}^{S_k} \frac{v'_{\widetilde{F}_n}(y_i)}{\xi_k v_{\widetilde{F}_n}(y_i) - y_i} + \sum_{i=1+S_{k-1}}^{S_k} \int_0^{y_i} \frac{t v'_{\widetilde{F}_n}(t)}{(\xi_k v_{\widetilde{F}_n}(t) - t)^2} dt = 0, \quad k = 1, 2, \ldots, p. \tag{14}$$

The above equations can be solved using the numerical Newton–Raphson method of iteration. In Table 1, we perform a simulation study for solving the system of Equations (14) by the so-called Newton–Raphson method with six iterations.

Table 1. Estimates of $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ for $N = 3000$ and $N = 6000$ data simulated from Exponential distribution with mean 2.

	$(N_1, N_2, N_3) = (1000, 1000, 1000)$			$(N_1, N_2, N_3) = (2000, 2000, 2000)$		
	$\xi_1 = 2$	$\xi_2 = 3$	$\xi_3 = 4$	$\xi_1 = 3$	$\xi_2 = 4$	$\xi_3 = 5$
Iteration = 0	2.4537	3.1101	2.2154	1.5987	5.7751	7.8574
1	1.6512	2.4589	3.3875	2.8573	3.8591	4.9517
2	1.8021	2.6096	3.4587	2.9512	3.9013	4.9664
3	1.9324	2.7584	3.6533	2.9815	3.9742	4.9910
4	1.9733	2.8911	3.7152	2.9933	3.9852	4.9933
5	1.9987	2.9128	3.7836	2.9981	3.9941	4.9986
6	1.999399	2.965466	3.834184	2.999698	3.998691	4.999185

4. Some Closure and Preservation Properties

In this section, the closure (preservation) property of some stochastic orders (aging classes) under the formation of the model is studied. The problem has attracted the attention of many researchers in the recent past decades (cf. Amini-Seresht and Zhang [28], Amini-Seresht and Khaledi [29], Barmalzan and Najafabadi [30]). The following result indicates that the model by restricting the domain of the parameter ξ induces the hazard rate (mean residual life) ordering property between the underlying random variables.

Theorem 2. *$X \leq_{hr} (\leq_{mrl}) Y$, if and only if, $\xi \geq 1$.*

Proof of Theorem 2. From the relationships among the HR and the MRL orders, it suffices to prove that $\xi \geq 1$ implies $X \leq_{hr} Y$, and that $X \leq_{mrl} Y$ implicates $\xi \geq 1$. Under the set up of the model, by (1)

$$\begin{aligned} h_X(t) - h_Y(t) &= \frac{v'_X(t)}{v_X(t) - t} - \frac{v'_Y(t)}{v_Y(t) - t} \\ &= \frac{v'_X(t)}{v_X(t) - t} - \frac{\xi v'_X(t)}{\xi v_X(t) - t}, \end{aligned}$$

which is non-negative for $\xi \geq 1$ as $v'_X(t) > 0$ for all $t \geq 0$ because $v_X(t)$ is increasing in $t \geq 0$ (see, e.g., Proposition 2.4 of Nanda and Jain [31]) and also $\xi v'_X(t)/(\xi v_X(t) - t)$ is decreasing in ξ, for any $t \geq 0$. This indicates that when $\xi \geq 1$, $X \leq_{hr} Y$. Now, we prove that $X \leq_{mrl} Y$ implies that $\xi \geq 1$. From (2), for random variables X and Y that satisfy the PVIT model, we can get $m_Y(t) = \xi m_X(t) + (\xi - 1)t$, for all $t \geq 0$ and thus $\xi = (t + m_Y(t))/(t + m_X(t))$, which is greater than 1 by definition. □

However, the following example is an indication that the hazard rate order in Theorem 2 cannot be replaced by the likelihood ratio order.

Example 2. *Let X and Y have, respectively, exponential distribution and Lomax distribution with survival functions $\bar{F}(t) = \exp(-t), t \geq 0$ and $\bar{G}(t) = \left(\frac{2}{t+2}\right)^2, t \geq 0$. It can be seen that $v_Y(t) = 2v_X(t)$ for all $t \geq 0$,*

i.e., vitality of Y has been evolved with respect to vitality of X according to the PVIT model with a frailty growth identified by $\xi = 2$. *By some routine calculations, it is perceived that* $X_{l_r}Y$.

In the recent past decade, Righter et al. [32] propounded some fresh ageing notions on the basis of scaled conditional lifetime. The scaled conditional life of a random variable X is the total life relative to the current age, conditioned on the current age, and is given by

$$X_{SC}(t) \stackrel{d}{=} \frac{1}{t}\{X \mid X > t\} = 1 + \frac{X_t}{t},\ t > 0, \tag{15}$$

where $X_t \stackrel{d}{=} (X - t \mid X > t)$, for all t with $F(t) < 1$, is called the residual life of an item with age t, in which $\stackrel{d}{=}$ means equality in distribution. For further studies of the family of distributions of $X_{SC}(t)$, see Righter et al. [32], Belzunce et al. [33] and the references therein. A new class of life distributions could be as the following. Recall that a function η is anti-star-shaped on (a, b), if $(1/t)\eta(t)$ is non-increasing in $t \in (a, b)$.

Definition 2. *The non-negative random variable X is said to have decreeing mean scaled conditional life (denoted as* $X \in DMSCL$*) if* $E[X_{SC}(t)]$ *is non-increasing in* $t > 0$, *or equivalently,* $v_X(t)$ *is an anti-star-shaped function in* $t > 0$.

It has been investigated by the authors that some ageing properties of X are inherited by the random variable Y in the context of the model. Before stating our findings about that problem, we need to prove the following definition and key lemma.

Definition 3. *The non-negative random variable X is said to have*

(i) increasing failure rate (IFR) whenever the hazard rate function h_X *of X is non-decreasing.*
(ii) increasing mean residual life (IMRL) whenever the mean residual life function m_X *of X is non-decreasing.*
(iii) increasing failure rate in average (IFRA) whenever $(1/x)\int_0^x h_X(u)du$ *is non-decreasing in* $x > 0$.
(iv) new worse than used (NWU) property whenever $P(X > x)P(X > t) \leq P(X > x + t)$, *for all* $x \geq 0$ *and for all* $t \geq 0$.
(v) new worse than used in expectation (NWUE) whenever $E(X) \leq E(X - t \mid X > t)$, *for all* $t \geq 0$
(vi) increasing generalized failure rate (IGFR) whenever $xh_X(x)$ *is non-decreasing in* $x > 0$.

The decreasing failure rate (DFR) and the decreasing failure rate in average (DFRA) aging paths are defined by reversing the required monotonicity behaviour in Definition 3(i) and Definition 3(ii), respectively. The new better than used in expectation (NBUE) property is also defined by revering the side of the inequality in Definition 3(v).

Lemma 1. *Let* $X \in DMSCL$ *such that* $\xi \in (0, 1]$ $(\xi \in [1, \infty))$. *Then the function* δ *given by*

$$\delta(x, \xi) = \frac{\xi v_X(x) - \xi x}{\xi v_X(x) - x}$$

is non-decreasing (non-increasing) in $x \geq 0$ *for all* ξ *in the specified intervals.*

Proof of Lemma 1. Take v_X as differentiable over $(0,\infty)$. We prove that $(\partial/\partial x)\delta(x,\xi) \geq (\leq)0$, for all $x \geq 0$ and for any $\xi \in (0,1]$ ($\xi \in [1,\infty)$). One has

$$\begin{aligned}
\frac{\partial \delta(x,\xi)}{\partial x} &= \frac{\xi(v'_X(x)-1)(\xi v_X(x)-x)-(\xi v'_X(x)-1)(\xi v_X(x)-\xi x)}{(\xi v_X(x)-x)^2} \\
&\overset{sign}{=} v_X(x)-xv'_X(x)+x\xi v'_X(x)-\xi v_X(x) \\
&= (1-\xi)v_X(x)-(1-\xi)xv'_X(x) \\
&= (1-\xi)(v_X(x)-xv'_X(x)) \\
&\overset{sign}{=} (1-\xi)(d/dx)(1/E(X_{SC}(x))) \\
&\geq (\leq) \quad 0, \text{ for all } x \geq 0,
\end{aligned}$$

where $a \overset{sign}{=} b$ means that a and b have the same sign. The proof is completed. □

In the situations where X and Y with common support $(0,\infty)$ satisfy the IFR (or even the ultimately IFR) property, the assumption of Theorem 1 is fulfilled. Furthermore, when X and Y possess the (ultimately) DMRL ageing property, then trivially both $m_X(t)/t$ and $m_Y(t)/t$ tend to zero when t is closing to infinity and X and Y are equal in distribution in this case either. It can be shown that when X and Y have IFRA property then for all $t > 0$:

$$th_X(t) \geq \int_0^t h_X(x)dx, \text{ and } th_Y(t) \geq \int_0^t h_Y(y)dy, \tag{16}$$

and the requirement $\lim_{t\to\infty}\int_0^t h_X(x)dx = \lim_{t\to\infty}\int_0^t h_Y(y)dy = \infty$ together with (16) makes the assumption of Theorem 1 satisfied. Therefore, in all of these cases the model is valid if and only if distributions of X and Y coincide each other and as a result preservation property for the foregoing positive aging classes is an ineffective study. In spite of that, we can prove the following results. For the IGFR class of distributions, we refer the readers to Lariviere and Porteus [34].

Theorem 3. *Suppose that X and Y are as in the model. Then,*

(i) For any $\xi \geq 1$, $X \in$ IMRL implies that $Y \in$ IMRL.
(ii) For any $\xi \geq 1$, $X \in$ NWUE implies that $Y \in$ NWUE.
(iii) For any $\xi \leq 1$, $X \in$ NBUE implies that $Y \in$ NBUE.
(iv) For any $\xi \leq 1$, $X \in$ IGFR implies that $Y \in$ IGFR.

Proof of Theorem 3. To prove (i), from the relationship between m_Y and m_X in Theorem 2, we have $m'_Y(t) = \xi m'_X(t) + (\xi - 1)$ which is non-negative for all $t \geq 0$, by assumption. For proving (ii) [(iii)], note that $X \in$ NWUE [NBUE] yields $v_X(t) - v_X(0) \geq [\leq]t$, for all $t \geq 0$ which concludes by assumption that $\xi(v_X(t) - v_X(0)) \geq [\leq]t$, for all $t \geq 0$, considering that $v_X(t) - v_X(0) \geq 0$ for all $t \geq 0$, since v_X is a non-decreasing function. It thus follows that $\xi v_X(t) - t \geq [\leq]\xi v_X(0)$, for all $t \geq 0$, which is equivalent to $m_Y(t) \geq [\leq]m_Y(0)$, for all $t \geq 0$, i.e, $Y \in$ NWUE [NBUE]. To prove the assertion (iv), denote first by $l_Y(t) = th_Y(t)$ and $l_X(t) = th_X(t)$ the PHRs of X and Y, respectively. From the construction of the model it follows that

$$\begin{aligned}
l_Y(t) &= t\frac{\xi v'_X(t)}{\xi v_X(t)-t} \\
&= \frac{\xi v_X(t)-\xi t}{\xi v_X(t)-t}\frac{tv'_X(t)}{v_X(t)-t} \\
&= \delta(t,\xi)l_X(t), \text{ for all } t \geq 0,
\end{aligned}$$

where δ is as defined in Lemma 1. From assumption, since $X \in$ IGFR thus l_X is non-decreasing, and according to Theorem 20 in Kayid et al. [35], $X \in$ DMSCL. On using Lemma 1, when $\xi \leq 1$, $\delta(t,\xi)$ is non-decreasing in $t \geq 0$, which results $Y \in$ IGFR. □

The next result establishes preservation of some other ageing properties.

Theorem 4. *Let X and Y be related as in the model such that $X \in$ DMSCL. Then,*

(i) For any $\xi \geq 1$, $X \in$ DFR implies that $Y \in$ DFR.
(ii) For any $\xi \geq 1$, $X \in$ DFRA implies that $Y \in$ DFRA.
(iii) For any $\xi \geq 1$, $X \in$ NWU implies that $Y \in$ NWU.

Proof of Theorem 4. To prove (i), observe that $h_Y(t) = \delta(t,\xi)h_X(t)$ for all $t \geq 0$. By applying Lemma 1, the result follows immediately. For the sake of proving (ii), notice that $X \in$ DFRA, if and only if, $\int_0^t [h_X(x) - h_X(t)]dx \geq 0$, for all $t \geq 0$. Since $\xi \geq 1$ and $X \in$ DMSCL thus Lemma 1 concludes that $\delta(x,\xi)$ is non-negative and non-increasing in $x \geq 0$. Therefore,

$$\begin{aligned}
\int_0^t [h_Y(x) - h_Y(t)]dx &= \int_0^t [\delta(x,\xi)h_X(x) - \delta(t,\xi)h_X(t)]dx \\
&\geq \int_0^t [\delta(t,\xi)h_X(x) - \delta(t,\xi)h_X(t)]dx \\
&= \delta(t,\xi)\int_0^t [h_X(x) - h_X(t)]dx \\
&\geq 0, \text{ for all } t \geq 0,
\end{aligned}$$

which clarifies that $Y \in$ DFRA. To demonstrate (iii), be mindful of that $X \in$ NWU if and only if, $\int_0^x [h_X(u) - h_X(u+t)]du \geq 0$, for all $x \geq 0$ and for all $t \geq 0$. Now, set $\xi_{t,x,\xi}(u) = \delta(u+t,\xi)I[u \leq x]$, where δ is as given in Lemma 1 and $I[u \leq x]$ is the indicator function of the set $[u \leq x]$. It is clear from assumption by using Lemma 1 that $\xi_{t,x,\xi}(u)$ is non-increasing in $u \geq 0$, for all $x \geq 0$ and for all $t \geq 0$, with $\xi \geq 1$. On that account, Lemma 7.1(b) in Barlow and Proschan [21] implies that

$$\int_0^\infty \xi_{t,x,\xi}(u)dW_t(u) \geq 0, \text{ for all } t \geq 0, x \geq 0, \tag{17}$$

where

$$dW_t(u) = [h_X(u) - h_X(u+t)]du, \ \ u,t \geq 0.$$

In contrast, since $\delta(x,\xi)$ is non-negative and non-increasing by assumption, thus

$$\begin{aligned}
\int_0^x [h_Y(u) - h_Y(u+t)]du &= \int_0^x [\delta(u,\xi)h_X(u) - \delta(u+t,\xi)h_X(u+t)]du \\
&\geq \int_0^x \delta(u+t,\xi)[h_X(u) - h_X(u+t)]du \\
&= \int_0^\infty \xi_{t,x,\xi}(u)[h_X(u) - h_X(u+t)]du, \\
&= \int_0^\infty \xi_{t,x,\xi}(u)dW_t(u), \text{ for all } x,t \geq 0,
\end{aligned}$$

which is non-negative from (17) and the result follows. □

5. Conclusions

The PVIT model has some complementary role with respect to the so called PHR model. The random varaibles X and Y are said to have PHR when $h_Y(t) = \theta h_X(t)$ for all $t > 0$ and for some constant $\theta > 0$ (see, e.g., Nanda and Das [36]). As in the PHR model, the data from lifetime of units

that deteriorate with age could be used but in the PVIT model those units that recuperate with age can be modelled. In a regression analysis problem, the vitality growth parameter ξ may be expressed in terms of some covaraites, i.e., $\xi = \exp(\mathbf{b}^t\mathbf{z})$ where $\mathbf{b}$ is a $p \times 1$ vector of regression parameters, and $\mathbf{z}$ is a $p \times 1$ vector of covariates. On the ith individual is available a certain vector of observed covariates $\mathbf{z}_i$. As a result, the factor $\mathbf{z}_i$ in the study can be used to quantify some positive/negative events. For example, in the PHR model $\mathbf{z}_i$, as a covariate, has two levels for the ith individual, $\mathbf{z}_i = 0$ describes a non-smoker and $\mathbf{z}_i = 1$ a smoker. In the PVIT model, $\mathbf{z}_i = 0$ if the ith individual does not get enough exercises weekly, and $\mathbf{z}_i = 1$ if he/she does have bodily activity weekly enough. It is well-known that if somebody is a smoker, the hazard rate of his/her lifetime increases, but if the person is an athlete, his/her lifetime is improved in the sense of vitality function, relatively.

Specifically, in the current investigation, some simulation studies were carried out to detect the accuracy of the PVIT model based on some data sets. The numerical iteration method of Newton–Raphson has been adopted to estimate the parameters of the model. Several reliability properties in the PVIT model, including some stochastic orders and some aging properties, are discussed. In the future of this study, time dependent proportional vitalities model will be considered.

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Appendix A

t_i	$v_X(t_i)$	ϵ_i	z_i	$v_Y(t_i)$
0.055118644	0.5826780	-1.469364×10^{-2}	1.733340	1.748034
0.143187920	0.7147819	-9.217576×10^{-3}	2.135128	2.144346
0.155038920	0.7325584	6.771378×10^{-4}	2.198352	2.197675
0.351670888	1.0275063	-4.836144×10^{-3}	3.077683	3.082519
0.069925950	0.6048889	1.185490×10^{-2}	1.826522	1.814667
0.235963384	0.8539451	1.350878×10^{-2}	2.575344	2.561835
7.510316903	11.7654754	6.811129×10^{-5}	35.296494	35.296426
0.288851648	0.9332775	-9.279590×10^{-3}	2.790553	2.799832
0.301022689	0.9515340	7.740785×10^{-3}	2.862343	2.854602
0.060425372	0.5906381	-2.384972×10^{-3}	1.769529	1.771914
0.152692596	0.7290389	1.978634×10^{-2}	2.206903	2.187117
0.632615584	1.4489234	-2.864548×10^{-3}	4.343906	4.346770
0.473336937	1.2100054	-4.182921×10^{-3}	3.625833	3.630016
3.562609349	5.8439140	-1.673617×10^{-3}	17.530068	17.531742
1.347293897	2.5209408	4.364296×10^{-3}	7.567187	7.562823
0.334250766	1.0013761	2.298771×10^{-3}	3.006427	3.004128
0.166813852	0.7502208	2.516934×10^{-3}	2.253179	2.250662
0.168756079	0.7531341	1.889646×10^{-2}	2.278299	2.259402
0.076176952	0.6142654	2.181841×10^{-3}	1.844978	1.842796
1.226728920	2.3400934	1.151333×10^{-2}	7.031793	7.020280
0.230970644	0.8464560	6.008524×10^{-3}	2.545376	2.539368
1.107529567	2.1612944	7.590414×10^{-3}	6.491473	6.483883
3.417582958	5.6263744	-6.868880×10^{-3}	16.872254	16.879123
0.268723365	0.9030850	-1.086299×10^{-2}	2.698392	2.709255
0.396640979	1.0949615	-1.364035×10^{-2}	3.271244	3.284884
1.083457155	2.1251857	5.666859×10^{-3}	6.381224	6.375557
0.337826748	1.0067401	7.819611×10^{-3}	3.028040	3.020220
0.075206569	0.6128099	-1.076559×10^{-2}	1.827664	1.838430
0.071492330	0.6072385	3.152931×10^{-3}	1.824868	1.821715

0.282092216	0.9231383	-1.456236×10^{-2}	2.754853	2.769415
0.020369897	0.5305548	-1.092742×10^{-3}	1.590572	1.591665
0.328080191	0.9921203	-8.981199×10^{-3}	2.967380	2.976361
0.714095647	1.5711435	6.633057×10^{-4}	4.714094	4.713430
0.005927773	0.5088917	-2.410771×10^{-3}	1.524264	1.526675
0.053527833	0.5802917	-2.033584×10^{-2}	1.720539	1.740875
0.007627633	0.5114414	1.000426×10^{-2}	1.544329	1.534324
0.086416609	0.6296249	1.238910×10^{-2}	1.901264	1.888875
0.059584329	0.5893765	-2.509592×10^{-2}	1.743034	1.768129
0.441831858	1.1627478	1.118656×10^{-2}	3.499430	3.488243
0.173709168	0.7605638	-1.220161×10^{-2}	2.269490	2.281691
0.263953279	0.8959299	2.713221×10^{-3}	2.690503	2.687790
0.277541623	0.9163124	-7.574604×10^{-3}	2.741363	2.748937
0.016202953	0.5243044	5.493229×10^{-3}	1.578407	1.572913
0.077373355	0.6160600	-6.131611×10^{-3}	1.842048	1.848180
0.921859508	1.8827893	1.077780×10^{-3}	5.649446	5.648368
0.946334782	1.9195022	1.481825×10^{-3}	5.759988	5.758507
0.490957913	1.2364369	3.594307×10^{-3}	3.712905	3.709311
0.712953679	1.5694305	2.116783×10^{-4}	4.708503	4.708292
0.731077502	1.5966163	-9.413487×10^{-3}	4.780435	4.789849
0.122441762	0.6836626	-4.718849×10^{-4}	2.050516	2.050988

References

1. Nanda, A.K.; Maiti, S.S.; Kundu, C.; Kundu, A. Parameter estimates of general failure rate model: A Bayesian approach. *J. Comput. Appl. Math.* **2019**, *351*, 317–330. [CrossRef]
2. Henley, E.J.; Kumamoto, H. *Reliability Engineering and Risk Assessment*; Prentice-Hall: Englewood Cliffs, NJ, USA, 1981; Volume 568.
3. Fleming, T.R.; Harrington, D.P. *Counting Processes and Survival Analysis*; John Wiley and Sons: Hoboken, NJ, USA, 2011; Volume 169.
4. Miller, R.G., Jr. *Survival Analysis*; John Wiley and Sons: Hoboken, NJ, USA, 2011; Volume 66.
5. Cox, D.R. Regression models and life tables. *J. R. Stat. Soc. Ser. (Methodol.)* **1972**, *34*, 187–202. [CrossRef]
6. Oakes, D.; Dasu, T. A note on residual life. *Biometrika* **1990**, *77*, 409–410. [CrossRef]
7. Nanda, A.K.; Bhattacharjee, S.; Alam, S.S. Properties of proportional mean residual life model. *Stat. Probab. Lett.* **2006**, *76*, 880–890. [CrossRef]
8. Gupta, R.C.; Gupta, P.L.; Gupta, R.D. Modeling failure time data by Lehman alternatives. *Commun. Stat. Theory Methods* **1998**, *27*, 887–904. [CrossRef]
9. Gupta, R.C.; Gupta, R.D. Proportional reversed hazard rate model and its applications. *J. Stat. Plan. Inference* **2007**, *137*, 3525–3536. [CrossRef]
10. Asadi, M.; Berred, A. Properties and estimation of the mean past lifetime. *Statistics* **2012**, *46*, 405–417. [CrossRef]
11. Rezaei, M. On proportional mean past lifetimes model. *Commun. Stat. Theory Methods* **2016**, *45*, 4035–4047. [CrossRef]
12. Navarro, J.; Franco, M.; Ruiz, J.M. Characterization through moments of the residual life and conditional spacings. *Sankhya A* **1998**, *60*, 48.
13. Sunoj, S.M.; Sankaran, P.G.; Maya, S.S. Characterizations of life distributions using conditional expectations of doubly (interval) truncated random variables. *Commun. Stat. Theory Methods* **2009**, *38*, 1441–1452. [CrossRef]
14. Abdul-Sathar, E.I.; Rajesh, G.; Nair, K.M. Bivariate geometric vitality function and some characterization results. *Calcutta Stat. Assoc. Bull.* **2010**, *62*, 207–228. [CrossRef]
15. Vaupel, J.W.; Baudisch, A.; Dölling, M.; Roach, D.A.; Gampe, J. The case for negative senescence. *Theor. Popul. Biol.* **2004**, *65*, 339–351. [CrossRef] [PubMed]
16. Yashin, A.I.; Wu, D.; Arbeev, K.G.; Stallard, E.; Land, K.C.; Ukraintseva, S.V. How genes influence life span: The biodemography of human survival. *Rejuvenation Res.* **2012**, *15*, 374–380. [CrossRef] [PubMed]
17. Sharrow, D.J.; Anderson, J.J. Quantifying intrinsic and extrinsic contributions to human longevity: Application of a two-process vitality model to the human mortality database. *Demography* **2016**, *53*, 2105–2119. [CrossRef]

18. Hosmer, D.W., Jr.; Lemeshow, S. *Applied Survival Analysis: Time-to-Event*; Wiley-Interscience: New York, NY, USA, 1999; Volume 317.
19. Liu, X. *Survival Analysis: Models and Applications*; John Wiley and Sons: Hoboken, NJ, USA, 2012.
20. Shaked, M.; Shanthikumar, J.G. *Stochastic Orders*; Springer Science and Business Media: Boston, NY, USA, 2007.
21. Barlow, R.E.; Proschan, F. *Statistical Theory of Reliability and Life Testing: Probability Models*; Florida State Univ Tallahassee: Tallahassee, FL, USA, 1975.
22. Karlin, S. *Total Positivity*; Stanford University Press: Stanford, CA, USA, 1968; Volume 1.
23. Nanda, A.K.; Bhattacharjee, S.; Balakrishnan, N. Mean residual life function, associated orderings and properties. *IEEE Trans. Reliab.* **2009**, *59*, 55–65. [CrossRef]
24. Gupta, R.C.; Keating, J.P. Relations for reliability measures under length biased sampling. *Scand. Stat.* **1986**, *13*, 49–56.
25. Scheaffer, R.L. Size-biased sampling. *Technometrics* **1972**, *14*, 635–644. [CrossRef]
26. Izadkhah, S.; Rezaei Roknabadi, A.H.; Borzadaran, G.M. Aspects of the mean residual life order for weighted distributions. *Statistics* **2014**, *48*, 851–861. [CrossRef]
27. Hall, W.J.; Wellner, J.A. Mean residual life. *Stat. Relat. Top.* **1981**, *169*, 184.
28. Amini-Seresht, E.; Zhang, Y. Stochastic comparisons on two finite mixture models. *Oper. Res. Lett.* **2017**, *45*, 475–480. [CrossRef]
29. Amini-Seresht, E.; Khaledi, B.E. Multivariate stochastic comparisons of mixture models. *Metrika* **2015**, *78*, 1015–1034. [CrossRef]
30. Barmalzan, G.; Najafabadi, A.T.P. On the convex transform and right-spread orders of smallest claim amounts. *Insur. Math. Econ.* **2015**, *64*, 380–384. [CrossRef]
31. Nanda, A.K.; Jain, K. Some weighted distribution results on univariate and bivariate cases. *J. Stat. Inference* **1999**, *77*, 169–180. [CrossRef]
32. Righter, R.; Shaked, M.; Shanthikumar, J.G. Intrinsic aging and classes of nonparametric distributions. *Probab. Eng. Inf. Sci.* **2009**, *23*, 563–582. [CrossRef]
33. Belzunce, F.; Candel, J.; Ruiz, J.M. Ordering of truncated distributions through concentration curves. *Sankhya Ser. A* **1995**, *57*, 375–383.
34. Lariviere, M.A.; Porteus, E.L. Selling to the newsvendor: An analysis of price-only contracts. *Manuf. Serv. Oper. Manag.* **2001**, *3*, 293–305. [CrossRef]
35. Kayid, M.; Izadkhah, S.; Alhalees, H. Reliability analysis of the proportional mean residual life order. *Math. Eng.* **2014**. [CrossRef]
36. Nanda, A.K.; Das, S. Dynamic proportional hazard rate and reversed hazard rate models. *J. Stat. Plan. Inference* **2011**, *141*, 2108–2119. [CrossRef]