PHYSICS 404 –Fall 2017
2nd HOMEWORK
Dr. V. Lempesis

Hand in: Sunday 29th of October 2017

1. Using the result of problem 6 of handout 5 to show that:

$$\text{(a) } \cos x = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x),$$

$$\text{(b) } \sin x = 2\sum_{n=1}^{\infty}(-1)^{n-1} J_{2n-1}(x).$$

Solution:

$$e^{iz\cos\theta} = \sum_{n=-\infty}^{\infty} i^n J_n(z) e^{in\theta} \underset{\theta=0}{\Rightarrow} e^{iz} = \sum_{n=-\infty}^{\infty} i^n J_n(z) \Rightarrow e^{i(x+iy)} = \sum_{n=-\infty}^{\infty} i^n J_n(x+iy) \underset{y=0}{\Rightarrow}$$

$$e^{ix} = \sum_{n=-\infty}^{\infty} i^n J_n(x) \Rightarrow \cos x + i\sin x = \sum_{n=-\infty}^{\infty}\left(e^{i\pi/2}\right)^n J_n(x) \Rightarrow$$

$$\cos x + i\sin x = \sum_{n=-\infty}^{\infty} e^{in\pi/2} J_n(x) \Rightarrow \cos x + i\sin x = \sum_{n=-\infty}^{\infty}\left(\cos(n\pi/2) + i\sin(n\pi/2)\right) J_n(x)$$

From the above relation we have that:

$$\left.\begin{aligned} \cos x &= \sum_{n=-\infty}^{\infty}\cos(n\pi/2) J_n(x) \\ \sin x &= \sum_{n=-\infty}^{\infty}\sin(n\pi/2) J_n(x) \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \cos x &= \sum_{n=-\infty}^{-1}\cos(n\pi/2) J_n(x) + J_0(x) + \sum_{n=1}^{\infty}\cos(n\pi/2) J_n(x) \\ \sin x &= \sum_{n=-\infty}^{-1}\sin(n\pi/2) J_n(x) + \sum_{n=1}^{\infty}\sin(n\pi/2) J_n(x) \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} \cos x &= \sum_{n=1}^{\infty}\cos(-n\pi/2) J_{-n}(x) + J_0(x) + \sum_{n=1}^{\infty}\cos(n\pi/2) J_n(x) \\ \sin x &= \sum_{n=1}^{\infty}\sin(-n\pi/2) J_{-n}(x) + \sum_{n=1}^{\infty}\sin(n\pi/2) J_n(x) \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} \cos x &= \sum_{n=1}^{\infty}\cos(n\pi/2)(-1)^n J_n(x) + J_0(x) + \sum_{n=1}^{\infty}\cos(n\pi/2) J_n(x) \\ \sin x &= \sum_{n=1}^{\infty}-\sin(n\pi/2)(-1)^n J_n(x) + \sum_{n=1}^{\infty}\sin(n\pi/2) J_n(x) \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned}\cos x = J_0(x) + \sum_{n=1}^{\infty}\left[1+(-1)^n\right]\cos\left(n\pi/2\right)J_n(x)\\ \sin x = \sum_{n=1}^{\infty}\sin\left(n\pi/2\right)[1-(-1)^n]J_n(x)\end{aligned}\right\} \Rightarrow \left.\begin{aligned}\cos x = J_0(x) - 2\sum_{n=1}^{\infty}\left(-1\right)^n J_{2n}(x)\\ \sin x = 2\sum_{n=1}^{\infty}\left(-1\right)^{n-1} J_{2n-1}(x)\end{aligned}\right\}$$

Be careful the quantity $\sin\left(n\pi/2\right)$ is equal to 1 if n=1, 5, 9 ,… and -1 if n =3, 7, 11, …

2. You are given that $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\sin x$ and $J_{-1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\cos x$. Find $J_{5/2}(x)$.

We know the recurrence relation:

$$J_{\nu-1}(x) + J_{\nu+1}\left(x\right) = \frac{2\nu}{x}J_\nu(x) \underset{\nu=3/2}{\Rightarrow} J_{\frac{3}{2}-1}(x) + J_{\frac{3}{2}+1}\left(x\right) = \frac{2\frac{3}{2}}{x}J_{\frac{3}{2}}(x) \Rightarrow$$

$$J_{1/2}(x) + J_{5/2}\left(x\right) = \frac{3}{x}J_{3/2}(x) \Rightarrow J_{5/2}\left(x\right) = \frac{3}{x}J_{3/2}(x) - J_{1/2}(x)$$

In the class we showed in the problem 2 of handout 6 that:

$$J_{3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x - x\cos x}{x}\right).$$

Thus we get:

$$J_{5/2}\left(x\right) = \frac{3}{x}J_{3/2}(x) - J_{1/2}(x) = \frac{3}{x}\sqrt{\frac{2}{\pi x}}\left(\frac{\sin x - x\cos x}{x}\right) - \sqrt{\frac{2}{\pi x}}\sin x =$$

$$\sqrt{\frac{2}{\pi x}}\left[\frac{3}{x}\left(\frac{\sin x - x\cos x}{x}\right) - \sin x\right] = \sqrt{\frac{2}{\pi x}}\left[\frac{3\sin x - 3x\cos x - x^2\sin x}{x^2}\right]$$

3. Use the result of problem 4 of handout 6 and calculate the integral $\int_0^1 xJ_{1/2}^2\left(\frac{\pi}{2}x\right)dx$.

Solution:

We have here that $\nu = 1/2$, $\lambda = \pi/2$. We also need the following quantities:

$$J_{1/2}(\pi/2)=\sqrt{\frac{2}{\pi\pi/2}}\sin\left(\pi/2\right)=0=\frac{2}{\pi}$$

$$J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x\Rightarrow J_{1/2}^{'}(x)=\sqrt{\frac{2}{\pi}}\frac{\sqrt{x}\cos x-(\sin x/2\sqrt{x})}{x}$$

thus

$$J_{1/2}^{'}(\pi/2)=\sqrt{\frac{2}{\pi}}\frac{\sqrt{\pi/2}\cos\left(\pi/2\right)-(\sin\left(\pi/2\right)/2\sqrt{\pi/2})}{\pi/2}=-\frac{\sqrt{2}}{\sqrt{\pi}}\frac{2}{\pi}\frac{1}{\sqrt{2\pi}}=-\frac{2}{\pi^2}$$

So the answer is:

$$\int_0^1 xJ_{1/2}^2\left(\pi x/2\right)dx=\frac{1}{2}\left[J_{1/2}^{'2}\left(\pi/2\right)+\left(1-\frac{(1/2)^2}{\left(\pi/2\right)^2}\right)J_{1/2}^2\left(\pi/2\right)\right]=$$

$$\frac{1}{2}\left[\left(-\frac{2}{\pi^2}\right)^2+\left(1-\frac{1}{\pi^2}\right)\left(\frac{2}{\pi}\right)^2\right]=\frac{1}{2}\left[\frac{4}{\pi^4}+\left(\frac{\pi^2-1}{\pi^2}\right)\frac{4}{\pi^2}\right]=\frac{1}{2}\frac{4}{\pi^4}\pi^2=\frac{2}{\pi^2}$$