

**PHYS 501**  
**2nd Midterm Exam - FALL 2019**  
**Tuesday 19<sup>th</sup> November 2019**

**Instructor: Prof. V. Lempesis**

**Duration: 3 hours**

*Please answer all questions*

1. A conducting wire along the z-axis carries a current  $I$ . The resulting magnetic vector potential in cylindrical coordinates is given by  $\mathbf{A} = \hat{\mathbf{k}} \frac{\mu I}{2\pi} \ln\left(\frac{1}{\rho}\right)$ . Show the magnetic induction  $\mathbf{B} = \vec{\nabla} \times \mathbf{A}$  is given by  $\mathbf{B} = \hat{\phi}_0 \frac{\mu I}{2\pi\rho}$ .

(5 marks)

**Solution:**

$$\mathbf{B} = \vec{\nabla} \times \mathbf{A} = \frac{1}{\rho} \begin{vmatrix} \hat{\rho}_0 & \rho \hat{\phi}_0 & \hat{\mathbf{k}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & A_\phi & A_z \end{vmatrix} =$$

$$\frac{1}{\rho} \begin{vmatrix} \hat{\rho}_0 & \rho \hat{\phi}_0 & \hat{\mathbf{k}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{\mu_0 I}{2\pi} \ln\left(\frac{1}{\rho}\right) \end{vmatrix} =$$

$$\frac{1}{\rho} \left\{ -\rho \hat{\phi}_0 \left[ \frac{\partial}{\partial \rho} \left( \frac{\mu_0 I}{2\pi} \ln\left(\frac{1}{\rho}\right) \right) \right] \right\} = -\hat{\phi}_0 \frac{\mu_0 I}{2\pi} \frac{\partial}{\partial \rho} \ln\left(\frac{1}{\rho}\right) =$$

$$-\frac{\mu_0 I}{2\pi} \frac{1}{\left(\frac{1}{\rho}\right)} \left(-\frac{1}{\rho^2}\right) \hat{\phi}_0 =$$

$$\frac{\mu_0 I}{2\pi\rho} \hat{\phi}_0$$

2. A certain force field is given (in spherical polar coordinates) by

$$\mathbf{F} = \hat{\mathbf{r}}_0 \frac{2P \cos\theta}{r^3} + \hat{\theta}_0 \frac{P}{r^3} \sin\theta, \quad r \geq P/2$$

Calculate  $\vec{\nabla} \times \mathbf{F}$ .

(5 marks)

**Solution:**

$$\vec{\nabla} \times F = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{\mathbf{r}}_0 & r\hat{\theta}_0 & r\sin\theta\hat{\phi}_0 \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{\mathbf{r}}_0 & r\hat{\theta}_0 & r\sin\theta\hat{\phi}_0 \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 2P\cos\theta/r^3 & P\sin\theta/r^2 & 0 \end{vmatrix}$$

$$\frac{1}{r^2 \sin \theta} \left\{ \hat{\mathbf{r}}_0 \left[ \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial \phi} \left( \frac{P\sin\theta}{r^2} \right) \right) - \frac{\partial}{\partial \phi} \left( \frac{\partial}{\partial r} \left( \frac{2P\cos\theta}{r^3} \right) \right) \right] - r\hat{\theta}_0 \left[ \frac{\partial}{\partial r} (0) - \frac{\partial}{\partial \phi} \left( \frac{2P\cos\theta}{r^3} \right) \right] + r\sin\theta\hat{\phi}_0 \left[ \frac{\partial}{\partial r} \left( \frac{P\sin\theta}{r^2} \right) - \frac{\partial}{\partial \theta} \left( \frac{2P\cos\theta}{r^3} \right) \right] \right\}$$

$$\frac{1}{r^2 \sin \theta} \left\{ Pr\sin\theta\hat{\phi}_0 \left[ -\frac{2\sin\theta}{r^3} + 2\frac{\sin\theta}{r^3} \right] \right\} = 0$$

3. Working in cylindrical coordinates we know for the unit vectors the following relations:

$$\hat{\rho}_0 = \mathbf{i} \cos \phi + \mathbf{j} \sin \phi, \quad \hat{\phi}_0 = -\mathbf{i} \sin \phi + \mathbf{j} \cos \phi, \quad \hat{\mathbf{k}} = \hat{\mathbf{k}}$$

Calculate the quantities:  $\frac{\partial \hat{\rho}_0}{\partial \phi}$ ,  $\frac{\partial \hat{\phi}_0}{\partial \phi}$ ,  $\frac{\partial \hat{\rho}_0}{\partial \rho}$ .

**Solution:**

$$\frac{\partial \hat{\rho}_0}{\partial \phi} = \frac{\partial}{\partial \phi} (\mathbf{i} \cos \phi + \mathbf{j} \sin \phi) = \mathbf{i} \frac{\partial \cos \phi}{\partial \phi} + \mathbf{j} \frac{\partial \sin \phi}{\partial \phi} = -\mathbf{i} \sin \phi + \mathbf{j} \cos \phi = \hat{\phi}_0$$

$$\frac{\partial \hat{\phi}_0}{\partial \phi} = \frac{\partial}{\partial \phi} (-\mathbf{i} \sin \phi + \mathbf{j} \cos \phi) = -\mathbf{i} \frac{\partial \sin \phi}{\partial \phi} + \mathbf{j} \frac{\partial \cos \phi}{\partial \phi} = -\mathbf{i} \cos \phi - \mathbf{j} \sin \phi = -\hat{\rho}_0$$

$$\frac{\partial \hat{\rho}_0}{\partial \rho} = 0$$

4. Evaluate the integral  $\oint \mathbf{r} \cdot d\mathbf{r}$  by Stoke's theorem.

**Solution:**

$$\oint \mathbf{V} \cdot d\mathbf{r} = \int_S (\vec{\nabla} \times \mathbf{V}) \cdot (\mathbf{n}dA)$$

Let  $\mathbf{V} = \mathbf{r}$  then we have

$$\oint \mathbf{r} \cdot d\mathbf{r} = \int_S (\vec{\nabla} \times \mathbf{r}) \cdot (\mathbf{n}dA)$$

but

$$\vec{\nabla} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} =$$

$$\mathbf{i} \left( \frac{\partial z}{\partial y} - \frac{\partial y}{\partial z} \right) - \mathbf{j} \left( \frac{\partial z}{\partial x} - \frac{\partial x}{\partial z} \right) + \mathbf{k} \left( \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} \right) = 0$$

So

$$\oint \mathbf{r} \cdot d\mathbf{r} = 0$$

**Mathematical Supplement:**

$$\vec{\nabla} \Phi = \frac{\partial \Phi}{\partial x} \mathbf{i} + \frac{\partial \Phi}{\partial y} \mathbf{j} + \frac{\partial \Phi}{\partial z} \mathbf{k}, \quad \vec{\nabla} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\vec{\nabla} \times \mathbf{F} = \frac{1}{\rho} \begin{vmatrix} \hat{\rho}_0 & \rho \hat{\phi}_0 & \hat{\mathbf{k}} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_\rho & F_\phi & F_z \end{vmatrix}$$

$$\nabla^2 f(x, y, z) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

**Mathematical Supplement:**

$$\vec{\nabla} \cdot \mathbf{V} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho V_\rho) + \frac{1}{\rho} \frac{\partial V_\varphi}{\partial \varphi} + \frac{\partial V_z}{\partial z}, \quad \vec{\nabla} \times \mathbf{V} = \frac{1}{\rho} \begin{vmatrix} \hat{\rho}_0 & \rho \hat{\varphi}_0 & \mathbf{k} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ V_\rho & \rho V_\varphi & V_z \end{vmatrix}$$

$$\oint \mathbf{V} \cdot d\mathbf{r} = \int_S (\vec{\nabla} \times \mathbf{V}) \cdot (n dA)$$

$$\vec{\nabla} \times \mathbf{V} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{\mathbf{r}}_0 & r \hat{\theta}_0 & r \sin \theta \hat{\phi}_0 \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ V_r & r V_\theta & r \sin \theta V_\phi \end{vmatrix}$$

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