PHYS 501
1st Midterm Exam - FALL 2019
Sunday 20th October 2019

Instructor: Prof. V. Lempesis
Duration: 2 hours

Please answer all questions

1. Show that $\vec{\nabla}(uv) = v\vec{\nabla}u + u\vec{\nabla}v$, where u and v are differentiable scalar functions of x, y, and z.

(5 marks)

Solution:

$$\vec{\nabla}(uv) = \mathbf{i}\frac{\partial}{\partial x}(uv) + \mathbf{j}\frac{\partial}{\partial y}(uv) + \mathbf{k}\frac{\partial}{\partial z}(uv) =$$

$$\mathbf{i}\left[u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial x}u\right] + \mathbf{j}\left[u\frac{\partial}{\partial y}v + v\frac{\partial}{\partial y}u\right] + \mathbf{k}\left[u\frac{\partial}{\partial z}v + v\frac{\partial}{\partial z}u\right] =$$

$$u\left[\mathbf{i}\left(\frac{\partial v}{\partial x}\right) + \mathbf{j}\left(\frac{\partial v}{\partial y}\right) + \mathbf{k}\left(\frac{\partial v}{\partial z}\right)\right] + v\left[\mathbf{i}\left(\frac{\partial u}{\partial x}\right) + \mathbf{j}\left(\frac{\partial u}{\partial y}\right) + \mathbf{k}\left(\frac{\partial u}{\partial z}\right)\right] = u\vec{\nabla}v + v\vec{\nabla}u$$

2. From the Navier-Stokes equation for the steady flow of an incompressible viscous fluid we have the term

$$\vec{\nabla}\times\left[\mathbf{v}\times\left(\vec{\nabla}\times\mathbf{v}\right)\right],$$

where **v** is the fluid velocity. Show that $\vec{\nabla}\times\left[\mathbf{v}\times\left(\vec{\nabla}\times\mathbf{v}\right)\right]$=0 when **v** =**i**$v(y, z)$.

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Comment [1]: The vector **v** has only x-component. Some of you considered that it has y- and z-component and that x-component is zero. This is absolutely wrong

(5 marks)

$$\vec{\nabla}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v(y,z) & 0 & 0 \end{vmatrix} = -\mathbf{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ v(y,z) & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ v(y,z) & 0 \end{vmatrix} =$$

$$\mathbf{j}\frac{\partial v(y,z)}{\partial z} - \mathbf{k}\frac{\partial v(y,z)}{\partial y}$$

$$\mathbf{v}\times(\vec{\nabla}\times\mathbf{v}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v(y,z) & 0 & 0 \\ 0 & \dfrac{\partial v(y,z)}{\partial z} & -\dfrac{\partial v(y,z)}{\partial y} \end{vmatrix} =$$

$$= -\mathbf{j}\begin{vmatrix} v(y,z) & 0 \\ 0 & -\dfrac{\partial v(y,z)}{\partial y} \end{vmatrix} + \mathbf{k}\begin{vmatrix} v(y,z) & 0 \\ 0 & \dfrac{\partial v(y,z)}{\partial z} \end{vmatrix} =$$

$$\mathbf{j}v(y,z)\frac{\partial v(y,z)}{\partial y} + \mathbf{k}v(y,z)\frac{\partial v(y,z)}{\partial z}$$

$$\vec{\nabla}\times[\mathbf{v}\times(\vec{\nabla}\times\mathbf{v})] = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & v(y,z)\dfrac{\partial v(y,z)}{\partial y} & v(y,z)\dfrac{\partial v(y,z)}{\partial z} \end{vmatrix} =$$

$$\mathbf{i}\begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v(y,z)\dfrac{\partial v(y,z)}{\partial y} & v(y,z)\dfrac{\partial v(y,z)}{\partial z} \end{vmatrix}$$

$$-\mathbf{j}\begin{vmatrix} \partial/\partial x & \partial/\partial z \\ 0 & v(y,z)\dfrac{\partial v(y,z)}{\partial z} \end{vmatrix}$$

$$+\mathbf{k}\begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ 0 & v(y,z)\dfrac{\partial v(y,z)}{\partial y} \end{vmatrix} =$$

$$\mathbf{i}\left[\frac{\partial}{\partial y}\left(v(y,z)\frac{\partial v(y,z)}{\partial z}\right) - \frac{\partial}{\partial z}\left(v(y,z)\frac{\partial v(y,z)}{\partial y}\right)\right]$$

$$-\mathbf{j}\frac{\partial}{\partial x}\left(v(y,z)\frac{\partial v(y,z)}{\partial z}\right) - \mathbf{k}\frac{\partial}{\partial x}\left(v(y,z)\frac{\partial v(y,z)}{\partial y}\right) =$$

$$\mathbf{i}\left[\frac{\partial}{\partial y}v(y,z)\frac{\partial v(y,z)}{\partial z}+v(y,z)\frac{\partial^2 v(y,z)}{\partial y\partial z}-\frac{\partial}{\partial z}v(y,z)\frac{\partial v(y,z)}{\partial y} - v(y,z)\frac{\partial^2 v(y,z)}{\partial z\partial y}\right]=0$$

3. Calculate the x-component of the vector $(\hat{\mathbf{r}}\cdot\vec{\nabla})\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is the unit vector $\hat{\mathbf{r}}=\mathbf{r}/\mathrm{r}$.

(5 marks)

Solution:

$$\hat{\mathbf{r}}=\frac{\mathbf{r}}{\mathrm{r}}=\frac{1}{\sqrt{x^2+y^2+z^2}}\{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\}$$

$$\hat{\mathbf{r}}\cdot\vec{\nabla}=\frac{1}{\sqrt{x^2+y^2+z^2}}\left\{x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right\}$$

Vasileios Lembessis 27/10/2019 13:25
Comment [2]: a) Not that this quantity is a scalar and not a vector. Some of you wrote this as a vector which is great mistake. B) Also some of you calculated the quantity $(\vec{\nabla}\cdot\hat{\mathbf{r}})$ which we do not need it.

The quantity $(\hat{\mathbf{r}}\cdot\vec{\nabla})\hat{\mathbf{r}}$ is a vector because $\hat{\mathbf{r}}$ is a vector and $(\hat{\mathbf{r}}\cdot\vec{\nabla})$ a scalar. The x-component of it is the following:

$$\left[(\hat{\mathbf{r}}\cdot\vec{\nabla})\hat{\mathbf{r}}\right]_x=\frac{1}{\sqrt{x^2+y^2+z^2}}\left\{x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right\}\underbrace{\left\{\frac{x}{\sqrt{x^2+y^2+z^2}}\right\}}_{x-component\ of\ \hat{\mathbf{r}}}=$$

$$\frac{1}{\sqrt{x^2+y^2+z^2}}\left\{x\frac{\partial}{\partial x}\left[\frac{x}{\sqrt{x^2+y^2+z^2}}\right]+y\frac{\partial}{\partial y}\left[\frac{x}{\sqrt{x^2+y^2+z^2}}\right] + z\frac{\partial}{\partial z}\left[\frac{x}{\sqrt{x^2+y^2+z^2}}\right]\right\}=$$

$$\frac{1}{\sqrt{x^2+y^2+z^2}}\left\{x\left[\frac{\sqrt{x^2+y^2+z^2}-x\frac{2x}{2\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}\right] + y\left[\frac{0-x\frac{2y}{2\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}\right]+z\left[\frac{0-x\frac{2z}{2\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}\right]\right\}=$$

$$\frac{1}{(x^2+y^2+z^2)^{1/2}}\left\{x\left[\sqrt{x^2+y^2+z^2}-x\frac{x}{\sqrt{x^2+y^2+z^2}}\right] - x\frac{y^2}{\sqrt{x^2+y^2+z^2}}-x\frac{z^2}{\sqrt{x^2+y^2+z^2}}\right\}=$$

$$\frac{x}{(x^2+y^2+z^2)^{1/2}}\left\{\left[\frac{x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}}-\frac{x^2}{\sqrt{x^2+y^2+z^2}}\right]-\frac{y^2}{\sqrt{x^2+y^2+z^2}}-\frac{z^2}{\sqrt{x^2+y^2+z^2}}\right\}=$$

$$\frac{x}{(x^2+y^2+z^2)^{3/2}}\{[y^2+z^2]-y^2-z^2\}=0$$

4. Calculate the Laplacian of the function $f(x,y,z)=x^2+2xy+3z+4$

(5 marks)

Solution:

$$\nabla^2 f(x,y,z) = \frac{\partial^2}{\partial x^2}(x^2+2xy+3z+4)+\frac{\partial^2}{\partial y^2}(x^2+2xy+3z+4) + \frac{\partial^2}{\partial z^2}(x^2+2xy+3z+4)=2+0+0=2$$

Mathematical Supplement:

$$\vec{\nabla}\Phi=\frac{\partial\Phi}{\partial x}\mathbf{i}+\frac{\partial\Phi}{\partial y}\mathbf{j}+\frac{\partial\Phi}{\partial z}\mathbf{k}, \quad \vec{\nabla}\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ F_x & F_y & F_z\end{vmatrix}$$

$$\nabla^2 f(x,y,z)=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$